Chapter 2

Potential flows

O. Thual, September 20, 2010

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Introduction

The underground is made of a mix of earth and rock through which water infiltrates and circulates. This circulation is modelled by potential flows in a porous medium. These flows, which velocity field is the gradient of a potential, are encountered in fluid mechanics when the vorticity (curl of the velocity) can be neglected. This is the case of slow subsurface flows at scales which are large in front of the size of the gravels.

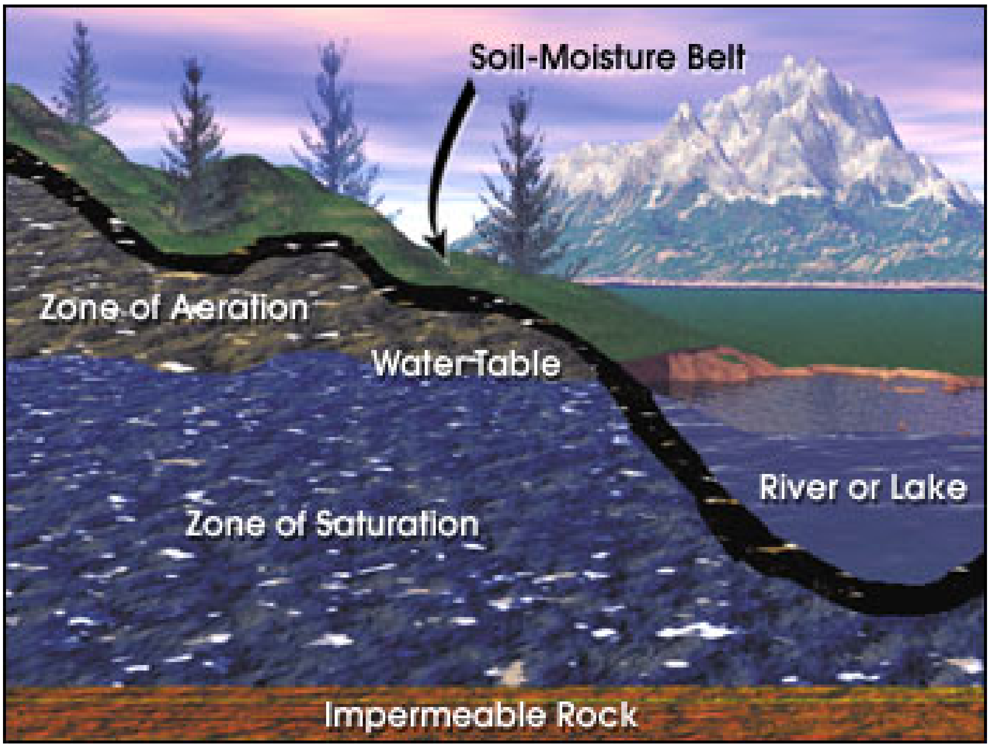

Figure **2**.1: *Water table in contact with a river or a lake [NASA GSFC, by Hailey King].*

Basic notions of ground water hydraulics are presented in this chapter with the help of simple examples which are representative of more complex subsurface problems. Only slow flows, that is with low Reynolds numbers, in isotropic and homogeneous porous media are considered.

The hydraulic head of porous media is presented with the Bernoulli equation derived out the laminar Navier-Stokes equations. Since the velocity of flows in water media is small, the hydraulic head is approximated by the piezometric height. The Darcy law, which postulate a linear relation between the discharge velocity and the head loss, is presented on the simple example of the confined aquifer flowing in a single direction. The generalization of the Darcy law to three dimensional flows in homogeneous porous media shows that the head can be viewed as the potential of the discharge velocity. By applying the mass conservation, one shows that the head loss satisfies the Laplace equation.

Understanding the nature of the boundary conditions used to solve this Laplace equation is one of the key points of this chapter. Several examples are presented.

1 Head loss

The one-dimensional Darcy law is presented here. It states that the discharge velocity of a flow in a porous media is proportional to the lineic head loss.

1.1 Bernoulli equation

We take as starting point the laminar incompressible Navier-Stokes equations

$$\mathrm{div}\ \underline{U} = 0 \quad , \quad \frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \underline{\mathrm{grad}}\ \underline{U} = \underline{F} - \frac{1}{\rho}\ \underline{\mathrm{grad}}\ p + \nu\, \Delta \underline{U}\ , \qquad (\mathbf{2}.1)$$

where the volume forces $\underline{F} = -g\ \underline{e}_z = -\underline{\mathrm{grad}}\ (g\, z)$ are due to gravity.

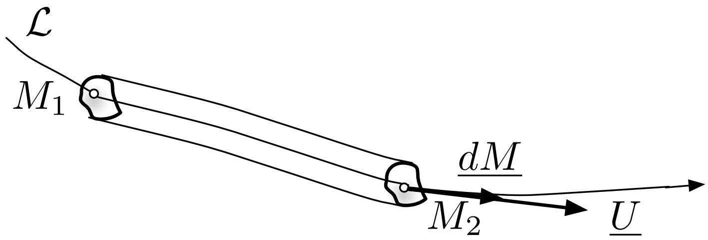

Figure **2**.2: *Stream $\mathcal{L}$ of a laminar flow.*

Let us consider a stream line $\mathcal{L}$ going from a point M_1 to a point M_2. By using the relation

