Chapter 3

Turbulence and friction

O. Thual, September 20, 2010

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Introduction

The goal of this chapter is to give an overview of the turbulence models which describe, from the engineering point of view, the friction of a walls on a flow.

This presentation of turbulence modelling is oriented towards the introduction of the empiric relations which express the turbulent friction coefficients as a function of global parameters of the flow, such as the average velocity and the section of the pipe (hydraulics in closed ducts) or the channel (open channel flows).

Simple notions on the decomposition between mean fields and turbulent fluctuations are given. By taking the average of transport equations or of the Navier-Stokes equations, the notions of turbulent fluxes and turbulence diffusivity are introduced. The mixing length model, which is helpful in a lot of engineering applications dealing with fluid mechanics, is explicited.

Figure **3**.1: *Turbulent flow at the exit of marine wharf.*

In the vicinity of a wall, such as the bottom of a channel flow or the inside boundaries of a pipe, the mixing length is the product of the distance to the wall and the Von Karman constant which value comes out of experiments. This robust law lead to identification of logarithm profiles for the velocity in the vicinity of the wall. The cases of smooth or rough boundaries are compared. They are limit cases of the "Moody diagram" which plot, for general roughnesses, the turbulent friction coefficient as a function of the Reynolds number and the dimensionless roughness.

1 Turbulence modelling

The decomposition of a turbulent field as the sum of its mean and its fluctuations enables to define the notion of turbulent fluxes as being the mean of the fluctuation products. One then parametrizes theses fluxes as functions of the mean fields with the help of coefficients such as the turbulent diffusivity or the turbulent viscosity.

1.1 Mean and fluctuations

When a flow is turbulent, any field $B(\underline{x}, t)$ (velocity components, pressure, temperature ...) is fluctuating in space and time on a large variety of scales.

Thus, one tries to decompose B as the sum of a mean field $\overline{B}$ which varies at the large space and time scales of interest (for instance for the engineer) and a rapidly fluctuating field B' which represents the motion of the smaller scales.

One considers the signal $b(t) = B(\underline{x}, t)$ measured at a given point $\underline{x}$. Its Fourier transform $\widehat{b}(\omega)$, defined by

$$b(t) = \int_{I\!R} \widehat{b}(\omega) e^{-i\,\omega\,t}\, d\omega \quad \Longleftrightarrow \quad \widehat{b}(\omega) = \frac{1}{2\pi} \int_{I\!R} b(t) e^{i\,\omega\,t}\, dt \;, \qquad (\mathbf{3}.1)$$

leads to the power spectrum $E_B(\omega) = \frac{1}{2}|\widehat{b}(\omega)|^2$ where ω is the frequency . A more exact definition of the power spectrum must be found in turbulence books ([**?**], [**?**]). Similarly, the three dimensional field $b(\underline{x}) = B(\underline{x}, t)$ measured at a given time t can be decomposed in Fourier modes through the relations

$$b(\underline{x}) = \iiint_{I\!R^3} \widehat{b}(\underline{K}) e^{i\,\underline{K}\cdot\underline{x}}\, dK^3 \Longleftrightarrow \widehat{b}(\underline{K}) = \frac{1}{(2\pi)^3} \iiint_{I\!R^3} b(\underline{x}) e^{-i\,\underline{K}\cdot\underline{x}}\, dx^3 \;, \quad (\mathbf{3}.2)$$

which lead to the power spectrum $E_B(K) = \frac{1}{2} \iint_{\|\underline{K}\|=K} |\widehat{b}(\underline{K})|^2 \, dS$ as a function of the wave number $K = \|\underline{K}\|$ and thus of the length scales.

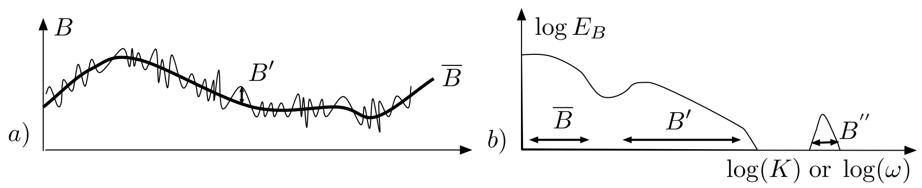

Figure **3**.2: *a) Decomposition $B = \overline{B} + B'$ in mean field and turbulent fluctuations. b) Power spectrum of B.*

These spectra show the repartition of the signal power as a function of the time and space scales (Figure **3**.2).

When considering the Brownian fluctuations B'' (at the molecular scales) of the field B, one can see (Figure **3**.2b) a spectral gap between the continuum mechanics scales ($\overline{B} + B'$) and the molecular scales (B''). The separation between these two scales is thus well defined.

Such a spectral gap between the large scales ($\overline{B}$) and the turbulent scales (B') is seldom present. Nevertheless, one assume that it is however possible to perform the decomposition

$$B(\underline{x}, t) = \overline{B}(\underline{x}, t) + B'(\underline{x}, t) \ , \qquad (\mathbf{3}.3)$$

and that the average operator owns nice properties such as $\overline{\overline{B}} = \overline{B}$ and $\overline{B'} = 0$. One thus has $\overline{B_1\, B_2} = \overline{B}_1\, \overline{B}_2 + \overline{B_1'\, B_2'}$. One also assumes $\overline{\frac{\partial B}{\partial t}} = \frac{\partial \overline{B}}{\partial t}$ and $\overline{\frac{\partial B}{\partial x_i}} = \frac{\partial \overline{B}}{\partial x_i}$. One says that $\overline{B}$ is the "Reynolds average" of B.

