Chapter 1

Incompressible flows

O. Thual, September 20, 2010

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Introduction

This chapter gives a summary of the essential steps of continuum mechanics which are needed to establish the incompressible Navier-Stokes equations

$$\operatorname{div} \underline{U} = 0 \qquad , \qquad \frac{d\underline{U}}{dt} = -\frac{1}{\rho}\underline{\operatorname{grad}}\, p + \underline{F} + \nu\, \Delta\underline{U}\,. \tag{1.1}$$

These equations come from the following ingredients

- the axioms of mechanics: mass and momentum conservation,
- the incompressibility constraint,
- the rheologic constitutive law of Newtonian fluids.

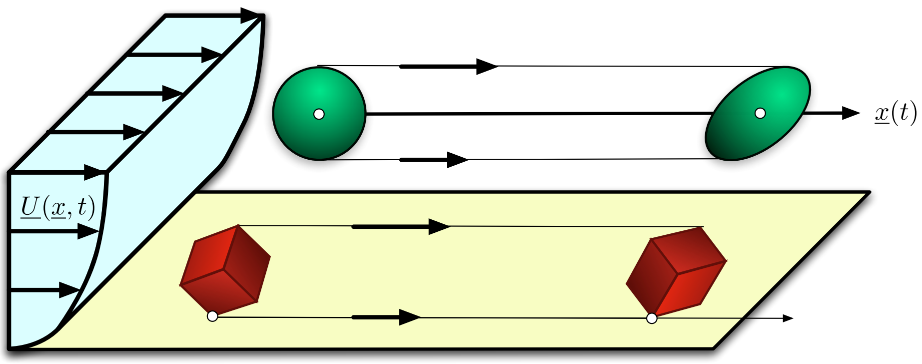

Figure **1**.1: *Particules motions in a parallel shear flow.*

The *axioms of mechanics* are expressed by saying that the mass of any volume of particules carried by the motion is constant and that the time derivative of its momentum is equal to the resultant of the applied forces. In order to obtain partial derivative equations out of these macroscopic principles, one has to derive triple integrals on domains which are varying with time while being carried by the flow. It is thus necessary to explore the "Lagrangian representation" of the considered fields in order to transform the moving domains integrals into fixed domains integrals which describe the initial positions of the moving particules.

The *incompressibility constraint* says that one can consider that the volume of any domain carried by the flow is constant. This is an approximation which is valid when the velocity of the flow velocity is small compared to the sound speed of the fluid. To express this constraint, which says that the divergence of the velocity field is zero, one needs to express the variation rate of the volumes as a function of the partial derivatives of this field.

The *rheologic constitutive laws of Newtonian fluids* are based on the fact that the contact forces depend on the pressure and on the spatial derivatives of the velocity constituting the "strain tensor" (deformation rates). One then needs to define the "stress tensor" in order to describe these contact forces. Its linear dependency with the normal to an elementary surface comes out from a general property of the surface forces involved in the momentum conservation law. The demonstration is classically obtained through the "tetrahedrons construction" which leads to the important concept of flux vector of scalar fields or of tensorial flux of vectorial fields.

In the present chapter, we try to explicit all the above concepts in the simplest manner as possible. Readers who already knows "continuum mechanics" will only have to acknowledge the set of notations before studying the examples developed at the end of the chapter.

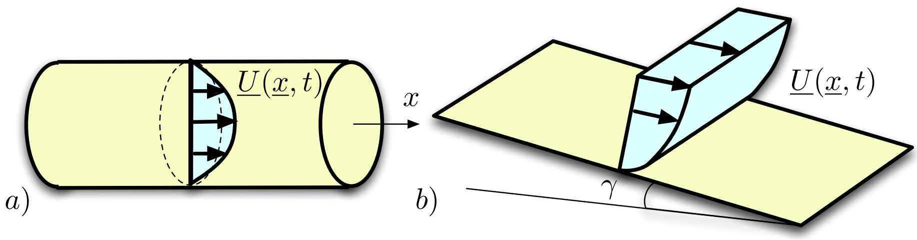

Figure **1**.2: *a) Circular Poiseuille flow in a pipe. b) Open Poiseuille flow on a tilted plane.*

In the last sections of this chapter, two examples of flows are presented. Beyond their interest as applications examples of the incompressible Navier-Stokes equations, these flows are here used to introduce the hydraulics of flows in pipes and the hydraulics of free surface flows in open channel flows. In both examples the important concept of "hydraulic head" is introduced. The relation between the "lineic head loss" and the "shear stress" due to the boundaries is put forward.

The first example, which introduces flows in pipes, is the circular Poiseuille in a pipe (Figure **1**.2a), forced by a pressure gradient. When the flow is slow enough, it is laminar and its mean velocity is proportional to the lineic head loss. This important property is at the root of the Darcy law which is valid for flows in porous media such as, for example, the subsurface flows.

The second example, which introduces free surface flows, is the plane Poiseuille flow in an open tilted channel (Figure **1**.2b). One introduces the friction coefficient which comes out of a dimensional analysis on the head loss. When the flow becomes turbulent, which is the case for realistic open channel flows, the parametrization of this friction coefficient is at the center of the modelling of these flows.

1 Kinematics

The motion of a continum media is defined by a velocity field $\underline{U}(\underline{x},t)$ or by the deformation family $\underline{X}(\underline{a},t)$ parametrized by the time. A first, Eulerian, point of vue, consists in expressing all the fields as functions of the points $\underline{x}$ of the distorted configuration. The Langrangian point of vue consists in expressing them as functions of the positions $\underline{a}$ of the particules in the reference configuration which is here the initial positions of the motion. We here express the mass conservation law in the two representations.

1.1 Eulerian ou Lagrangian

The Eulerian representation of a scalar field B is, simply, the function $(\underline{x},t) \mapsto B(\underline{x},t)$ which associates the scalar $B(\underline{x},t) \in I\!R$ to each point $\underline{x} = x\,\underline{e}_x + y\,\underline{e}_y + z\,\underline{e}_z \in I\!R^3$ and each time $t \in I\!R$.

The Eulerian representation $\underline{U}(\underline{x},t) = u\,\underline{e}_x + v\,\underline{e}_y + w\,\underline{e}_z \in I\!R^3$ of the velocity field of a flow is, similarly, a function of $I\!R^3 \times I\!R \to I\!R^3$. We also denote by $\underline{U} = (U_1, U_2, U_3)$ and $\underline{x} = (x_1, x_2, x_3)$ the components (u, v, w) and (x, y, z) when necessary.

The trajectories associated to the flow $\underline{U}(\underline{x},t)$ are time parametrized curves $t \mapsto \underline{x}(t)$ solutions of the equation $\frac{dx}{dt}(t) = \underline{U}[\underline{x}(t), t]$. Denoting by $\underline{x}(0) = \underline{a}$ the initial conditions of all the trajectories, one can construct the "Lagrangian description" of the flow $(\underline{a},t) \mapsto \underline{X}(\underline{a},t)$ where $\underline{X}(\underline{a},t) = \underline{x}(t)$ is the trajectory coming out the initial condition $\underline{x}(0) = \underline{a}$ (Figure **1**.3).

At each time t, on can consider the Jacobian matrix $\underline{\underline{F}}(\underline{a},t)$ of the function $\underline{a} \mapsto \underline{X}(\underline{a},t)$ which components are $F_{ij}(\underline{a},t) = \frac{\partial X_i}{\partial a_j}(\underline{a},t)$. On can thus write

$$\underline{X}(\underline{a} + \underline{\delta a}, t) = \underline{X}(\underline{a}) + \underline{\underline{F}}(\underline{a},t) \cdot \underline{\delta a} + \underline{O}(\|\underline{\delta a}\|^2) \ , \qquad (\mathbf{1}.2)$$

where $\underline{\delta a}$ is a small vector and $\underline{O}(\|\underline{\delta a}\|^2)$ an error of order two. By denoting $\underline{\delta x}(t) = \underline{X}(\underline{a} + \underline{\delta a}, t) - \underline{X}(\underline{a})$ the image at time t of the small vector $\underline{\delta a}$, one