$$\underline{U} \cdot \underline{\mathrm{grad}}\ \underline{U} = \frac{1}{2}\ \underline{\mathrm{grad}}\ \underline{U}^2 + \underline{\mathrm{rot}}\ \underline{U} \wedge \underline{U} \qquad (\mathbf{2}.2)$$

and the relation $(\underline{\mathrm{rot}}\ \underline{U} \wedge \underline{U}) \cdot \underline{dM} = \underline{\mathrm{rot}}\ \underline{U} \cdot (\underline{U} \wedge \underline{dM}) = 0$, one can derive the "Bernoulli equation"

$$\int_{\mathcal{L}} \underline{\mathrm{grad}}\ H \cdot \underline{dM} = \frac{1}{g} \int_{\mathcal{L}} \left(-\frac{\partial \underline{U}}{\partial t} + \nu\, \Delta \underline{U} \right) \cdot \underline{dM}\ , \qquad (\mathbf{2}.3)$$

where H is the "hydraulic head" defined by the relation

$$H = \frac{p}{\rho\, g} + z + \frac{1}{2\, g}\underline{U}^2 \, . \tag{2.4}$$

By integrating from the left hand side of the Bernoulli equation (2.3), one gets

$$H(M_2) = H(M_1) - \int_{\mathcal{L}} \left(\frac{1}{g}\frac{\partial \underline{U}}{\partial t} + \underline{J} \right) \cdot \underline{dM} \, , \qquad \underline{J} = \frac{1}{g}\left(-\nu\, \Delta \underline{U}\right) . \tag{2.5}$$

The term $\underline{J}$ is the lineic head loss due to viscous friction.

In this chapter, we only consider flows such that the acceleration term $\frac{\partial}{\partial t}\underline{U} + \underline{U} \cdot \underline{\text{grad}}\ \underline{U}$ is neglectable in front of the viscous force term $\nu\, \Delta \underline{U}$ (low Reynolds numbers flows). This is the case for subsurface flow in porous media. For such flow, on can write

$$H \sim \frac{p}{\rho\, g} + z \, , \qquad H(M_2) - H(M_1) \sim - \int_{\mathcal{L}} \underline{J} \cdot \underline{dM} \, . \tag{2.6}$$

1.2 Averaged head

TThe fluid particules of an undergroundwater in a porous medium follow complex trajectories between gravels. Let us consider a family of trajectories forming a tube of sections $\mathcal{A}(s)$ around a mean trajectory $\mathcal{L}$ parametrized by its curvilinear coordinate s (Figure 2.3).

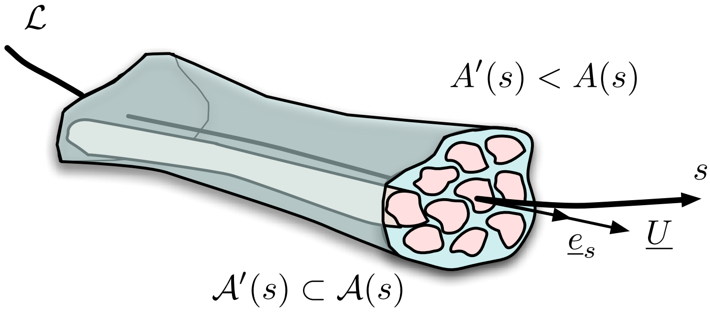

Figure 2.3: *Tube of trajectories in a porous medium.*

Since the medium is "porous", the fluid only crosses a section $\mathcal{A}'(s)$ smaller that $\mathcal{A}(s)$. If $A'(s)$ and $A(s)$ are, respectively, the area of these two sections, we denote by $m = A'/A \leq 1$ the "porosity" of the medium.

One denotes by $Q(s)$ the volumetric flow, or "seeping discharge", in the direction $\underline{e}_s$, where $\underline{e}_s$ is the unit vector tangent to the trajectory $\mathcal{L}$, and one defines it by

$$Q(s) = \iint_{\mathcal{A}'} \underline{U} \cdot \underline{e}_s \, dS \; . \qquad (\mathbf{2}.7)$$

The "discharge velocity" U is then defined by the relation

$$U(s) = \frac{Q(s)}{A(s)} = \frac{1}{A(s)} \iint_{\mathcal{A}'} \underline{U} \cdot \underline{e}_s \, dS \; . \qquad (\mathbf{2}.8)$$

We note that the real velocity of the fluid is, in average, greater than this discharge velocity since $A'(s) < A(s)$.

The averaged hydraulic head of the section $A(s)$ is defined by

$$H(s) \sim \frac{1}{A'} \iint_{\mathcal{A}'} \left(\frac{p}{\rho\, g} + z \right) dS = \frac{P_*(s)}{\rho\, g} = h_* \; , \qquad (\mathbf{2}.9)$$

where $P_*(s)$ is the "piezometric pressure" and $h_*(s)$ the "piezometric height".