1.2 Turbulent flux

If a flow is turbulent, one decomposes its velocity $\underline{U} = \overline{\underline{U}} + \underline{U}'$ into the sum of the mean and fluctuating vector fields. We consider here only incompressible flow and one has thus div $\underline{U} = 0$ and div $\overline{\underline{U}} = 0$.

The field B is a "passive scalar" convected by the velocity field $\underline{U}$ if it obeys to the equation

$$\frac{\partial B}{\partial t} + \underline{U} \cdot \underline{\text{grad}}\; B = k_B\, \Delta B \ , \qquad (\mathbf{3}.4)$$

where k_B is a molecular diffusion coefficient. Using the properties of the average operator and the relation div $\underline{U} = 0$, the average of this equation is

$$\frac{\partial \overline{B}}{\partial t} + \text{div}\,(\overline{\underline{U}}\,\overline{B}) = \frac{\partial \overline{B}}{\partial t} + \overline{\underline{U}} \cdot \underline{\text{grad}}\; \overline{B} = k_B\, \Delta \overline{B} - \text{div}\,(\overline{\underline{U}'\, B'}) \ . \qquad (\mathbf{3}.5)$$

The quantity $\underline{F}_{Bt} = \overline{\underline{U}'\, B'}$ is the "turbulent flux".

Similarly, the incompressible Navier-Stokes equation can be averaged, leading to

$$\text{div}\, \overline{\underline{U}} = 0 \ , \ \ \frac{\partial}{\partial t}\overline{\underline{U}} + \overline{\underline{U}} \cdot \underline{\text{grad}}\; \overline{\underline{U}} = -\frac{1}{\rho}\, \underline{\text{grad}}\; \overline{p} - \underline{\text{grad}}\,(g\, z) + \nu\, \Delta \overline{\underline{U}} - \underline{\text{div}}\, \underline{\underline{R}} \ , \qquad (\mathbf{3}.6)$$

where g is the gravity and $\underline{\underline{R}}$ the "Reynolds tensor" defined by its components

$$R_{ij} = \overline{U_i'\, U_j'} \ , \qquad (\mathbf{3}.7)$$

where U_i denotes the three components of $\underline{U}$.

A typical spectrum $E(K)$ of the kinetic energy $k = \frac{1}{2}\overline{\underline{U}'^2} = \frac{1}{2}\mathrm{tr}\ \underline{\underline{R}}$ of a turbulent flow is represented on Figure **3**.3. One can find in turbulence books ([**?**], [**?**]) the derivation of the $E(K) = C\,\epsilon^{3/2}\,K^{-5/3}$ law of the "Kolmogorov spectrum", where C_K is the "Kolmogorov constant" , valid between the scales K_0, where the energy is injected with the rate ϵ, and the dissipative scales K_d, where it is dissipated at the same rate, can be found in turbulence books ([**?**], [**?**]).

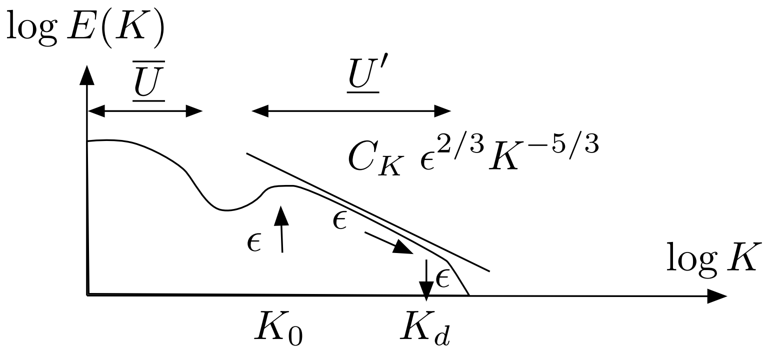

Figure **3**.3: *Example of a kinetic energy spectrum* $E(K)$ *and decomposition* $\underline{U} = \overline{\underline{U}} + \underline{U}'$.

The turbulence modelling problem is to express products such as $\overline{U_i'\,B'}$ or $\overline{U_i'\,U_j'}$, which are named "double correlations", as a function of the mean fields.

Order one models propose expressions of turbulent fluxes such as $\underline{F}_{Bt}$ or $\underline{\underline{R}}$ as functions of the mean fields $\overline{B}$ and $\overline{\underline{U}}$ or of their gradients.