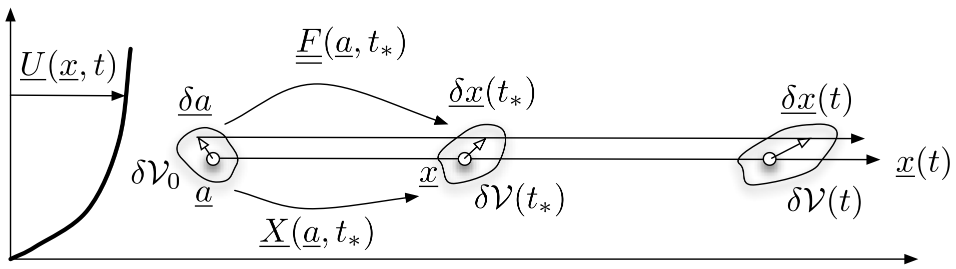

Figure **1**.3: *Trajectories* $\underline{x}(t) = \underline{X}(\underline{a}, t)$ *associated to the velocity* $\underline{U}(\underline{x}, t)$*. Small vector* $\underline{\delta x}(t)$ *carried by the flow with* $\underline{\delta x}(0) = \underline{\delta a}$ *and small volume* $\delta\mathcal{V}(t)$ *carried by the flow with* $\delta\mathcal{V}(0) = \delta\mathcal{V}_0$*.*

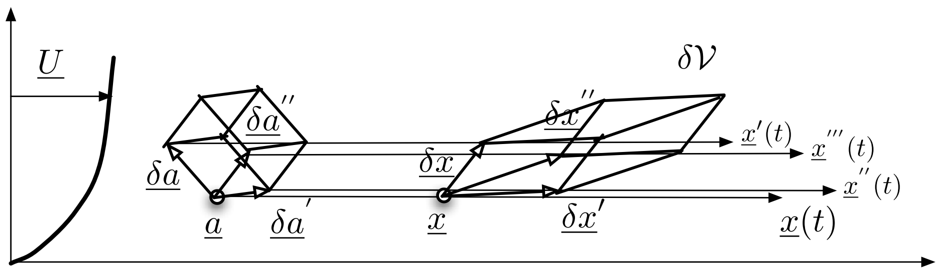

Figure **1**.4: *Deformation of small cube by the flow.*

can write

$$\underline{\delta x}(t) = \underline{\underline{F}}(\underline{a}, t) \cdot \underline{\delta a} + \underline{O}(\|\underline{\delta a}\|^2) \iff \delta x_i(t) = F_{ij}(\underline{a}, t)\, \delta a_j + O(\|\delta a\|^2)\ , \quad (\mathbf{1}.3)$$

where we have used the "Einstein summation convention" for which the $\sum_{j=1}^{3}$ symbol is omitted when the summation index j is repeated twice. One assimilates here the "order two tensor" $\underline{\underline{F}}$ and the matrix 3×3 of its components.

The variation of the volumes can be computed by considering three small vectors $\underline{\delta x}(t) = \underline{x}'(t) - \underline{x}(t)$, $\underline{\delta x}'(t) = \underline{x}''(t) - \underline{x}(t)$ and $\underline{\delta x}''(t) = \underline{x}'''(t) - \underline{x}(t)$ carried by the motion (figure **1**.4), which means that $\underline{x}(t')$, $\underline{x}''(t)$ and $\underline{x}''(t)$ are trajectories close to the trajectory $\underline{x}(t)$. They form a parallelepiped which volume $\delta\mathcal{V}$ is the "mixed product" of these three vectors is defined by

$$\delta\mathcal{V}(t) = \Big(\underline{\delta x}(t), \underline{\delta x}'(t), \underline{\delta x}''(t)\Big) = \begin{vmatrix} \delta x_1 & \delta x_1' & \delta x_1'' \\ \delta x_2 & \delta x_2' & \delta x_2'' \\ \delta x_3 & \delta x_3' & \delta x_3'' \end{vmatrix}\ , \quad (\mathbf{1}.4)$$

which is the determinant of the matrix made with the components of the three small vectors. If $\underline{\delta x}(0) = \delta a\, \underline{e}_x$, $\underline{\delta x}'(0) = \delta a\, \underline{e}_y$ and $\underline{\delta x}''(0) = \delta a\ \underline{e}_z$ at

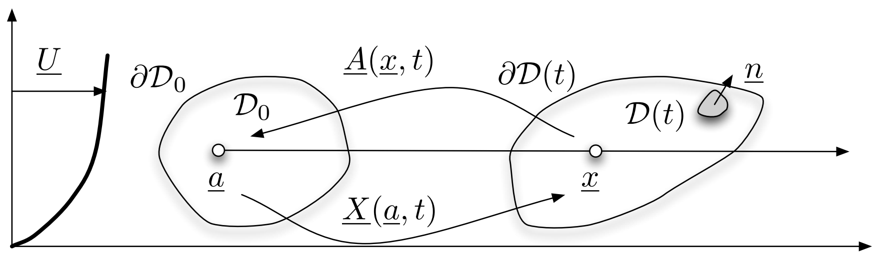

Figure **1**.5: *Domain $\mathcal{D}(t)$ carried by the flow with $\mathcal{D}(0) = \mathcal{D}_0$.*

$t = 0$, where δa is a small length, the parallelepiped is a cube at $t = 0$ which volume is $\delta\mathcal{V}(0) = (\delta a)^3 \left(\underline{e}_x, \underline{e}_y, \underline{e}_z\right) = (\delta a)^3$. Since $\underline{\delta x}(t) = \underline{\underline{F}}(\underline{a},t) \cdot \underline{\delta x}(0)$, $\underline{\delta x}'(t) = \underline{\underline{F}}(\underline{a},t) \cdot \underline{\delta x}'(0)$ and $\underline{\delta x}''(t) = \underline{\underline{F}}(\underline{a},t) \cdot \underline{\delta x}''(0)$, with an error of only $\underline{O}[(\delta a)^2]$, one can write

$$\begin{aligned}
\delta\mathcal{V}(t) &= \left(\underline{\underline{F}} \cdot \underline{\delta x}(0),\ \underline{\underline{F}} \cdot \underline{\delta x}'(0),\ \underline{\underline{F}} \cdot \underline{\delta x}''(0)\right) = (\delta a)^3 \left(\underline{\underline{F}} \cdot \underline{e}_x,\ \underline{\underline{F}} \cdot \underline{e}_y,\ \underline{\underline{F}} \cdot \underline{e}_z\right) \\
&= (\delta a)^3 \begin{vmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{vmatrix} = (\delta a)^3 \det \underline{\underline{F}} = \delta\mathcal{V}(0)\ J(\underline{a},t)\ , \qquad (\mathbf{1}.5)
\end{aligned}$$

where $J(\underline{a},t) = \det \underline{\underline{F}}(\underline{a},t)$ is the "Jacobian determinant" of the application $\underline{x} \mapsto \underline{X}(\underline{a},t)$ at time t for which one assumes $\det \underline{\underline{F}}(\underline{a},t) > 0$.

The relation $\delta\mathcal{V}(t) = \delta\mathcal{V}_0\, J(\underline{a},t)$, proven in the cas of a small cube, can be easily generalized to the cas of small volumes of arbitrary shape by cutting them into even smaller cubes.

For realistic flows, one has $J(\underline{a},t) > 0$ which implies that the inverse $\underline{a} = \underline{A}(\underline{x},t)$ of $\underline{x} = \underline{X}(\underline{a},t)$ at time t exists to determine the initial condition $\underline{a}$ of a trajectory which position $\underline{x}$ is known at time t. With this family of trajectories associated to the velocity field $\underline{U}(\underline{x},t)$, one can define the Lagrangian representation $B^{(L)}(\underline{a},t)$ of the field B through the relation

$$B(\underline{x},t) = B^{(L)}[\underline{A}(\underline{x},t),t] \quad\Longleftrightarrow\quad B[\underline{X}(\underline{a},t),t] = B^{(L)}(\underline{a},t)\ , \qquad (\mathbf{1}.6)$$

which is to say that $B(\underline{x},t) = B^{(L)}(\underline{a},t)$ when $\underline{x} = \underline{X}(\underline{a},t)$ or, equivalently, $\underline{a} = \underline{A}(\underline{x},t)$.