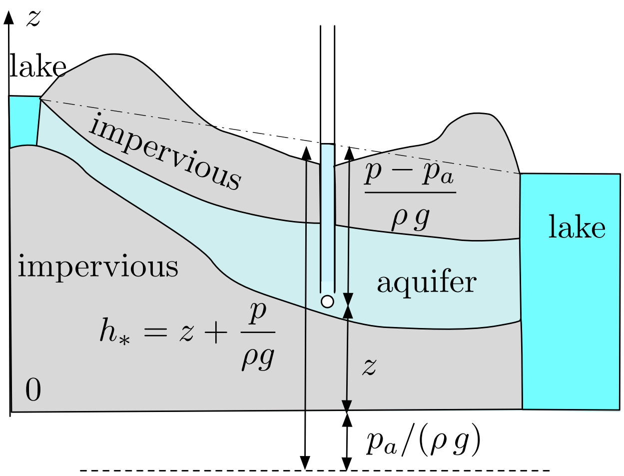

Figure **2**.4: *Piezometric height* $h_* = \frac{p}{\rho\, g} + z$ *in an aquifer.*

This piezometric height the altitude that would reach the water in a well, open to the atmospheric pressure, relatively to a plane located at the distance $p_a/(\rho g)$ below the (arbitrary) geographic zero $z = 0$ (see Figure **2**.4). In some books dealing with hydraulics, a gage is taken on the pressure scale in order to have $p_a = 0$. We do not make this choice in this presentation.

On One defines the "averaged lineic head loss" J by the relation

$$J(s) = \frac{1}{A'} \iint_{\mathcal{A}'} \underline{J} \cdot \underline{e}_s \, dS \; , \qquad (\mathbf{2}.10)$$

which thus satisfies, for a stationary flow, the relation

$$\frac{dH}{ds}(s) = -J(s) \ . \qquad (\mathbf{2}.11)$$

1.3 1D Darcy law

One considers a slow quasi-1D flow obtained by following a tube of trajectories in a porous media. If the section of this tube is large compared to the size of the porous medium gravels or cracks, one can, from experimental observations, model the head loss of this by the one dimensional Darcy which reads

$$J(s) = \frac{U(s)}{K_p(s)} \qquad \Longrightarrow \qquad U(s) = -K_p(s)\,\frac{dH}{ds}(s) \ , \qquad (\mathbf{2}.12)$$

where K_p is the "hydraulic conductivity" of the medium. This quantity has the dimension of a velocity. For instance, one can choose $K_p = 20$ m/day for water seeping in fine sand and $K_p = 2$ km/day for water flowing between gravels. The "intrinsic permeability" coefficient $K_0 = K_p\,\nu/g$ is often considered to characterize a porous medium since it only depends of its geometric properties. The porous media is homogeneous if K_p is independent of space.

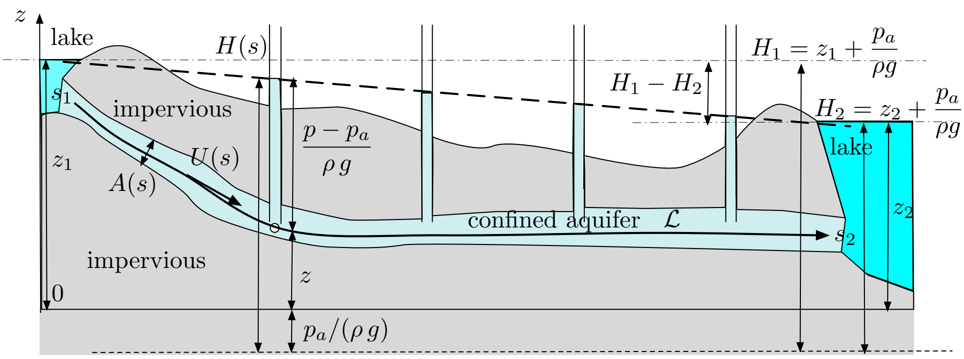

Figure **2**.5: *Head profile* $H(s)$.

As a first application example of the Darcy law, we consider the stationary flow by an aquifer confined in an impervious medium (for instance rock) and flowing between two lakes (Figure **2**.5) of respective altitudes $z_1 > z_2$. One denotes $A(s)$ the section of the gallery in which the fluid is flowing. The mass conservation implies that the discharge $Q = A(s)\,U(s)$ is constant. The model leads to the system of equations

$$\frac{d}{ds}(AU) = 0 \qquad , \qquad U = -K_p\,\frac{dH}{ds} \ . \qquad (\mathbf{2}.13)$$

In order to solve these equations, one must consider two boundary conditions which are here

$$H(s_1) = H_1 \qquad , \qquad H(s_2) = H_2 \ , \tag{2.14}$$

where s_1 and s_2 are the curvilinear coordinates at the two lakes.

If the porous medium is homogeneous (constant K_p) and the section $A(s)$ is constant, the solution is

$$H(s) = H_1 + \frac{H_1 - H_2}{s_1 - s_2}\,(s - s_2) \qquad , \qquad U = -K_p \frac{H_1 - H_2}{s_1 - s_2} \ . \tag{2.15}$$

2 Porous media

In a general porous media, one can define the "discharge velocity" in every point $\underline{x} = (x, y, z)$ in space, provided one looks at a scales larger than the one of gravels or cracks. Staying at these large scales, we denote, from now on, by $\underline{U}$ the "discharge velocity" and we ignore the "actual velocity". Contrarily to the actual velocity, the curl of the discharge velocity vanishes, the vorticity terms, associated with the walls boundary layers, being relegated in the modeling of the head losses. The flow is thus potential at large scales, which is traced back by the Dary law.