1.3 Turbulent viscosity

Nearly all the turbulent models starts with the empirical law

$$\underline{F}_{Bt} = -k_{Bt}\ \underline{\mathrm{grad}}\ \overline{B} \quad \Longleftrightarrow \quad \overline{\underline{U}'\,B'} = -k_{Bt}\ \underline{\mathrm{grad}}\ \overline{B}\ , \tag{3.8}$$

where k_{Bt} is called the "turbulent diffusivity" of B. The averaged convection equation for B is thus

$$\frac{\partial \overline{B}}{\partial t} + \overline{\underline{U}} \cdot \underline{\mathrm{grad}}\ \overline{B} = \mathrm{div}\ \left[(k_B + k_{Bt})\ \underline{\mathrm{grad}}\ \overline{B}\right]\ . \tag{3.9}$$

Similarly, the turbulent modelling of the Reynolds tensor $\underline{\underline{R}}$ as a function of the average components $\overline{U}_i$ assumes the empirical law

$$R_{ij} = -2\,\nu_t\,\overline{d}_{ij} + \frac{2}{3}\,k\,\delta_{ij} \quad \Longleftrightarrow \quad \underline{\underline{R}} = -2\,\nu_t\,\overline{\underline{\underline{d}}} + \frac{2}{3}\,k\,\underline{\underline{I}}\ , \tag{3.10}$$

where the "turbulent kinetic energy" k is defined by

$$k = \frac{1}{2}\text{tr}\ \underline{\underline{R}} = \frac{1}{2}\left(\overline{U_1'^2} + \overline{U_2'^2} + \overline{U_3^2}\right) \ , \qquad (\mathbf{3}.11)$$

and the components of the "deformation rate tensor" $\underline{\underline{d}}$ are $d_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$. The quantity ν_t is called the "turbulent viscosity".

For $i \neq j$, this law reads

$$\overline{U_i' U_j'} = -\nu_t \left(\frac{\partial \overline{U}_i}{\partial x_j} + \frac{\partial \overline{U}_j}{\partial x_i}\right) = -2\ \nu_t\ d_{ij}\ . \qquad (\mathbf{3}.12)$$

The expression of the diagonal components $i = j$ takes into account the incompressibility constraint tr $\underline{\underline{\overline{d}}} = \text{div}\ \underline{\overline{U}} = 0$.

Most of the time, the turbulent diffusion coefficient k_{Bt} and the turbulent viscosity ν_t are chosen as equal.

For a parallel flow $\underline{\overline{U}} = \overline{u}(z)\, \underline{e_x}$, the only non trivial component of this model is

$$\overline{u'w'} = -\nu_t\, \frac{\partial \overline{u}}{\partial z}\ . \qquad (\mathbf{3}.13)$$

2 Velocity profiles

The mixing length model is a turbulent parametrization often used in applications. Near the walls, it allows to compute the logarithmic velocity profiles taking into account the smooth or rough nature of the surface.

2.1 Mixing length

A very simple turbulent model is obtained by choosing a constant turbulent viscosity ν_t. This model can give realistic results when the turbulent fields are homogeneous in space. This is not the case of the wakes of obstacles or of the vicinity of boundaries such as the bottom of a river or the boundaries of a pipe. More complex turbulent models are thus required for such cases.

The mixing length model is used to describe inhomogeneous turbulence. Its assumes that the turbulent viscosity reads

$$\nu_t = l_m^2\ \sqrt{2\ \underline{\underline{\overline{d}}} : \underline{\underline{\overline{d}}}}\ , \qquad (\mathbf{3}.14)$$

where $\underline{\underline{\overline{d}}} : \underline{\underline{\overline{d}}} = \text{tr}\ (\underline{\underline{\overline{d}}} \cdot \underline{\underline{\overline{d}}}) = d_{ij}\, d_{ji} = d_{ij}\, d_{ij}$ is the sum of the squares of all the components of the (symmetric) deformation rate tensor $\underline{\underline{d}}$ (we have used here

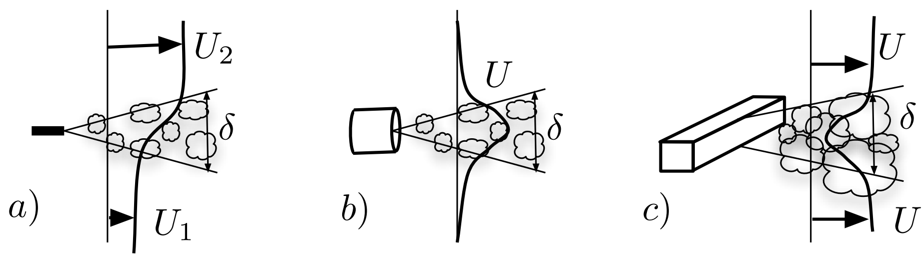

Figure **3**.4: *Mixing length values: a) mixing layer, $l_m \sim 0.07\,\delta$. b) round jet, $l_m \sim 0.075\,\delta$. c) plane wake, $l_m \sim 0.16\,\delta$.*

the Einstein convention which consists in summing the repeated indices i and j). The quantity $l_m(\underline{x}, t)$, is called the "mixing length". It must be fitted as a function of the flow geometry. For instance, it is the order of the thenth of the thickness $\delta(x)$ for a mixing layer, a jet, or a turbulent wake (see Figure **3**.4).

For the parallel flow $\underline{\overline{U}} = \overline{u}(z)\,\underline{e}_x$, the mixing length model leads to

$$\nu_t = l_m^2 \left|\frac{\partial \overline{u}}{\partial z}\right| \qquad , \qquad \overline{u'w'} = -l_m^2 \left|\frac{\partial \overline{u}}{\partial z}\right| \frac{\partial \overline{u}}{\partial z} \; . \tag{3.15}$$

Near a wall, experiments observations show that

$$l_m = \kappa\, z \qquad , \qquad \kappa = 0.41 \; , \tag{3.16}$$

where κ is the "Von Karman constant" (figure **3**.5).