Given a set $\mathcal{D}_0$ of particules $\underline{a}$ at time $t = 0$, let's denote $\mathcal{D}(t) = \underline{X}[\mathcal{D}_0,t]$ the set of their images $\underline{x} = \underline{X}(\underline{a},t)$ at time t (Figure **1**.5). The change of variable $\underline{x} = \underline{X}(\underline{a},t)$ for a triple integral on moving domain $\mathcal{D}(t)$ reads

$$\iiint_{\mathcal{D}(t)} B(\underline{x},t)\ d^3x = \iiint_{\mathcal{D}_0} B^{(L)}(\underline{a},t)\ J(\underline{a},t)\ d^3a\ , \qquad (\mathbf{1}.7)$$

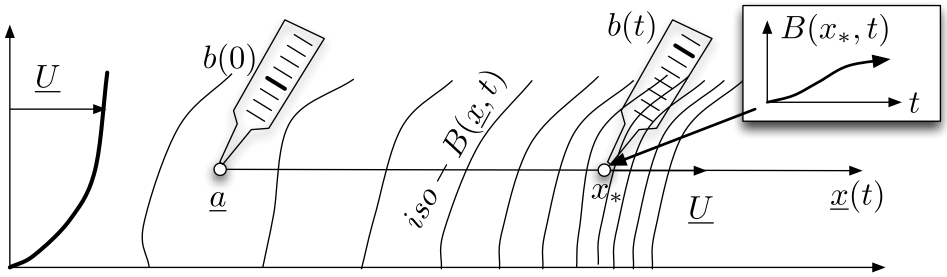

Figure **1**.6: *Function $b(t)$ obtained by measuring B along the trajectory $\underline{x}(t)$.*

where $d^3x = J(\underline{a},t)\, d^3a$ is the integration volume in $\mathcal{D}(t)$ which can be seen as the image of the integration volume d^3a in $\mathcal{D}_0$.

The mass conservation of a fluid of velocity field $\underline{U}(\underline{x},t)$ states that the quantity

$$m[\mathcal{D}(t)] = \iiint_{\mathcal{D}(t)} \rho(\underline{x},t)\; d^3x = \iiint_{\mathcal{D}_0} \rho^{(L)}(\underline{a},t)\; J(\underline{a},t)\; d^3a\;, \qquad (\mathbf{1}.8)$$

where ρ is the mass density, is independent of time. If $\rho[\underline{X}(\underline{a},t),0] = \rho^{(L)}(\underline{a},0) = \rho_0$ is uniform at time $t=0$, the mass conservation principle imposes that the mass density satisfies the relation

$$\rho^{(L)}(\underline{a},t)\; J(\underline{a},t) = \rho_0 \qquad (\mathbf{1}.9)$$

for all time t. This Lagrangian formulation of the mass conservation principle will be completed by an Eulerian formulation in the following.

1.2 Particular derivative

Given a scalar field $B(\underline{x},t)$ and a trajectory $\underline{x}(t)$, associated to the velocity field $\underline{U}(\underline{x},t)$ with $\underline{x}(0) = \underline{a}$, one can consider the function

$$b(t) = B^{(L)}(\underline{a},t) = B[\underline{x}(t),t] \qquad (\mathbf{1}.10)$$

which describes the time evolution of the quantity B seen by a particule which follows the trajectory $\underline{x}(t)$ (Figure **1**.6). Since $\frac{d\underline{x}}{dt}(t) = \underline{U}[\underline{x}(t),t]$, on can write

$$\frac{db}{dt}(t) = \frac{\partial B^{(L)}}{\partial t}(\underline{a},t) = \left(\frac{\partial B}{\partial t} + \underline{U}\cdot \underline{\text{grad}}\; B\right)[\underline{x}(t),t]\;. \qquad (\mathbf{1}.11)$$

One can then define the "total derivative" $\frac{dB}{dt}$ of a field B by its Lagrangian representation $\left(\frac{dB}{dt}\right)^{(L)}(\underline{a},t) = \frac{\partial B^{(L)}}{\partial t}(\underline{a},t)$ or, equivalently, by its Eulerian representation

$$\frac{dB}{dt}(\underline{x},t) = \frac{\partial B}{\partial t}(\underline{x},t) + \underline{U}(\underline{x},t)\cdot \underline{\mathrm{grad}}\ B(\underline{x},t)\ . \tag{1.12}$$

Similarly, the total derivative of a vector field $\underline{V}(\underline{x},t)$ is defined by its components

$$\frac{dV_i}{dt} = \frac{\partial V_i}{\partial t} + \underline{U}\cdot\underline{\mathrm{grad}}\ V_i = \frac{\partial V_i}{\partial t} + U_j\,\frac{\partial V_i}{\partial x_j} \tag{1.13}$$

which can also read, by using the differential operator $\underline{U}\cdot\underline{\mathrm{grad}}\ = U_j\,\frac{\partial}{\partial x_j}$:

$$\frac{d\underline{V}}{dt} = \frac{\partial \underline{V}}{\partial t} + (\underline{U}\cdot\underline{\mathrm{grad}}\ )\ \underline{V} = \frac{\partial \underline{V}}{\partial t} + \underline{U}\cdot\underline{\mathrm{grad}}\ \underline{V}\ . \tag{1.14}$$

The acceleration field $\underline{\Gamma}(\underline{x},t)$ is defined as the total derivative of the velocity field $\underline{U}$. Using some algebra, on can show the useful identity

$$\underline{\Gamma} = \frac{d\underline{U}}{dt} = \frac{\partial \underline{U}}{\partial t} + \underline{U}\cdot\underline{\mathrm{grad}}\ \underline{U} = \frac{\partial \underline{U}}{\partial t} + \frac{1}{2}\underline{\mathrm{grad}}\ \underline{U}^2 + \underline{\mathrm{rot}}\ \underline{U}\wedge\underline{U}\ . \tag{1.15}$$

where $\underline{\mathrm{rot}}\ \underline{U}$ is the vorticity of the flow.

1.3 Deformation rates

The gradient of the velocity field $\underline{U}(\underline{x},t)$ is the Jacobian matrix $\underline{\underline{K}} = \underline{\mathrm{grad}}\ \underline{U}$ of the application $\underline{x}\mapsto\underline{U}(\underline{x},t)$ at time t. Its components read $K_{ij} = \frac{\partial U_i}{\partial x_j}$ and one can write

$$\underline{U}(\underline{x}+\underline{\delta x},t) = \underline{U}(\underline{x},t) + \underline{\underline{K}}(\underline{x},t)\cdot\underline{\delta x} + \underline{O}(\|\underline{\delta x}\|^2)\ , \tag{1.16}$$

where $\underline{\delta x}$ is a small vector and $\underline{O}(\|\underline{\delta x}\|^2)$ an order 2 error. If one considers two trajectories $\underline{x}(t)$ and $\underline{x}'(t)$ such that $\underline{\delta x}(t) = \underline{x}'(t) - \underline{x}(t)$ is a small vector (figure **1**.7), one can write

$$\frac{d}{dt}\left[\underline{\delta x}(t)\right] = \underline{U}[\underline{x}'(t),t] - \underline{U}[\underline{x}(t),t] = \underline{\underline{K}}[\underline{x}(t),t]\cdot\underline{\delta x}(t) + \underline{O}[\|\underline{\delta x}(t)\|^2]\ . \tag{1.17}$$

The decomposition $\underline{\underline{K}} = \underline{\underline{\omega}} + \underline{\underline{d}}$ into its antisymmetric part $\underline{\underline{\omega}}$ and its symmetric part $\underline{\underline{d}}$ leads to

$$\underline{U}(\underline{x}',t) - \underline{U}(\underline{x},t) = \underline{\underline{\omega}}(\underline{x},t)\cdot\underline{\delta x} + \underline{\underline{d}}\cdot\underline{\delta x} + \underline{O}(\|\underline{\delta x}\|^2) \tag{1.18}$$

if $\underline{x}'$ and $\underline{x}$ are separated by a small vector $\underline{\delta x} = \underline{x}' - \underline{x}$.

For any antisymmetric matrix $\underline{\underline{\omega}}$, there exists a unique vector $\underline{\omega}$ such that $\underline{\underline{\omega}}\cdot\underline{\delta x} = \underline{\omega}\wedge\underline{\delta x}$ for any vector $\underline{\delta x}$. Its components satisfy $\omega_i + \omega_{jk} = 0$ if

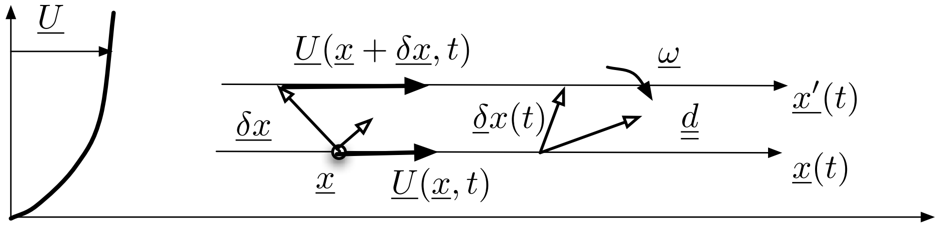

Figure **1**.7: *Comparison of the velocities of two close trajectories.*

the triplet (i,j,k) is a direct permutation of $(1,2,3)$. For the present case $\omega_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right)$, one shows that $\underline{\omega} = \frac{1}{2}\,\underline{\mathrm{rot}}\;\underline{U}$. By writing

$$\underline{U}(\underline{x}',t) = \underline{U}(\underline{x},t) + \underline{\omega}(\underline{x},t) \wedge (\underline{x}' - \underline{x}) + \underline{\underline{d}} \cdot \delta\underline{x} + \underline{O}(\|\delta\underline{x}\|^2) \;, \tag{1.19}$$

one sees that $\underline{\underline{\omega}}$, called the "rotation rate tensor", describes a rotation solid motion in the vicinity of $\underline{x}$ which rotation vector is, by definition, $\underline{\omega} = \frac{1}{2}\underline{\mathrm{rot}}\;\underline{U}$.