2.1 3D Darcy law

We only consider low Reynolds flows, that is such that the acceleration term $\frac{\partial}{\partial t}\underline{U} + \underline{U} \cdot \underline{\text{grad}}\ \underline{U}$ can be neglected in the Navier-Stokes equations which then read

$$\text{div}\ \underline{U} = 0\ , \quad \underline{\text{grad}} \left(\frac{p}{\rho\, g} + z \right) = \frac{1}{g}\,(\nu\, \Delta \underline{U}) \quad \Longleftrightarrow \quad \underline{\text{grad}}\ H = -\underline{J}\ , \tag{2.16}$$

where $H = \frac{p}{\rho\, g} + z$ is the hydraulic head and $\underline{J} = \frac{1}{g}\,(-\nu\, \Delta \underline{U})$ the lineic head loss vector due to the viscous friction.

We now settle at a "macroscopical" spatial scale, large in front of the porous medium gravels and the cracks size which defines the "microscopic" scale. henceforth, we denote by $\underline{U}$ the "discharge velocity" obtained by spatially averaging, at the macroscopic scale, the "real velocity" which we ignore from now on, excepted for saying that it is locally more intense at the microscopic scale.

The discharge velocity also satisfies div $\underline{U} = 0$. On the other hand, the lineic head loss vector averaged at the macroscopic scale, that we henceforth denote by $\underline{J}$ while ignoring the lineic head loss vector at the microscopic scale, is not linked to the velocity $\underline{U}$ by a Laplacian as it was the cas for its microscopic scale. Experimental observations allow to express it with the help of the tridimensional (3D) Darcy law which reads

$$\underline{J}(\underline{x},t) = \frac{1}{K_p(\underline{x},t)}\,\underline{U}(\underline{x},t)\ , \qquad (\mathbf{2}.17)$$

where K_p is the hydraulic conductivity of the porous medium. The modelling of anisotropic medium can be obtained by replacing $1/K_p$ by an order two tensor (a matrix). This case will not be considered in this presentation.

We henceforth denote by H the "averaged hydraulic head" defined by

$$H = \frac{p}{\rho\, g} + z\ , \qquad (\mathbf{2}.18)$$

where p henceforth denotes the averaged pressure at the macroscopic scale.

$$\text{div}\ \underline{U} = 0 \qquad , \qquad \underline{U} = -K_p\ \underline{\text{grad}}\ H\ . \qquad (\mathbf{2}.19)$$

When the porous medium is homogeneous (constant K_p), which we assume from now on, the elimination of the velocity $\underline{U}$ between the two relations div $\underline{U} = 0$ and $\underline{U} = -K_p\ \underline{\text{grad}}\ H$ leads to the Laplace equation

$$\Delta H = 0 \qquad , \qquad H = \frac{p}{\rho\, g} + z\ . \qquad (\mathbf{2}.20)$$

Since the Laplace equation is elliptic, one needs to specify boundary conditions on the whole border of the studied domain.

Trajectories are thus orthogonal to the iso-H surfaces (Figure **2**.6). One shows that if one can draw a circle in a "case" bounded by two iso-H and two trajectories, one can draw a circle in each of the others "cases". This property leads to a graphical solution of the Laplace equation with the "circle method".

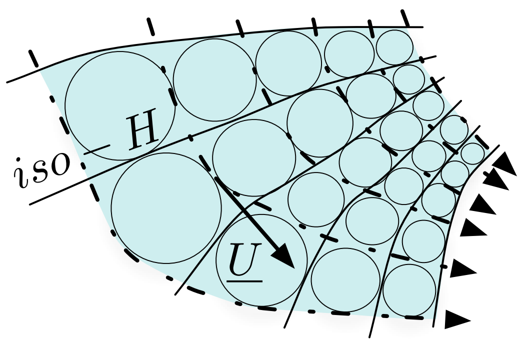

Figure **2**.6: *Orthogonality between the iso-H and the trajectories.*

2.2 Confined flows

We consider a stationary flow in an isotropic and homogeneous medium and we suppose here that the subsurface flow is confined between impervious boundaries or surface water layer such as a lake or a river.

At the interface between the aquifer and the impervious boundaries, the normal velocity vanished. The boundary conditions on the interface are thus

$$\underline{\text{grad}}\ H \cdot \underline{n} = \frac{\partial H}{\partial n} = 0 \quad / \text{ interface} \quad . \tag{2.21}$$

These are "Neumann" boundary conditions for the elliptic problem $\Delta H = 0$.

Since trajectories are crossing the interface between the surface water layer and the aquifer, the head must be continuous. The boundary conditions on this interface are thus

$$H = H_i \quad / \text{ interface} \quad , \tag{2.22}$$

where H_i is the head of the surface water layer at the interface. These are "Dirichlet" boundary conditions for the elliptic problem $\Delta H = 0$.

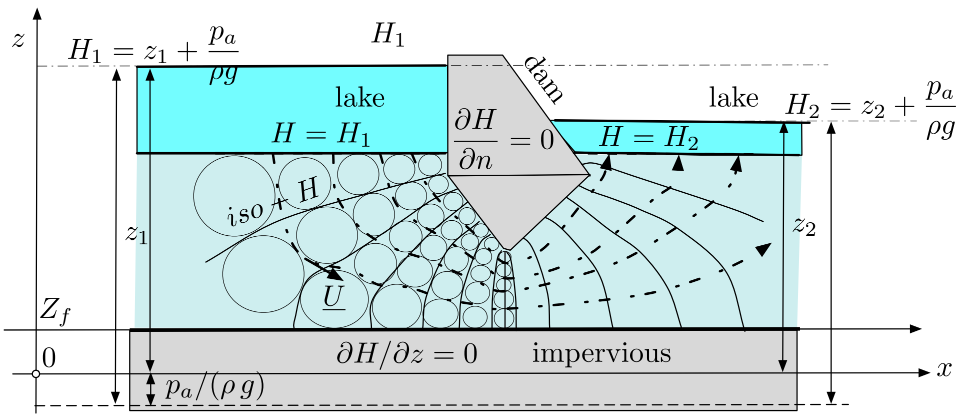

Figure 2.7: *Iso-H (solid lines) and trajectories (dot-dashed lines) of a groundwater flow under a dam.*