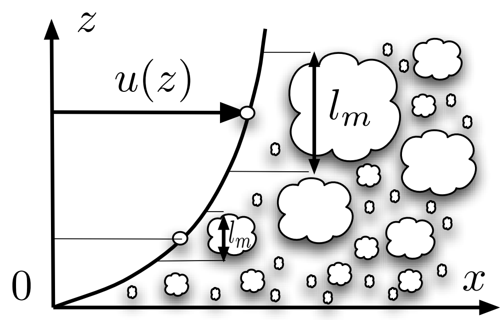

Figure **3**.5: *Mixing length near a wall: $l_m = \kappa\, z$ with $\kappa = 0.41$.*

The mixing length model can be compared to the expression $\nu \sim l_{mol}\, u_{mol}$ of the molecular velocity where l_{mol} is the free mean path of the molecules and u_{mol} their typical velocity. Indeed, one can write $\nu_t = l_m\, u_m$ where $u_m \sim l_m \sqrt{2\,\underline{\underline{\overline{d}}} : \underline{\underline{\overline{d}}}}$, which reads $u_m \sim l_m \left|\frac{\partial \overline{u}}{\partial z}\right|$ for a parallel flow. The length l_m can be seen as the free mean path of small vortexes and u_m as their characteristic velocities.

2.2 Flat bottom

Let us consider the layer of thickness R of a turbulent flow above a flat bottom. One assumes that the flow is forced by a constant pressure gradient $\frac{\partial \overline{p}}{\partial x} = -G$, in a gravity field $-g\, \underline{e}_z$.

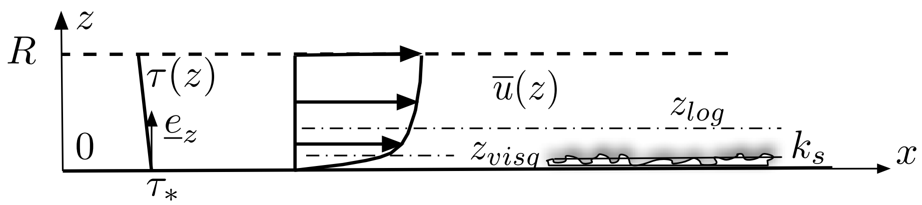

Figure **3**.6: *Free surface flow on a smooth or rough bottom.*

Saying that the averaged velocity $\underline{\overline{U}} = \overline{u}(z)\, \underline{e}_x$ satisfies the Reynolds averaged Navier-Stokes equations read

$$\begin{aligned} 0 &= -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \frac{\partial}{\partial z}\left(\nu \frac{\partial \overline{u}}{\partial z} - \overline{u'w'}\right) \\ 0 &= -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial z} - g - \frac{\partial}{\partial z}\left(\overline{w'w'}\right) . \end{aligned} \qquad (\mathbf{3}.17)$$

One denotes by τ the tangential stress $\tau(z) = \rho\left(\nu \frac{\partial \overline{u}}{\partial z} - \overline{u'w'}\right)$ and $\tau_* = \tau(0)$ its value at $z = 0$. One thus has $0 = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau}{\partial z}$. Since $\frac{\partial \overline{p}}{\partial x} = -G$ is constant $\frac{\partial \tau}{\partial z} = -G$ is so. One then deduces the relation

$$\tau(z) = \tau_* - G\, z\ , \qquad (\mathbf{3}.18)$$

where $\tau_* = \tau(0)$ is so far unknown.. One denotes by u_* the "friction velocity" defined by $\tau_* = \rho\, u_*^2$. The mixing length turbulence model $\overline{u'w'} = -l_m^2 \left|\frac{\partial \overline{u}}{\partial z}\right| \frac{\partial \overline{u}}{\partial z}$ thus leads to

$$\left(\nu + l_m^2 \left|\frac{\partial \overline{u}}{\partial z}\right|\right) \frac{\partial \overline{u}}{\partial z} = u_*^2 - \frac{G}{\rho}\, z\ . \qquad (\mathbf{3}.19)$$

The integration of this equation will allow the determination of the velocity profile $\overline{u}(z)$ whose expression depends on the match with the bottom boundary conditions.

2.3 Logarithmic profiles

One supposes that $\tau(z) \sim \tau_* = \rho\, u_*^2$ is nearly constant in the layer $z \in [0, R]$, and thus $R \ll \tau_*/G$. Assuming the law $l_m = \kappa\, z$, where $\kappa = 0.41$ is the Von Karman constant, the equations read

$$\left(\nu + \kappa^2\, z^2 \left|\frac{\partial \overline{u}}{\partial z}\right|\right) \frac{\partial \overline{u}}{\partial z} = u_*^2 \; . \qquad (\mathbf{3}.20)$$

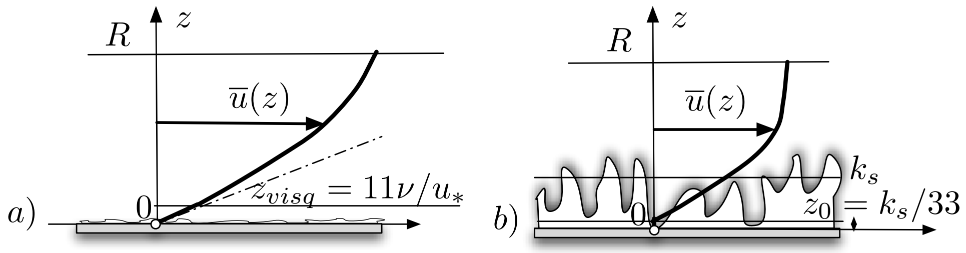

Figure **3**.7: *Boundary layer over a flat bottom: a) Smooth regime. b) rough regime.*