The symmetric part $\underline{\underline{d}}$, which components are $d_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$, is called the "deformation rate tensor" or the "strain tensor". It is such that

$$\frac{d}{dt}\left[\delta\underline{x}(t) \cdot \delta\underline{x}'(t)\right] = 2\,\delta\underline{x}(t) \cdot \underline{\underline{d}}[\underline{x}(t),t] \cdot \delta\underline{x}'(t) = 2\,\delta x_i(t)\, d_{ij}[\underline{x}(t),t]\,\delta x'_j(t) \;, \tag{1.20}$$

where $\delta\underline{x}(t) = \underline{x}'(t) - \underline{x}(t)$ and $\delta\underline{x}'(t) = \underline{x}''(t) - \underline{x}(t)$ are small vectors describing the separations between the three trajectories $\underline{x}(t)$, $\underline{x}'(t)$ and $\underline{x}''(t)$. Thus, the deformation rate tensor describes the variation rate of the distances and the angles of small vectors carried by the motion.

The rate of variation of the volumes car be computed by considering three small vectors $\delta\underline{x}(t) = \underline{x}'(t) - \underline{x}(t)$, $\delta\underline{x}'(t) = \underline{x}''(t) - \underline{x}(t)$ et $\delta\underline{x}''(t) = \underline{x}'''(t) - \underline{x}(t)$ carried by the motion (Figure **1**.8). They form a parallelepiped which volume is $\delta\mathcal{V}(t) = \left(\delta\underline{x}(t), \delta\underline{x}'(t), \delta\underline{x}''(t)\right)$.

If $\delta\underline{x}(t_*) = \delta x\;\underline{e}_x$, $\delta\underline{x}'(t_*) = \delta x\;\underline{e}_y$ and $\delta\underline{x}''(t_*) = \delta x\;\underline{e}_z$ at $t = t_*$, where δx is a small length, the parallelepiped is a cube at $t = t_*$ which volume is $\delta\mathcal{V}(t_*) = (\delta x)^3\left(\underline{e}_x, \underline{e}_y, \underline{e}_z\right) = (\delta x)^3$. Since $\frac{d}{dt}\delta\underline{x}(t_*) = \underline{\underline{K}}[\underline{x}(t_*),t_*] \cdot \delta\underline{x}(t_*)$, $\frac{d}{dt}\delta\underline{x}'(t_*) = \underline{\underline{K}}[\underline{x}(t_*),t_*] \cdot \delta\underline{x}'(t_*)$ and $\frac{d}{dt}\delta\underline{x}''(t_*) = \underline{\underline{K}}[\underline{x}(t_*),t_*] \cdot \delta\underline{x}''(t_*)$ with an error of only $\underline{O}[(\delta x)^2]$, one can write

$$\begin{aligned}\frac{d}{dt}\delta\mathcal{V}(t_*) &= \left(\underline{\underline{K}} \cdot \delta\underline{x}, \delta\underline{x}', \delta\underline{x}''\right) + \left(\delta\underline{x}, \underline{\underline{K}} \cdot \delta\underline{x}', \delta\underline{x}''\right) + \left(\delta\underline{x}, \delta\underline{x}', \underline{\underline{K}} \cdot \delta\underline{x}''\right) \\ &= (\delta x)^3\left[\begin{vmatrix} K_{11} & 0 & 0 \\ K_{21} & 1 & 0 \\ K_{31} & 0 & 1\end{vmatrix} + \begin{vmatrix} 1 & K_{12} & 0 \\ 0 & K_{22} & 0 \\ 0 & K_{32} & 1\end{vmatrix} + \begin{vmatrix} 1 & 0 & K_{13} \\ 0 & 1 & K_{23} \\ 0 & 0 & K_{33}\end{vmatrix}\right]\end{aligned}$$

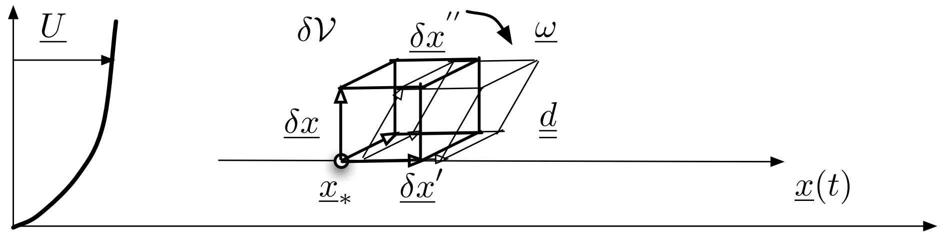

Figure 1.8: *Deformation of a small cube between two closed times.*

$$= (\delta x)^3 \, (K_{11} + K_{22} + K_{33}) = \delta\mathcal{V}(t_*) \text{ div } \underline{U}[\underline{x}(t_*), t_*] \; . \tag{1.21}$$

The relation $\frac{d}{dt}\delta\mathcal{V}(t) = \delta\mathcal{V}(t) \,\text{div}\, \underline{U}[\underline{x}(t), t]$, proven in the cas of a small cube, can be easily generalized to the cas of small volumes of arbitrary shape by cutting them into even smaller cubes.

One then sees that the divergence div $\underline{U}$ of the velocity field is equal to the relative deformation rate of the volumes. In particular, if the volume of any particules domain is conserved by the flow, one has div $\underline{U}(\underline{x}, t) = 0$ for all points $\underline{x}$ and at each time t. One then says that the flow is isochoric.

2 Conservation laws

The "little thetrahedrons demonstration" shows that the surface flux involved in a conservation law is linearly dependant of the unit vector normal to the boundary of the considered mouving domain. For the momentum conservation law, this linear dependancy of the surface contact forces defines the stress tensor.

2.1 Transport theorems

A conservation law for a field $c(\underline{x}, t)$ is the statement saying that for each domain $\mathcal{D}(t) = \underline{X}(\mathcal{D}_0, t)$ of particules of trajectories $\underline{x}(t) = \underline{X}(\underline{a}, t)$, with $\mathcal{D}(0) = \mathcal{D}_0$, thus carried by the velocity field $\underline{U}$, one can write

$$\frac{d}{dt} \iiint_{\mathcal{D}(t)} c(\underline{x}, t) \; d^3x + \iint_{\partial\mathcal{D}(t)} q_c(\underline{x}, \underline{n}, t) \, dS = \iiint_{\mathcal{D}(t)} f_c(\underline{x}, t) \, d^3x \; , \tag{1.22}$$

where $\partial\mathcal{D}(t)$ is the boundary of $\mathcal{D}(t)$, $\underline{n}$ its normal at $(\underline{x}, t)$ pointing towards the exterior, $q_c(\underline{x}, \underline{n})$ a field representing the "outgoing surface flux" of c and

$f_c(\underline{x}, t)$ a field representing the "internal production" of c (Figure **1**.9).