As an example, let's consider the groundwater flow under an impervious dam surrounded by two lakes which free surfaces are at the respective altitudes z_1 and z_2 (see Figure 2.7). We suppose that the lakes are at rest so that their pressure are hydrostatic and their head constant. We assume that an impervious bottom is located at the altitude $z = Z_f$ with Z_f constant.

We suppose that the aquifer is bounded at the bottom by a horizontal impervious medium and that the problem is invariant by a translation in the

y direction (2D flow). The flow is seeping from the first lake, with a head equal to $H_1 = z_1 + p_a/(\rho\, g)$, to the second lake at a head equal to $H_2 = z_2 + p_a/(\rho\, g) < H_1$.

We must thus solve $\Delta H = 0$ with the Dirichlet condition $H = H_1$ or $H = H_2$, at the bottom of the lakes, and with the Neuman conditions $\frac{\partial H}{\partial n} = 0$ on all the impervious interfaces.

Accurate solutions of this problem are obtained through numerical simulations, the literature on solving elliptic problem being very large. But graphical methods, developed at the time when computers where not available, are helpful to get a first hint of the solution. This is the case of the "circle method" that can be applied for two-dimensional geometries (see Figure **2**.7).

2.3 Unconfined flows

We now consider that an aquifer whose upper part is unconfined and whose lower part is delimited by an impervious boundary of equation $z = Z_f$. The free surface, located inside the porous medium, is called the "water-table" and the aquifer is said to be "phreatic". We ignore here the capillarity layer which separates the fluid and the dry porous media and we suppose that the water-table is a surface on which the pressure is the atmospheric pressure p_a. We consider, in this presentation, that Z_f is constant.

The existence of this water-table leads to a new boundary conditions. Indeed, the location of this surface, which we denote by the equation $z = Z_f + h(x, y)$, is unknown. In order to find this new function, two boundary conditions instead of one are imposed at this interface, which read

$$\frac{\partial H}{\partial n} = 0 \quad , \quad H = Z_f + h(x,y) + \frac{p_a}{\rho\, g} \quad / \text{ interface } z = Z_f + h(x,y), \qquad (\mathbf{2}.23)$$

since no flow is crossing the stationary water-table and the pressure is equal to the atmospheric pressure. We thus see that the water-table is made of trajectories.

As an example, we consider the flow in a phreatic aquifer between two lakes at rest which free surface are at the respective altitudes z_1 and z_2. The water flows in a porous dam bounded by the planes $x = 0$ and $z = z_b(x)$. (see Figure **2**.8). The boundary conditions $\frac{\partial H}{\partial n}$ on the impervious interface and the boundary conditions $H = H_1$ and $H = H_2$ at the interfaces with the lakes are easy to understand. The location of the water-table $z = Z_f + h(x)$ is obtained by solving the whole family of trajectories starting from the condition $H = H_1$ on the Oz axis and choosing the one which starts at $z = z_1$.

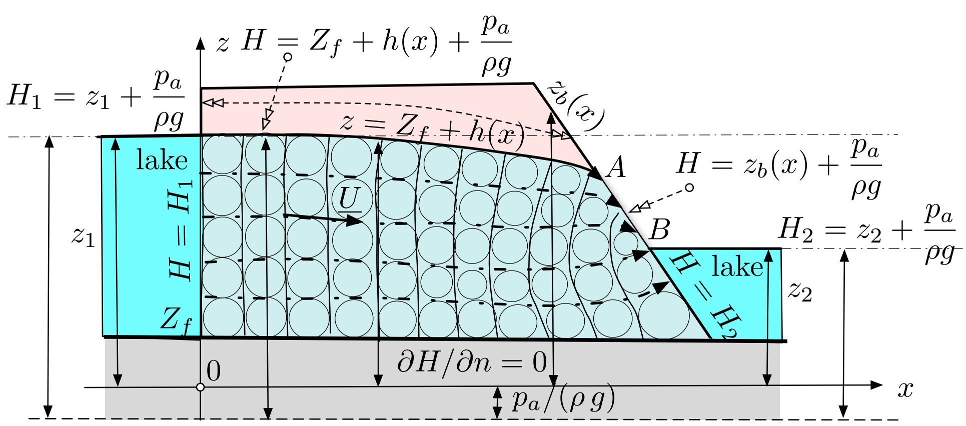

Figure **2**.8: *Iso-H (solid lines) and trajectories (dot-dashed lines) of in a phreatic aquifer between two lakes.*

The peculiarity of this problem is the fact that there must a "seeping face", represented by the non zero line AB in Figure **2**.8. This face is in contact with the atmosphere and fluid is emerging out of it and trickling down along it. Indeed, the is no reason that the trajectory coming from $z = z_1$ on the Oz axis cut the oblique dam surface in a point A equal to the point B, excepted for a very particular value of z_1. Along the AB line, the boundary condition is $H = z_b(x) + p_a/(\rho\, g)$ since the pressure is equal to the atmospheric pressure p_a.