One says that the bottom is "smooth" if there is a boundary layer $z \in [0, z_{vis}]$ in which the molecular viscosity ν is dominant with respect with the turbulent viscosity $\nu_t = l_m \left|\frac{\partial \overline{u}}{\partial z}\right|$. This hypothesis can be stated as $l_m = 0$ in this layer. In that case, the equations $\nu\, \frac{\partial \overline{u}}{\partial z} = u_*^2$ with $\overline{u}(0) = 0$ leads to

$$\frac{\overline{u}(z)}{u_*} = \frac{u_*\, z}{\nu} \; . \qquad (\mathbf{3}.21)$$

One says that the bottom is "rough" if there is a layer $z \in [0, z_0]$ in which the velocity $u(z)$ is zero. If k_s is the mean height of the roughness, experiment observations show that

$$z_0 = k_s/33 \; . \qquad (\mathbf{3}.22)$$

The "smooth" or "rough" nature of the bottom depend on k_s and ν but also of the turbulent flow through its friction velocity u_*. Indeed, experimental observations shows that the criteria for the smooth and rough regime are respectively

$$\frac{u_*\, k_s}{\nu} < 5 \qquad , \qquad \frac{u_*\, k_s}{\nu} > 70 \; . \qquad (\mathbf{3}.23)$$

When the dimensionless number $u_*\, k_s/\nu$ is between 5 and 70, the regime is neither smooth nor rough and the boundary layer analysis is more complex.

When z is sufficiently large, the molecular viscosity ν is negligible in front of the turbulent viscosity and the equation read

$$\kappa^2 \, z^2 \left| \frac{\partial \overline{u}}{\partial z} \right| \frac{\partial \overline{u}}{\partial z} = u_*^2 \; . \tag{3.24}$$

In this case, the velocity profile read

$$\frac{\overline{u}}{u_*} = \frac{1}{\kappa} \ln \left(\frac{z}{\delta} \right) + \zeta \; , \tag{3.25}$$

where the two constants δ and C represent in fact a single integration constant $-\frac{1}{\kappa} \ln(\delta) + C$ which is the only one degree of freedom of the velocity profile family.

When the bottom is smooth, experimental observations show that the logarithmic profile matches with the viscous profile at z_{vis} defined by

$$z_{vis} = 11 \, \frac{\nu}{u_*} \; . \tag{3.26}$$

The velocity profile of the logarithmic layer is thus

$$\frac{\overline{u}_{sth}(z)}{u_*} = \frac{1}{\kappa} \, \ln \left(\frac{u_* \, z}{\nu} \right) + 5.2 = \frac{1}{\kappa} \ln \left(\frac{z}{\delta_{sth}} \right) + \zeta_{sth} \; , \tag{3.27}$$

with $\delta_{sth} = \nu / u_*$ and $\zeta_{sth} = 11 - \ln(11)/\kappa = 5.2$.

When the bottom is rough, experimental observations show how the logarithmic profile matches with the zero velocity profile at z_0 with $k_s = 33 \, z_0$. The velocity profile of the logarithmic layer is thus

$$\frac{\overline{u}_{rgh}(z)}{u_*} = \frac{1}{\kappa} \, \ln \left(\frac{z}{k_s} \right) + 8.5 = \frac{1}{\kappa} \ln \left(\frac{z}{\delta_{rgh}} \right) + \zeta_{rgh} \; , \tag{3.28}$$

with $\delta_{rgh} = k_s$ and $\zeta_{rgh} = \ln(33)/\kappa = 8.5$.

3 Moody diagram

By averaging the logarithmic velocity profiles that we have determined on the layer of thickness $R \ll \tau_*/G$, one can link the friction force to the averaged velocity of the flow. The Moody diagram extend this relatioon to the cas of the walls which nature is intermediary between the smooth and rough limit cases. By linking the friction forces to the lineic head loss in a pipe, one gets a model usefull for the applications, which can be completed by the parametrizations of the singular head losses.

3.1 Average friction

We consider the velocity profil $\overline{u}/u_* = \frac{1}{\kappa}\ln(z/\delta) + \zeta$, with $(\delta, \zeta) = (\delta_{sth}, \zeta_{sth})$ in the smooth case and $(\delta, \zeta) = (\delta_{rgh}, \zeta_{rgh})$ in the rough case. We then define the average velocity of the layer $z \in [0, R]$ by the relation $U = \frac{1}{R}\int_0^R \overline{u}\, dz$.

If z_{vis}/R, in the smooth case, and z_0/R, in the rough case, are small enough, one can write

$$\frac{U}{u_*} = \frac{1}{R}\int_0^R \frac{\overline{u}}{u_*}\, dz = \frac{1}{\kappa}\ln\left(\frac{R}{\delta}\right) + \zeta - \frac{1}{\kappa}\ , \tag{3.29}$$

neglecting the integral on $[0, z_{vis}]$ in the smooth case and $[0, z_0]$ in the rough case and the primitive $(z/\delta)\ \ln(z/\delta) - (z/\delta)$ of the function $\ln(z/\delta)$ respectively at $z = z_{vis}$ and $z = z_0$.