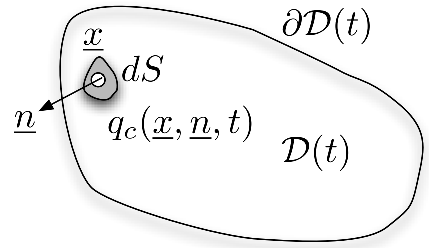

Figure **1**.9: *Outgoing surface flux* $q_c(\underline{x}, \underline{n}, t)$*, outgoing normal* $\underline{n}$ *and border* $\partial\mathcal{D}$ *of the domain* $\mathcal{D}$*.*

To compute the first term of this generic conservation law, one can use the change variable $\underline{x} = \underline{X}(\underline{a}, t)$ and write

$$\begin{aligned} \frac{d}{dt} \iiint_{\mathcal{D}(t)} c(\underline{x}, t)\, d^3x &= \frac{d}{dt} \iiint_{\mathcal{D}_0} c^{(L)}(\underline{a}, t)\, J(\underline{a}, t)\, d^3a \\ &= \iiint_{\mathcal{D}_0} \left(\frac{\partial c^{(L)}}{\partial t}\, J + c^{(L)}\, \frac{\partial J}{\partial t} \right) (\underline{a}, t)\, d^3a\,. \qquad (\mathbf{1}.23) \end{aligned}$$

To compute $\frac{\partial J}{\partial t}(\underline{a}, t)$, one can combine the relations $\delta\mathcal{V}(t)/\delta\mathcal{V}(0) = J(\underline{a}, t)$ and $\frac{d\mathcal{V}}{dt}(t)/\mathcal{V}(t) = \text{div}\ \underline{U}[\underline{x}(t), t]$ to obtain $\frac{\partial J}{\partial t}(\underline{a}, t)/J(\underline{a}, t) = \text{div}\ \underline{U}[\underline{X}(\underline{a}, t), t]$. By recognizing that $\left(\frac{dc}{dt}\right)^{(L)}(\underline{a}, t) = \frac{\partial c^{(L)}}{\partial t}(\underline{a}, t)$ is the Lagrangian representation of the total derivative $\frac{dc}{dt}$, one can write

$$\begin{aligned} \frac{d}{dt} \iiint_{\mathcal{D}(t)} c(\underline{x}, t)\, d^3x &= \iiint_{\mathcal{D}_0} \left[\left(\frac{dc}{dt} \right)^{(L)} + c^{(L)}\ (\text{div}\ \underline{U})^{(L)} \right] J(\underline{a}, t)\, d^3a \\ &= \iiint_{\mathcal{D}(t)} \left(\frac{dc}{dt} + c\ \text{div}\ \underline{U} \right) (\underline{x}, t)\, d^3x\,. \qquad (\mathbf{1}.24) \end{aligned}$$

By using the relations $\frac{dc}{dt} = \frac{\partial c}{\partial t} + \underline{U} \cdot \underline{\text{grad}}\ c$, $\underline{U} \cdot \underline{\text{grad}}\ c + c\ \text{div}\ \underline{U} = \text{div}\ (c\ \underline{U})$ as well as the divergence theorem, on can write

$$\frac{d}{dt} \iiint_{\mathcal{D}(t)} c\, d^3x = \iiint_{\mathcal{D}(t)} \frac{\partial c}{\partial t}\, d^3x + \iint_{\partial\mathcal{D}(t)} c\ \underline{U} \cdot \underline{n}\ dS\,, \qquad (\mathbf{1}.25)$$

where $c\ \underline{U}$ is called the "kinematic flux" of the quantity c.

2.2 Fluxes and local budget

We now show that the "outgoing surface flux" $q_c(\underline{x}, \underline{n}, t)$ which appears in a conservation law must obey to a very strong property: a linear dependency with the unit vector $\underline{n}$.

Indeed, since the conservation law is valid for all the domains $\mathcal{D}(t)$, one can write that $\frac{1}{\mathcal{V}[\mathcal{D}(t)]}\iint_{\partial\mathcal{D}(t)} q_c(\underline{x},\underline{n},t)\,dS$ is bounded when the volume $\mathcal{V}[\mathcal{D}(t)]$ is taken arbitrarily small.

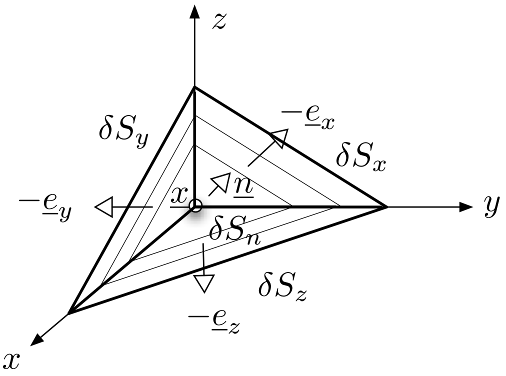

Figure **1**.10: *Tetrahedron used for demonstrating the linearity of the flux, with:* $\underline{n} = (\delta S_x\,\underline{e}_x + \delta S_y\,\underline{e}_x + \delta S_z\,\underline{e}_x)\,/\delta S_n$

By applying this property to a family of shrinking tetrahedrons (Figure **1**.10), one can show that the dependency of $q_c(\underline{x},\underline{n},t)$ with $\underline{n}$ is linear and deduce that there exists a "flux vector" $\underline{Q}_c(\underline{x},t)$ such that

$$q_c(\underline{x},\underline{n},t) = \underline{Q}_c(\underline{x},t)\cdot\underline{n}\;. \qquad (\mathbf{1}.26)$$

By applying the divergence theorem, the conservation law now writes

$$\frac{d}{dt}\iiint_{\mathcal{D}(t)} c(\underline{x},t)\,d^3x + \iint_{\partial\mathcal{D}(t)} \underline{Q}_c(\underline{x},t)\cdot\underline{n}\,dS = \iiint_{\mathcal{D}(t)} f_c(\underline{x},t)\,d^3x\;. \qquad (\mathbf{1}.27)$$

Using the results obtained previously for the first term of this relation, the conversation law also read

$$\iiint_{\mathcal{D}(t)} \frac{\partial c}{\partial t}\,d^3x + \iint_{\partial\mathcal{D}(t)} \left(c\,\underline{U} + \underline{Q}_c\right)\cdot\underline{n}\,dS = \iiint_{\mathcal{D}(t)} f_c\,d^3x\;, \qquad (\mathbf{1}.28)$$

where the kinematic flux $c\,\underline{U}$ and the conservation law flux $\underline{Q}_c$ add together. Since the conservation law is true for all the domains $\mathcal{D}$, in particular arbitrarily small ones, on can write the local budget

$$\frac{\partial c}{\partial t} + \text{div}\,(c\,\underline{U} + \underline{Q}_c) = \frac{dc}{dt} + c\,\text{div}\,\underline{U} + \text{div}\,\underline{Q}_c = f_c\;. \qquad (\mathbf{1}.29)$$

The mass conservation is a simple example of law with $c(\underline{x},t) = \rho(\underline{x},t)$, $\underline{Q}_c = \underline{0}$ and $f_c = 0$. The local budget then reads

$$\frac{\partial\rho}{\partial t} + \text{div}\,(\rho\,\underline{U}) = \frac{d\rho}{dt} + \rho\,\text{div}\,\underline{U} = 0\;. \qquad (\mathbf{1}.30)$$

On sees that isochoric flows (div $\underline{U} = 0$) correspond to fluids considered as incompressible ($\frac{d\rho}{dt} = 0$) and inversly. For iso-volume flows, encountered in incompressible flows, one has $\frac{d\rho}{dt} = 0$ since div $\underline{U} = 0$. In this case, if $\rho(\underline{x}, 0) = \rho_0$ is homogeneous at $t = 0$, one has $\rho = \rho_0$ for all times.

2.3 Momentum

The first part of the momentum conversation law states that

$$\frac{d}{dt} \iiint_{\mathcal{D}(t)} \rho\, \underline{U}\, d^3x - \iint_{\partial\mathcal{D}(t)} \underline{T}(\underline{x}, \underline{n}, t)\, dS = \iiint_{\mathcal{D}} \rho\, \underline{F}\, d^3x\ , \qquad (\mathbf{1}.31)$$

where $\underline{T}(\underline{x}, \underline{n}, t)$ is the surface density of the contact forces applied on the boundary $\partial\mathcal{D}$ (Figure **1**.11), by the exterior of the domain $\mathcal{D}$, and $\rho\, \underline{F}(\underline{x}, t)$ the density of the volume forces. Contact forces $\underline{T}$ represent interactions on distances smaller than the scale of the continuous description of the fields. Volume forces $\rho\, \underline{F}$ represent interactions at the macroscopic continuous scale. Most of the time, the volume forces are reduced to the gravity forces $\underline{F} = \underline{g}$ where $\underline{g}$ is the gravity vector.