3 Subsurface flows

We apply the Darcy law to the case of artesian aquifers and wells. For phreatic water tables, the Dupuit approximation enables the modelling of free surfaces with small slopes.

3.1 Artesian well

On denote by "artesian aquifer" and aquifer confined between two impervious media. We consider here an artesian aquifer is fed by its contact with a lake at head H_0. We suppose that the fluid is initially at rest so that its head is also equal to H_0 every where.

We then dig a well at some point far from the lake. If the head in the aquifer is strong enough, the fluid will go up naturally along the well up to the land surface or beyond. In that case, one says that the well is "artesian". There is no need pumping to get the water out of such a well.

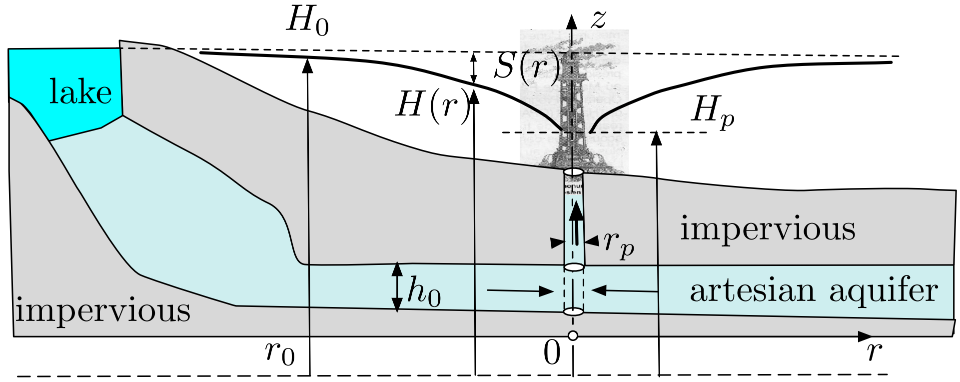

Figure **2**.9: *Depression curve $S(r)$ for an* **artesian well**.

We suppose that the artesian aquifer is confined between two horizontal impervious plane separated by a distance h_0 (see Figure **2**.9). We suppose that the well is a vertical cylinder of radius r_p which can absorb the water out of the whole thickness h_0 of the layer.

When a discharge flux Q is allowed in the well, the head is no longuer constant and we assume a radial distribution $H(r)$ where r is the distance to the well axis. The boundary condition $\frac{\partial H}{\partial z} = 0$ is thus satisfied on the impervious interfaces.

One must then solve the Laplace equation

$$\Delta H = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H}{\partial r}\right) = 0 \qquad (\mathbf{2}.24)$$

which leads to $\frac{\partial H}{\partial r} = C/r$ where the integration constant C must be expressed in function of Q. Using the Darcy law $\underline{U} = -K_p \underline{\text{grad}}\ H = -K_p \frac{\partial H}{\partial r}\ \underline{e}_r$ and integrating the flow discharge on a cylinder of radius r and of height h_0, one finds $C = Q/(2\,\pi\,K_p\,h_0)$.

One then deduces the "depression curve" $S(r)$ defined by the "Thiem equation"

$$S(r) = H_0 - H(r) = \frac{Q}{2\,\pi\,T}\ \text{Ln}\left(\frac{r_0}{r}\right) \qquad , \qquad T = K_p\ h_0\ , \qquad (\mathbf{2}.25)$$

where r_0 is the distance between the well and the lake. One also deduces a relation between the flow discharge of the well and the head H_p at the center of the well which reads

$$S_p = H_0 - H_p = \frac{Q}{2\,\pi\,T}\,\mathrm{Ln}\left(\frac{r_0}{r_p}\right) \ . \qquad (\mathbf{2}.26)$$

3.2 Dupuit approximation

We consider a phreatic aquifer is contained between a horizontal impervious plane of equation $z = Z_f$ and its water-table surface of equation $z = Z_f + h(x, y)$. On this water-table surface the pressure is equal to the atmospheric pressure p_a, which is equivalent to say that the head $H(x, y, z) = z + p/(\rho\,g)$ is equal to $H = Z_f + h(x, y) + p_a/(\rho\,g)$ for $z = Z_f + h(x, y)$. Between these two surfaces, the discharge velocity is given by the Darcy law $\underline{U}(x, y, z) = -K_p\,\underline{\mathrm{grad}}\,H(x, y, z)$.

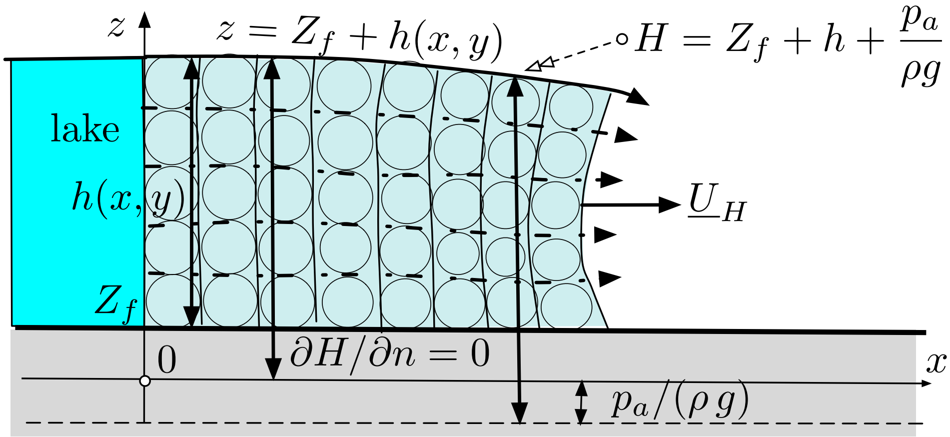

Figure **2**.10: *Dupuit approximation valid for nearly vertical iso-H.*

The "Dupuit approximation" applies to configurations where the slope of the water-table is small enough to consider that the iso-H are vertical (figure **2**.10). In that case we have