For practical studies, it is useful to look at the relation between the vertically averaged velocity U and the tangential stress τ_*. A dimensionless analysis leads to the relation $\tau_* = \frac{1}{2}\, C_f\, \rho\, U^2$ where C_f is a dimensionless coefficient, called "drag coefficient", which can depend on U, R and of other parameter of the flow such as ν or k_s.

Very often, the "friction coefficient" λ, defined by the relation $\lambda = 4\ C_f$, is preferred. Using the definition of the friction velocity u_* from the relation $\tau_* = \rho\, u_*^2$, the relation between τ_* and U or, equivalently, between u_* and U, reads

$$\tau_* = \frac{1}{8}\,\lambda\,\rho\, U^2 \quad \Longleftrightarrow \quad \frac{U}{u_*} = \sqrt{\frac{8}{\lambda}}\ . \tag{3.30}$$

It is also common to define the "hydraulic diameter" $D_H = 4\ R$ associated to the considered domain $z \in [0, R]$. With these definitions, the relation between the friction coefficient λ and the velocity U now reads

$$\frac{1}{\sqrt{\lambda}} = a\ \log_{10}\left(\frac{D_H}{\delta}\right) + b\ , \tag{3.31}$$

where $a = \ln(10)/(\kappa\sqrt{8}) = 2.0$ and $b = [\zeta - (1 + \ln 4)/\kappa]/\sqrt{8}$.

Using the expression $(\delta_{rgh}, \zeta_{rgh}) = (k_s, 8.5)$ for the rough regime and $(\delta_{sth}, \zeta_{sth}) = (\nu/u_*, 5.2)$ for the smooth regime, the friction coefficients of both regimes respectively read

$$\frac{1}{\sqrt{\lambda_{ru}}} = -2.0\ \log_{10}\left(\frac{Ru}{\alpha_f}\right)\ ,\quad Ru = \frac{k_s}{D_H} \tag{3.32}$$

with $2.0\ \log_{10}(\alpha_f) = b_{rgh} = [\zeta_{rgh} - (1 + \ln 4)/\kappa]/\sqrt{8} = 0.9$ and

$$\frac{1}{\sqrt{\lambda_{sth}}} = -2.0\ \log_{10}\left(\frac{\beta_f}{Re\,\sqrt{\lambda_{sth}}}\right)\ ,\quad Re = \frac{U\, D_H}{\nu}, \tag{3.33}$$

with $2.0 \log_{10}\left(\beta_f/\sqrt{8}\right) = b_{sth} = [\zeta_{sth} - (1+\ln 4)/\kappa]/\sqrt{8} = -0.2$. The Reynolds number Re and the number Ru are two dimensionless numbers. The numerical values of the coefficients coming out of this analysis are $(\alpha_f, \beta_f) = (2.8, 2.2)$.

Contrarily to κ and thus $a = 2.0$, the coefficients (α_f, β_f) depend on the geometry of the flow which can be confined in a pipe of mundane section or own a free surface in a chanel of mundane section. One has to estimate these coefficients with models of the type of the one that we have just presented in the case of a flat bottom or to measure them experimentally. Typical values are $\alpha_f \in [2, 4]$ and $\beta_f \in [0, 6]$

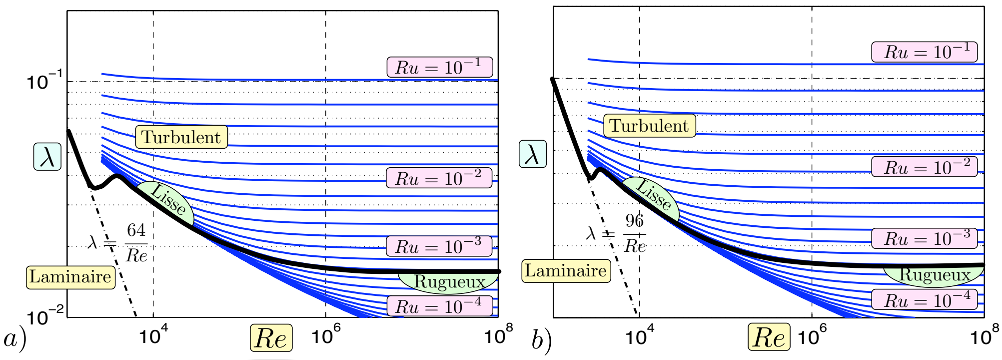

Figure 3.8: *Moody diagram for the Colebrook formula. a) Flows in pipes with* $(\alpha_f, \beta_f) = (3.7, 2.51)$. *b) Open channel flows with* $(\alpha_f, \beta_f) = (3, 2.5)$.

For "intermediate regimes", which are neither smooth nor rough, these formula are completed by the Colebrook formula which reads

$$\frac{1}{\sqrt{\lambda}} = -2.0 \log_{10}\left(\frac{Ru}{\alpha_f} + \frac{\beta_f}{Re\sqrt{\lambda}}\right) . \qquad (3.34)$$

For confined flows in pipes, the choice $(\alpha_f, \beta_f) = (3.7, 2.51)$ is very common. In this las case, the Haaland formula $\frac{1}{\sqrt{\lambda}} = -1.8 \log_{10}\left[\frac{6.9}{Re} + \left(\frac{Ru}{3.7}\right)^{1.11}\right]$, which has the advantage to be explicit, only depart by 2% of the Colebrook one.