As for scalar conservation laws, the tetrahedrons demonstration implies that $\underline{T}(\underline{x}, \underline{n}, t)$ depends linearly of $\underline{n}$. Thus, there exists a tensor $\underline{\underline{\sigma}}(\underline{x}, t)$, called the "strain tensor", which allows to write

$$\underline{T}(\underline{x}, \underline{n}, t) = \underline{\underline{\sigma}}(\underline{x}, t) \cdot \underline{n}\ . \qquad (\mathbf{1}.32)$$

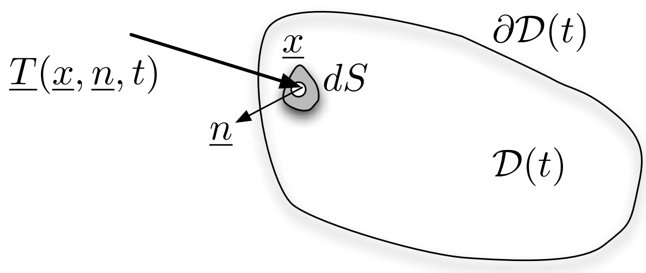

Figure **1**.11: *Surface forces applied on the boundary $\mathcal{D}(t)$ by its exterior.*

Assuming that the mass conservation is satisfied, one can show that the momentum conservation law reads

$$\iiint_{\mathcal{D}(t)} \rho\, \frac{d\underline{U}}{dt}\, d^3x - \iint_{\partial\mathcal{D}(t)} \underline{\underline{\sigma}} \cdot \underline{n}\, dS = \iiint_{\mathcal{D}} \rho\, \underline{F}\, d^3x\ . \qquad (\mathbf{1}.33)$$

By applying the divergence theorem and the mass conservation, the local budget for the momentum is then is

$$\rho \frac{d\underline{U}}{dt} = \rho \left(\frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \underline{\mathrm{grad}}\ \underline{U} \right) = \rho\, \underline{F} + \underline{\mathrm{div}}\ \underline{\underline{\sigma}}\ , \qquad (1.34)$$

where the i^{th} component of $\underline{\mathrm{div}}\ \underline{\underline{\sigma}}$ is $\frac{\partial \sigma_{ij}}{\partial x_j}$ in which the Einstein summation convention is used (summation on the repeated index j).

The second part of the momentum conservation laws states that the integral of angular momentum $-\rho\, \underline{x} \wedge \underline{U}$ is associated to the output flux $\underline{x} \wedge \underline{T}$ and the production term $\rho\, \underline{x} \wedge \underline{F}$. Combined with the firt part of the laws, algebraic manipulation shows that this statement reduced to the fact that $\underline{\underline{\sigma}}$ est symmetric.

3 Newtonian fluids

The Navier-Stokes equations are obtained by introducing the rheological law of Newtonian fluids in the momentum conservation law and by considering the mass conversation law which we suppose here to be the one of the incompressible flows. We then detail two examples of applications which are the circular or plane Poiseuille flows.

3.1 Navier-Stokes equations

The rheologic law of a Newtonian fluid reads

$$\underline{\underline{\sigma}} = -p\, \underline{\underline{I}} + \underline{\underline{\tau}} = -p\, \underline{\underline{I}} - \frac{2\,\mu}{3}\, \mathrm{div}\, \underline{U}\ \underline{\underline{I}} + 2\,\mu\, \underline{\underline{d}}\ , \qquad (1.35)$$

where $\underline{\underline{I}}$ is the unit tensor, $p(\underline{x}, t)$ the pressure, $\underline{\underline{\tau}}(\underline{x}, t) = -\frac{2\mu}{3}\, \mathrm{div}\, \underline{U}\ \underline{\underline{I}} + 2\,\mu\, \underline{\underline{d}}$ the viscous stress tensor and μ the dynamic viscosity. The form of the viscous stress tensor is obtained by imposing its linear and isotropic dependency with $\underline{\underline{d}}$ and by assuming there is no dissipation in an isotropic compression (Stokes hypothesis).

The "incompressible flows" approximation is obtained by imposing the constraint $\mathrm{div}\, \underline{U} = 0$ in addition to the mass conservation laws. If the mass density is uniform at some time, the mass conservation $\frac{d\rho}{dt} = 0$ implies that $\rho = \rho_0$ stays uniform and constant for all time, which we assume in the following.

For incompressible fluids, the Newtonian rheologic law leads to

$$\underline{\underline{\sigma}} = -p\, \underline{\underline{I}} + 2\,\mu\, \underline{\underline{d}} \qquad \Longrightarrow \qquad \underline{\mathrm{div}}\ \underline{\underline{\sigma}} = -\underline{\mathrm{grad}}\ p + \mu\, \Delta \underline{U}\ , \qquad (1.36)$$

where the i^{th} component of the Laplacian $\Delta \underline{U}$ is, in Cartesian coordinates, $\Delta U_i = \frac{\partial^2 U_i}{\partial x_j\,\partial x_j}$ (one use the Einstein summation convention which consists in summing the repeated indices). This relation can be obtained by expressing the i^{th} component of $\underline{\text{div}}\,\underline{\underline{d}}$ under the form $\frac{\partial d_{ij}}{\partial x_j} = \frac{1}{2}\frac{\partial^2 U_i}{\partial x_j\,\partial x_j} + \frac{1}{2}\frac{\partial^2 U_j}{\partial x_i\,\partial x_j}$ and using the incompressible constraint $\text{div}\,\underline{U} = \frac{\partial U_j}{\partial x_j} = 0$.

All the above results lead to the incompressible Navier-Stokes equations

$$\text{div}\,\underline{U} = 0 \qquad , \qquad \frac{d\underline{U}}{dt} = -\frac{1}{\rho}\underline{\text{grad}}\;p + \underline{F} + \nu\,\Delta\underline{U}\;, \tag{1.37}$$

where $\nu = \mu/\rho$ is called the "kinematic viscosity".

To solve practical problem, one has to consider boundary conditions. Due to the "parabolic nature" of these Navier-Stokes equations, three boundary conditions must be specified on the whole border of the domain.

On no-slip and motionless boundaries, these boundary conditions read $\underline{U} = \underline{0}$. On slipping and motionless boundaries, they can read $\underline{U}\cdot\underline{n} = 0$ and $\underline{\underline{\sigma}}\cdot\underline{n} - (\underline{n}\cdot\underline{\underline{\sigma}}\cdot\underline{n})\;\underline{n} = \underline{0}$, meaning that the normal velocity and the tangential constraints are zero.

On a free and moving boundary in contact with a gaz at constant pressure p_a, for instance the atmosphere, one describes the free surface with the unknown equation $F(\underline{x}, t) = 0$. The boundary conditions on this interface read $\frac{dF}{dt}(\underline{x}, t) = 0$ and $\underline{\underline{\sigma}}(\underline{x}, t)\cdot\underline{n} = -p_a\;\underline{n}$, where there are four conditions instead of three to balance the new unknown field $F(\underline{x}, t)$.

Before going the application examples, we mention the important Bernoulli equation which comes from the Navier-Stokes equations. Its simplest formulation says that a stationary flow ($\frac{\partial}{\partial t} = 0$ and thus $\frac{d}{dt} = \underline{U}\cdot\underline{\text{grad}}$)) and non viscous ($\nu = 0$), with gravity forces $\underline{F} = -\underline{\text{grad}}\;(g\,z)$, one can write

$$\frac{d}{dt}\left(p + \rho\,g\,z + \frac{1}{2}\rho\,\underline{U}^2\right) = \underline{U}\cdot\underline{\text{grad}}\;\left(p + \rho\,g\,z + \frac{1}{2}\rho\,\underline{U}^2\right) = 0\;. \tag{1.38}$$

One then denotes by "hydraulic load" the quantity $H = \frac{p}{\rho g} + z + \frac{\underline{U}^2}{2\,g}$ which is thus invariant along the trajectories. When the fluid is viscous ($\nu > 0$), this quantity decreases along a trajectory and one then speaks of "load loss". One then denotes by $\underline{J} = -\frac{1}{\rho\,g}\,\underline{\text{div}}\,\underline{\underline{\tau}} = \frac{1}{g}(-\nu\,\Delta\underline{U})$ the "lineic head loss" vector.

3.2 Circular Poiseuille in a pipe

As a first example of application of the incompressible Navier-Stokes equations, we consider the viscous stationary flow in a horizontal tube of circular section.

We denote by D the diameter of this section and we suppose that the flow is forced by a constant pressure gradient $\frac{\partial p}{\partial x} = -G$. The volume forces $\underline{F} = -g\,\underline{e}_z$ are due to gravity.