$$H(x, y, z) \sim H(x, y) = Z_f + h(x, y) + \frac{p_a}{\rho\,g} \ , \qquad (\mathbf{2}.27)$$

by applying the the boundary condition at the surface. The discharge velocity $\underline{U} = -K_p\,\underline{\mathrm{grad}}\ h$ is then equal approximatively to the horizontal velocity $\underline{U}_H = -K_p\,\underline{\mathrm{grad}}\ h$.

The lineic flow discharge vector, integrated from the bottom of equation $z = Z_f$, to the water-table of equation $z = Z_f + h$, is then approximatively equal to

$\underline{q} = h\,\underline{U}_H$. By integrating the equation div $\underline{U} = 0$ from $z = Z_f$ to $z = Z_f + h$ and by writing that the velocity normal to these boundaries vanishes, one shows that div $\underline{q} = 0$.

By denoting U and V the two components of $\underline{U}$, the equations of a stationary flow obtained under the Dupuit approximation are the two equations div $\underline{q} = 0$ and $\underline{U}_H = -K_p\,\underline{\text{grad}}\; h$, which reads

$$\frac{\partial}{\partial x}(h\,U) + \frac{\partial}{\partial y}(h\,V) = 0\,, \qquad U = -K_p\,\frac{\partial h}{\partial x}\,, \qquad V = -K_p\,\frac{\partial h}{\partial y}\,. \tag{2.28}$$

If the porous medium is homogeneous (constant K_p), which we assume here, the square of the height h is solution of the horizontal Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\, h^2 = 0\,. \tag{2.29}$$

3.3 Applications and limitations

As an example of the application for the Dupuit approximation, we first consider a phreatic aquifer flowing from a lake to a **prismatic ditch** (Figure 2.11). We assume that the bottom previous interface is the horizontal plane $z = Z_f$ with Z_f constant.

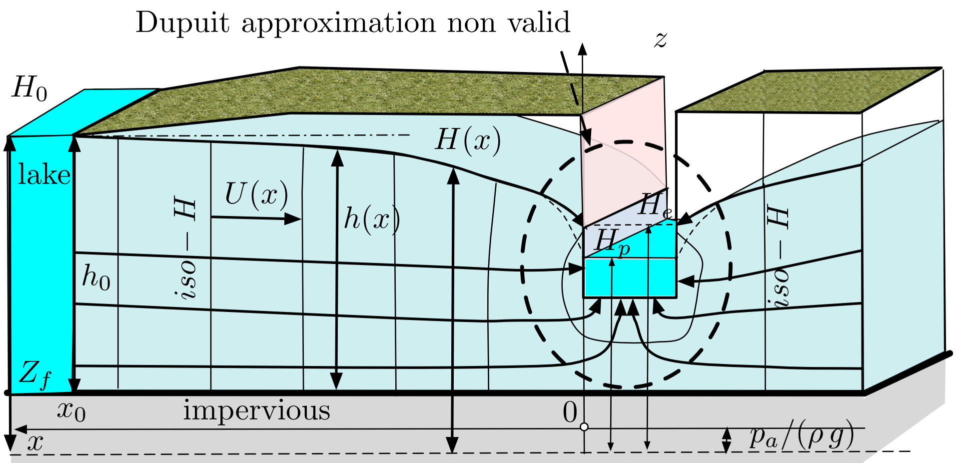

Figure 2.11: *Flow towards a* **prismatic ditch** *in a phreatic aquifer.*

We denote by q the lineic flow discharge of the ditch in the y direction. We assume that the lake is at the load H_0 and we denote by x_0 its distance from the ditch. We denote by h_0 be the distance between the impervious plane and the free surface of the lake. We thus have $H_0 = Z_f + h_0 + \frac{p_a}{\rho g}$.

By comparing with the solutions of the the exact equations, it can be shown that the Dupuit approximation is everywhere valid excepted near the ditch where the discharge velocity $\underline{U}$ can no longer be considered as horizontal.

For the points where the Dupuit hypothesis is valid, the load is equal to $H = h + p_a/(\rho\, g)$ equations (**2**.28) read

$$\frac{d}{dx}\left[U(x)\, h(x)\right] = 0 \qquad , \qquad U(x) = -K_p\, \frac{dH}{dx}(x) = -K_p\, \frac{dh}{dx}(x) \qquad (\mathbf{2}.30)$$

and can be integrated, imposing a symmetry $x \to -x$, into

$$h_0^2 - h^2(x) = \frac{2\, q}{K_p}\, |x_0 - x| \; . \qquad (\mathbf{2}.31)$$

One then deduces the head H_p at the ditch obtained in the framework of the Dupuit approximation. Even though this approximation is not valid close to the well, this value of H_p can be used to determine the level of the water in the ditch by saying that $z = Z_f + h_p = H_p - p_a/(\rho\, g)$ is the altitude of its free surface. The solution obtained with the Dupuit approximation put forwards the existence of a "seeping face" so that the water-table does not coincide with the free surface of the ditch water.