A very common set of values for open channel flows is $(\alpha_f, \beta_f) = (3, 2.5)$, particularily valid for channels with trapezoid sections. The choice $(\alpha_f, \beta_f) = (3, 3.4)$ is preferred for large sections.

The dependency of $\lambda(Re, Ru)$ with respect to Re and Ru constitutes the "Moody diagram" which shown on Figure 3.8. At low Reynolds, when the flow is laminar, one finds the analytic relation $\lambda = 64/Re$ for the circular

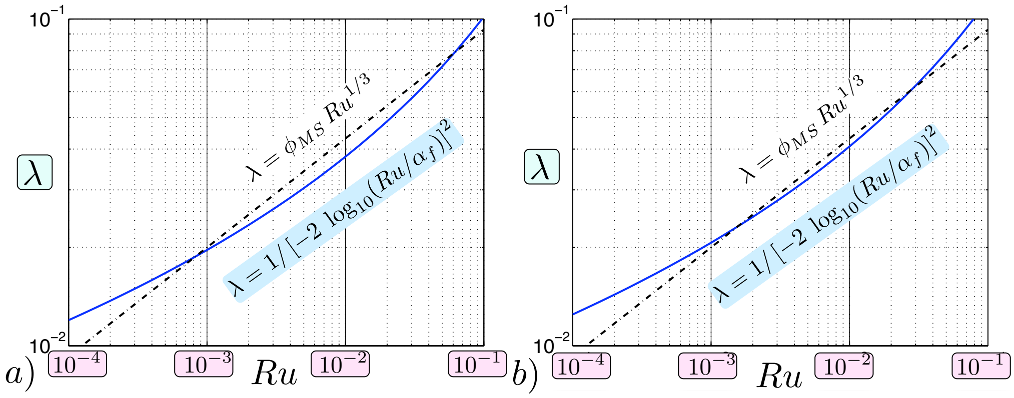

Figure **3**.9: *Comparison between the function $\lambda(Ru)$ of the Colebrook formula at large Re and the Manning- Strickler parametrization $\lambda = \phi_{MS}\, Ru^{1/3}$ with $\phi_{MS} = 0.2$. a) $\alpha_f = 3.7$, b) $\alpha_f = 3$.*

Poiseuille flow and $\lambda = 96/Re$ for the plane Poiseuille flow on a tilted plane. Between the laminar and the fully turbulent regime, a "transitional regime" is observed with a friction coefficient λ which can be growing on a small interval with Re at fixed Ru.

Rough regimes are obtained in the limit of large Reynolds numbers Re such that λ only depends on Ru. The function $\lambda(Ru)$ for infinite Re is shown on Figure **3**.9 for $\alpha_f = 3.7$ et $\alpha_f = 3$. For the values $Ru \in [10^{-4}, 10^{-1}]$ that are useful for practical applications, this function can be replaced by the Manning-Strickler parametrization $\lambda = \phi_{MS}\, Ru^{1/3}$, where the value $\phi_{MS} = 0.2$ can be chosen for the two considered values.

3.2 Hydraulic in pipes

The use of the Moody diagram is very common to model the flows confined in pipes. One first writes the incompressible and turbulent Navier-Stokes equations under the form

$$\text{div}\, \underline{\overline{U}} = 0 \quad , \quad \frac{\partial \underline{\overline{U}}}{\partial t} + \underline{\overline{U}} \cdot \underline{\text{grad}}\, \underline{\overline{U}} = -\frac{1}{\rho} \underline{\text{grad}}\, \overline{p} - \underline{\text{grad}}\, (g\, z) + \frac{1}{\rho}\, \underline{\text{div}}\, \underline{\underline{\overline{\tau}}}\ , \quad (\mathbf{3}.35)$$

where $\underline{\underline{\overline{\tau}}} = \rho(2\,\nu\, \underline{\underline{\overline{d}}} - \underline{\underline{R}})$ and $\underline{\underline{\overline{d}}}$ are respectively the Reynolds averages of the viscous and turbulent stress tensor and of the deformation rate tensor. One considers a current line $\mathcal{L}$ going from the point M_1 toward the point M_2. Using the identity $\underline{\overline{U}} \cdot \underline{\text{grad}}\, \underline{\overline{U}} = \frac{1}{2}\, \underline{\text{grad}}\, \underline{\overline{U}}^2 + \underline{\text{rot}}\, \underline{\overline{U}} \wedge \underline{\overline{U}}$, one shows the "Bernoulli