One seeks a symmetric stationary solution under the form $\underline{U} = u(r)\,\underline{e}_x$ where r is the distance to the axe. The incompressible Navier-Stokes equations lead to

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u \quad , \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad , \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\ . \tag{1.39}$$

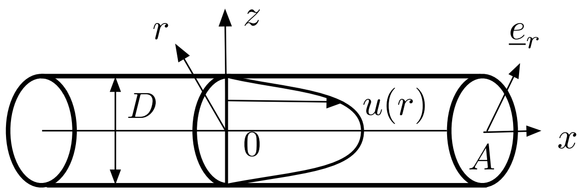

Figure **1**.12: *Circular Poiseuille flow in a pipe.*

One deduces $p(\underline{x}) = p_r - G\,x - \rho\,g\,z$, where p_r is an arbitrary constant. The profile $u(r)$ must satisfy

$$\Delta u = \frac{1}{r}\frac{d}{dr}\left(r\,\frac{du}{dr}\right) = -\frac{G}{\rho\,\nu}\ , \tag{1.40}$$

with the no-slip boundary condition $u(D/2) = 0$. One then deducts that

$$u(r) = \frac{G}{\rho\,\nu}\left(\frac{D^2}{16} - \frac{r^2}{4}\right)\ . \tag{1.41}$$

The discharge flux Q is thus

$$Q = \int_0^{D/2}\int_0^{2\pi} u(r)\,r\,d\theta\,dr = \frac{\pi\,D^4}{128\,\rho\,\nu}\,G\ . \tag{1.42}$$

Denoting by $A = \pi\,D^2/4$ the section area of the tube, one can define the mean velocity U in the direction x by

$$U = \frac{Q}{A} = \frac{D^2\,G}{32\,\rho\,\nu}\ . \tag{1.43}$$

The "hydraulic load" is the

$$H(x,r) = \frac{p}{\rho\,g} + z + \frac{U^2}{2\,g} = \frac{p_r - G\,x}{\rho\,g} + \frac{U^2}{2\,g}\ . \tag{1.44}$$

The lineic head loss J, due to viscous friction on the walls, is defined by $J = \frac{1}{A}\int_0^{D/2}\int_0^{2\pi}\frac{1}{g}(-\nu\,\Delta u)\; r\,d\theta\,dr$ which allows to write

$$\frac{\partial H}{\partial x} = -J \qquad \Longrightarrow \qquad \frac{G}{\rho\, g} = J\ . \tag{1.45}$$

This relation traduces the equilibrium between the pressure forcing and the friction.

We denote by $P = \pi\, D$ the "wet perimeter", $R_H = A/P$ the "hydraulic radius" and $D_H = 4\, R_H$ the "hydraulic diameter". Here, we have $D_H = D$ and $R_H = D/2$. The friction coefficient λ is a dimensionless quantity defined by the Darcy-Weissbach equation

$$J = \lambda\, \frac{U^2}{2gD_H}\ . \tag{1.46}$$

Using the analytic expression (**1**.43) one has here

$$\lambda = \frac{64}{Re} \qquad , \qquad Re = \frac{U\ D_H}{\nu}\ , \tag{1.47}$$

where Re is the "Reynolds number". Another presentation of the lineic head loss is obtained by writing the "Darcy law"

$$U = -K_p\, \frac{\partial H}{\partial x} \qquad , \qquad K_p = \frac{gD^2}{32\,\nu}\ , \tag{1.48}$$

where K_p has the dimension of a velocity.

We denote by $\tau(r) = -\underline{e}_x \cdot \underline{\underline{\sigma}} \cdot \underline{e}_r$ the tangential shear stress defined from the stress tensor

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + 2\,\rho\,\nu\,\underline{\underline{d}} \qquad , \qquad \underline{\underline{d}} = \frac{1}{2}\frac{du}{dr}\left(\underline{e}_r \otimes \underline{e}_x + \underline{e}_x \otimes \underline{e}_r\right)\ , \tag{1.49}$$

where $\underline{\underline{d}}$ is the deformation rate tensor and $\underline{e}_r$ the radial unit vector (the tensorial product $\underline{a} \otimes \underline{b}$ of two vecteurs is the ordre two tensor of components $a_i\, b_j$). Denoting by τ_* the absolute value of the shear stress applied by the fluid on the pipe, one can write

$$\tau_* = \rho\, g\, R_H\, J\ . \tag{1.50}$$

For a horizontal pipe of 100 m length connecting a water tank (mass density $\rho = 1\ 000$ kg/m^3, kinematic viscosity $\nu = 10^{-6}$ m^2/s) under 3 bar of pressure (about 30 m of water height) to the atmosphere (1 bar), the mean velocity U, computed with the help of (**1**.43), is about 25 cm/s for a 2 mm diameter and of 25 m/s (about 100 km/h) for a 2 cm diameter.

If the first estimation of the velocity, corresponding to a Reynolds number of about 500, seems realistic, the second, which corresponds to a Reynolds number of about $5\ 10^5$, is not correct. This is due to the fact that the "laminar" solution that we have computed is no longer pertinent when the Reynolds number is higher that a critical value beyond which the flow becomes "turbulent".

3.3 Open plane Poiseuille flow

We now consider the stationary, viscous, free surface flow of a water layer of depth h on a tilted plane (Figure 1.13). The is no imposed pressure gradient, but the flow forced by the volume forces $\underline{F} = -g\,\underline{e}_z$ of gravity, where z is the vertical coordinate.

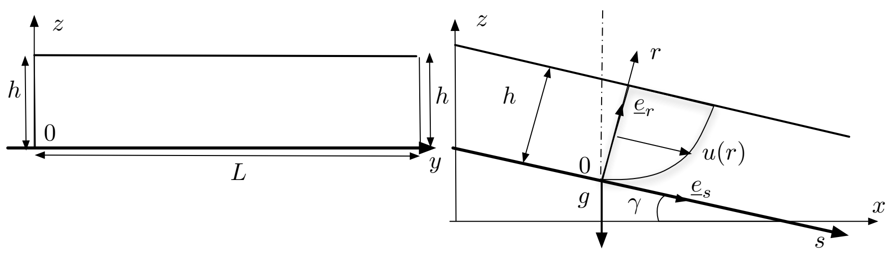

Figure 1.13: *Free surface Poiseuille flow on a tilted plane.*

We denote by γ the angle of the tilted plan with the horizontal plane and p_a the atmospheric pressure, which we assume constant, in contact with the free surface.

We denote by s the coordinate in the direction of the slope and $\underline{e}_s$ the unit vector parallel to it. We denote by $z = Z_f(s)$ the equation of the bottom and $I = -Z_f'(s) = \sin\gamma$ its "slope" (by misnomer).

We assume that the section of the channel is rectangular and we denote by L its width. We defined the "wet perimeter" as the length of fluid in contact with the wall in a given section, which reads here $P = L + 2\,h$. The area of the section is $A = L\,h$. The "hydraulic radius" R_H is defined by the relation $R_H = A/P = L\,h/(L + 2\,h)$. We suppose $h \ll L$ which implies $R_H \sim h$.

We now looks at a stationary solution of the form $\underline{U} = u(r)\,\underline{e}_s$ where $\underline{e}_s$ is in the direction of the slope and r is the coordinate normal to the bottom. By symmetry, on can assume that $\frac{\partial p}{\partial s}(r,s) = 0$. By projecting the incompressible Navier-Stokes equation on the axes $\underline{e}_s$ and $\underline{e}_z$ (they are not orthogonal), one obtains

$$0 = g\, I + \nu\, \frac{d^2 u}{dr^2}(r) \qquad , \qquad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}(x,z) - g \;, \qquad (\mathbf{1}.51)$$

the velocity $u(r)$ being independent of s for a fixed r. The no-slip boundary condition at the bottom reads $u(0) = 0$. The deformation rate tensor $\underline{\underline{d}} = \frac{1}{2}\frac{du}{dr}\,(\underline{e}_r \otimes \underline{e}_s + \underline{e}_s \otimes \underline{e}_r)$ is used to express the stress tensor $\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + 2\,\rho\,\nu\,\underline{\underline{d}}$.

The continuity of the stress through the free surface reads $\underline{\underline{\sigma}} \cdot \underline{e}_r = -p_a\,\underline{e}_r$ where p_a is the atmospheric pressure. This leads to $p(x,z) = p_a$ for $r = h$ and $\frac{du}{dr}(h) = 0$. From this result and the relation and the change of variable $[x,z] = [s\,\cos\gamma + r\,\sin\gamma, Z_f(s) + r\,\cos\gamma]$ with $Z_f(s) = -s\,\sin\gamma$, one deduces that

$$\frac{p}{\rho\, g} + z = \frac{p_a}{\rho\, g} + h\;\cos\gamma + Z_f(s) \qquad , \qquad u = \frac{gI}{\nu}\left(h\,r - r^2/2\right)\;. \qquad (\mathbf{1}.52)$$

The discharge flux Q and the vertically averaged velocity U are then

$$Q = \frac{g\,I\,h^3\,L}{3\nu} \qquad , \qquad U = \frac{Q}{A} = \frac{g\,I\,h^2}{3\nu}\;. \qquad (\mathbf{1}.53)$$

The "hydraulic head", defined by, $H = \frac{p}{\rho\, g} + z + \frac{U^2}{2\,g}$, is thus

$$H(s,r) = \frac{p_a}{\rho\, g} + Z_f(s) + h\;\cos\gamma + \frac{u^2(r)}{2\,g}\;. \qquad (\mathbf{1}.54)$$

The lineic head loss J due to the viscous friction on the bottom is defined by $J = \frac{1}{h}\int_0^h \frac{1}{g}\left(-\nu\,\frac{d^2u}{dr^2}\right)\,dr$ which allows to write

$$\frac{\partial H}{\partial s} = -J \qquad \Longrightarrow \qquad I = J\;. \qquad (\mathbf{1}.55)$$

This relation traduces the equilibrium between the forcing due to the gravity and the friction.