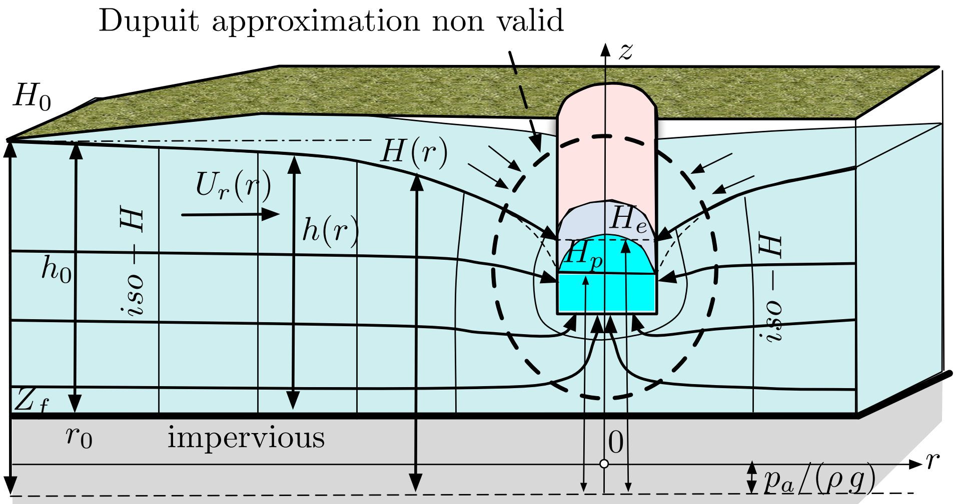

Figure **2**.12: *Velocity* $\underline{U} = U_r\, \underline{e}_r$ *of a phreatic aquifer around* **vertical and cylindrical well**.

If we now replace the ditch by a **vertical and cylindrical well** (figure **2**.12) in which a flow discharge Q is pumped out, the radial discharge velocity $U_r(r)$ and the head $H(r)$ only depend of the radius r. For the points where the

Dupuit approximation is valid, one has $H = Z_f + h$ and the equations (**2**.28) read

$$\frac{d}{dr}\left[r\, U_r(r)\, h(r)\right] = 0 \qquad , \qquad U_r(r) = -K_p\, \frac{dH}{dr}(r) = -K_p\, \frac{dh}{dr}(r)\ . \qquad (\mathbf{2}.32)$$

By integrating these equations, one obtains

$$h_0^2 - h^2(r) = \frac{Q}{\pi\, K_p}\ \mathrm{Ln}\left(\frac{r_0}{r}\right)\ . \qquad (\mathbf{2}.33)$$

FORMULAS

Head loss

Hydraulic head :

$$H = \frac{p}{\rho\, g} + z + \frac{1}{2\, g}\underline{U}^2 \sim \frac{p}{\rho\, g} + z\ .$$

Averaged Navier-Stokes équations :

$$\mathrm{div}\ \underline{U} = 0 \qquad , \qquad \mathrm{\underline{grad}}\ H = -\underline{J}\ .$$

Porous media

Darcy law:

$$\underline{J}(\underline{x}, t) = \frac{1}{K_p(\underline{x}, t)}\ \underline{U}(\underline{x}, t)\ .$$

Slow flows:

$$\mathrm{div}\ \underline{U} = 0 \qquad , \qquad \underline{U} = -K_p\ \mathrm{\underline{grad}}\ H\ .$$

Homogeneous porous media:

$$\Delta H = 0 \qquad , \qquad H = \frac{p}{\rho\, g} + z\ .$$

Impervious boundary conditions:

$$\frac{\partial H}{\partial n} = 0 \; .$$

Free surface boundary conditions:

$$\frac{\partial H}{\partial n} = 0 \qquad , \qquad H = Z_f + h + \frac{p_a}{\rho\, g} \quad / \text{ interface } z = Z_f + h(\underline{x}) \; .$$

Subsurface flows

Artesian well:

$$S(r) = H_0 - H(r) = \frac{Q}{2\,\pi\,T}\,\text{Ln}\left(\frac{r_0}{r}\right) \qquad , \qquad T = K_p\, h_0 \; .$$

Dupuit approximation:

$$H(x,y,z) \sim H(x,y) = Z_f + h(x,y) + \frac{p_a}{\rho\, g} \; .$$

Dupuit and discharge:

$$U = -K_p\, \frac{\partial h}{\partial x} \; , \qquad V = -K_p\, \frac{\partial h}{\partial y} \; , \qquad \frac{\partial}{\partial x}\,(h\, U) + \frac{\partial}{\partial y}\,(h\, V) = 0 \; .$$

Homogeneous Dupuit:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\, h^2 = 0 \; .$$

Prismatic ditch:

$$h_0^2 - h^2(x) = \frac{2\, q}{K_p}\, |x_0 - x| \; .$$

Cylindrical well:

$$h_0^2 - h^2(r) = \frac{Q}{\pi\, K_p}\, \mathrm{Ln}\left(\frac{r_0}{r}\right) .$$

EXERCISES

See exercices in French language in the book:

O. THUAL, Hydrodynamique de l'Environnement, Éditions de l'École Polytechnique, 2010.

or at `http://thual.perso.enseeiht.fr/xsee`