The lineic head loss obeys the Darcy-Weissbach relation

$$J = \lambda\,\frac{U^2}{2gD_H}\;, \qquad (\mathbf{1}.56)$$

where λ is the friction coefficient and D_H the hydraulic diameter is defined by $D_H = 4\,R_H = 4\,h$. Using the analytic expression (**1**.53), one has here

$$\lambda = \frac{96}{Re} \qquad , \qquad Re = \frac{U\,D_H}{\nu}\;, \qquad (\mathbf{1}.57)$$

where Re is the Reynolds number.

The stress tensor $\underline{\underline{\sigma}} = -p\,\underline{\underline{I}} + 2\,\rho\,\nu\,\underline{\underline{d}}$, where $\underline{\underline{d}} = \frac{1}{2}\frac{du}{dr}\,(\underline{e}_r \otimes \underline{e}_s + \underline{e}_s \otimes \underline{e}_r)$ is the deformation rate tensor, allows the definition of the tangential stress $\tau(r) = \underline{e}_s \cdot \underline{\underline{\sigma}} \cdot \underline{e}_r$. Denoting by τ_* the absolute value of the shear stress applied by the fluid on the bottom, one can then write

$$\tau_* = \rho\, g\, R_H\, J \; . \tag{1.58}$$

For a slope I of 3%, a mass density $\rho = 1\,000$ kg/m^3 and a molecular viscosity ν of the order of 10^{-6}m^2/s, one finds that U is of the order of 10 cm/s for a height of the order of 1 mm and that U is of the order of 10^5m/s for a height h of 1m. The first result, which corresponds to a Reynolds number $Re \sim 400$ seems realistic, while the second, corresponding to a Reynolds number of $Re \sim 4\,10^{11}$ is not correct. This is due to the fact that the "laminar" solution that we have computed is no longer pertinent to describe the flow when it becomes turbulent beyond a critical Reynolds number.

A modelling of turbulence is thus necessary to describe the high Reynolds numbers flows.

FORMULAS

Kinematics

Differential of the deformation $\underline{X}(\underline{a}, t)$:

$$\boxed{\underline{\delta x}(t) = \underline{\underline{F}}(\underline{a}, t) \cdot \underline{\delta a} + \underline{O}(\|\underline{\delta a}\|^2) \qquad , \qquad F_{ij}(\underline{a}, t) = \frac{\partial X_i}{\partial a_j}(\underline{a}, t) \; .}$$

Volume transport:

$$\boxed{\delta\mathcal{V}(t) = \delta\mathcal{V}(0)\; J(\underline{a}, t) \qquad , \qquad J(\underline{a}, t) = \det \underline{\underline{F}}(\underline{a}, t) > 0 \; .}$$

Eulerian and Lagrangian representations:

$$\boxed{B(\underline{x}, t) = B^{(L)}(\underline{a}, t) \quad , \quad \underline{x} = \underline{X}(\underline{a}, t) \quad \Longleftrightarrow \quad \underline{a} = \underline{A}(\underline{x}, t) \; .}$$

Mass conservation law:

$$\boxed{\rho^{(L)}(\underline{a}, t)\; J(\underline{a}, t) = \rho_0 \; .}$$

Praticular derivative:

$$\frac{dB}{dt}(\underline{x},t) = \frac{\partial B}{\partial t}(\underline{x},t) + \underline{U}(\underline{x},t)\cdot \underline{\text{grad}}\ B(\underline{x},t)\ .$$

Gradient of the velocity field $\underline{U}(\underline{x},t)$:

$$\frac{d}{dt}\left[\underline{\delta x}(t)\right] = \underline{\underline{K}}[\underline{x}(t),t]\cdot \underline{\delta x}(t) \quad , \quad K_{ij} = \frac{\partial U_i}{\partial x_j}\ .$$

strain tensor:

$$\frac{d}{dt}\left[\underline{\delta x}(t)\cdot \underline{\delta x}'(t)\right] = 2\ \underline{\delta x}(t)\cdot \underline{\underline{d}}[\underline{x}(t),t]\cdot \underline{\delta x}'(t) \quad , \quad d_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\ .$$

Volumes transport by a trajectory $\underline{x}(t)$:

$$\frac{d}{dt}\delta\mathcal{V}(t) = \delta\mathcal{V}(t)\ \text{div}\ \underline{U}[\underline{x}(t),t]\ .$$

Conservation laws

Conservation law for the quantity c:

$$\frac{d}{dt}\iiint_{\mathcal{D}(t)} c(\underline{x},t)\ d^3x + \iint_{\partial\mathcal{D}(t)} \underline{Q}_c(\underline{x},t)\cdot \underline{n}\, dS = \iiint_{\mathcal{D}(t)} f_c(\underline{x},t)\, d^3x\ .$$

Mass conservation law:

$$\frac{\partial\rho}{\partial t} + \text{div}\ (\rho\,\underline{U}) = \frac{d\rho}{dt} + \rho\ \text{div}\ \underline{U} = 0\ .$$

Stress tensor:

$$\underline{T}(\underline{x},\underline{n},t) = \underline{\underline{\sigma}}(\underline{x},t)\cdot \underline{n}\ .$$

Momentum conservation:

$$\rho\,\frac{d\underline{U}}{dt} = \rho\ \left(\frac{\partial \underline{U}}{\partial t} + \underline{U}\cdot \underline{\text{grad}}\ \underline{U}\right) = \rho\,\underline{F} + \underline{\text{div}}\ \underline{\underline{\sigma}}\ .$$

Newtonian fluids

Incompressible Newtonian fluids:

$$\boxed{\underline{\underline{\sigma}} = -p\, \underline{\underline{I}} + 2\, \mu\, \underline{\underline{d}}\ .}$$

Incompressible Navier-Stokes equations:

$$\boxed{\operatorname{div} \underline{U} = 0 \qquad , \qquad \frac{d\underline{U}}{dt} = -\frac{1}{\rho} \underline{\operatorname{grad}}\, p + \underline{F} + \nu\, \Delta \underline{U}\ .}$$

Hydraulic head:

$$\boxed{H = \frac{p}{\rho\, g} + z + \frac{\underline{U}^2}{2\, g}\ .}$$

Circular Poiseuille flow in a pipe

Hydraulic head:

$$\boxed{H(x, r) = \frac{p_r - G\, x}{\rho\, g} + \frac{u^2(r)}{2\, g}\ .}$$

Forcing-friction equilibrium:

$$\boxed{\frac{dH}{dx} = -J \qquad \Longrightarrow \qquad \frac{G}{\rho\, g} = J\ .}$$

Darcy-Weissbach equation:

$$\boxed{J = \lambda\, \frac{U^2}{2gD_H} \quad , \quad Re = \frac{U\ D_H}{\nu} \qquad \Longrightarrow \qquad \lambda = \frac{64}{Re}\ .}$$

Darcy law:

$$\boxed{U = -K_p\, \frac{dH}{dx} \qquad \Longrightarrow \qquad K_p = \frac{gD^2}{32\, \nu}\ .}$$

Open plane Poiseuille

Hydraulique head:

$$\boxed{H(s,r) = \frac{p_a}{\rho\, g} + Z_f(s) + h\, \cos\gamma + \frac{u^2(r)}{2\, g}\ .}$$

Forcing-friction equilibrium:

$$\boxed{\frac{\partial H}{\partial x} = -J \qquad \Longrightarrow \qquad I = J\ .}$$

Darcy-Weissbach equation:

$$\boxed{J = \lambda\, \frac{U^2}{2gD_H} \quad , \quad Re = \frac{U\ D_H}{\nu} \qquad \Longrightarrow \qquad \lambda = \frac{96}{Re}\ .}$$

Bottom shear stress:

$$\boxed{D_H = 4\, R_H = 4\, h \qquad \Longrightarrow \qquad \tau_* = \rho\, g\, R_H\, J\ .}$$

EXERCISES

See exercices in French language in the book:

O. THUAL, Hydrodynamique de l'Environnement, Éditions de l'École Polytechnique, 2010.

or at `http://thual.perso.enseeiht.fr/xsee`