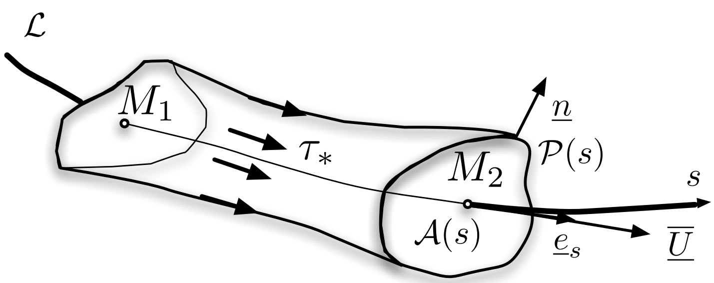

Figure 3.10: *Flow in a pipe.*

relation"

$$H(M_2) - H(M_1) = \int_{\mathcal{L}} \underline{\text{grad}}\, H \cdot \underline{dM} = -\int_{\mathcal{L}} \left(\frac{1}{g} \frac{\partial \overline{U}}{\partial t} + \overline{\underline{J}} \right) \cdot \underline{dM} \,, \qquad (3.36)$$

where $\overline{\underline{J}} = -\frac{1}{\rho\, g} \underline{\text{div}}\,(\underline{\underline{\overline{\tau}}})$ and where H is the "hydraulic head" defined by the relation

$$H(\underline{x}, t) = \frac{\overline{p}}{\rho\, g} + z + \frac{1}{2\, g} \overline{\underline{U}}^2 \,. \qquad (3.37)$$

The term $\frac{1}{g}\frac{\partial \overline{U}}{\partial t}$ is the lineic head loss due to the instationarity of the flow while $\overline{\underline{J}}$ is the lineic head loss due to the viscous and turbulent friction.

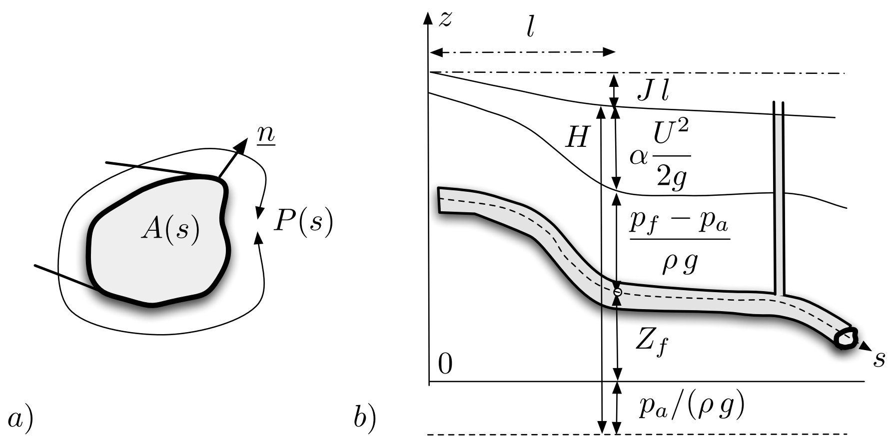

Figure 3.11: *Flow confined in a pipe. a) Section A and wet perimeter P. b) Représentation de la charge moyenne H_{charge} where p_a is the atmospheric pressure.*

For fluid flows filling the whole volume of a pipe, called "loaded flows", one defines the hydraulic radius $R_H(s) = A(s)/P(s)$ as the ratio between the area $A(s)$ of the section $\mathcal{A}(s)$ and its "wet perimeter" $P(s)$ of its boundary $\mathcal{P}(s)$ (see Figure **3**.11a). One denotes by $z = Z_f(s)$ the equation of the pipe axis $\mathcal{L}$ and $\overline{p}_f(s)$ the average pressure on this axis.

In a lot of cases, one notices that Reynolds averaged pressure is hydrostatic. In this cas, the quantity $\frac{\overline{p}}{\rho g} + z = \frac{p_f}{\rho g} + Z_f$ is constante in a section normal to the axis. The hydraulic head averaged on such a section then reads

$$H(s) = \frac{p_f(s)}{\rho g} + Z_f(s) + \alpha(s)\,\frac{U^2(s)}{2\,g} \; . \qquad (\mathbf{3}.38)$$

where the averaged velocity $U(s)$ and the coefficient $\alpha(s)$ are defined by the relations

$$U(s) = \frac{1}{A(s)} \iint_{\mathcal{A}(s)} \underline{\overline{U}} \cdot \underline{e}_s \, dS \quad , \qquad \alpha(s) = \frac{1}{U^2(s)}\,\frac{1}{A(s)} \iint_{\mathcal{A}(s)} \underline{U}^2 \, dS \; . \quad (\mathbf{3}.39)$$

One defines the averaged lineic head loss $J(s)$ due to the friction by the relation

$$J(s) = \frac{1}{A(s)} \iint_{\mathcal{A}(s)} \underline{\overline{J}} \cdot \underline{e}_s \, dS = -\frac{1}{\rho\, g}\frac{1}{A(s)} \iint_{\mathcal{A}(s)} \underline{\mathrm{div}\, \underline{\underline{\overline{\tau}}}} \cdot \underline{e}_s \, dS \; . \qquad (\mathbf{3}.40)$$

One then defines the averaged mean shear constraint $\tau_*(s)$ applied by the fluid on the wall by the relation

$$\tau_*(s) = -\frac{1}{P(s)} \int_{\mathcal{P}(s)} \underline{e}_s \cdot \underline{\underline{\overline{\tau}}} \cdot \underline{n} \, dl \; , \qquad (\mathbf{3}.41)$$

where $\underline{n}$ is the unit vector normal to the wall of the pipe pointing towards the exterior.

One now assumes that the turbulence of the flow is stationary and fully developed and that the shape of the pipe varies slowly as a function of s (gradually varied flow).

One then notices that the velocity profile $\underline{\overline{U}}$ is nearly flat, such that the coefficient $\alpha(s)$ is close to the value $\alpha = 1$ and that the unit vector $\underline{e}_s$ is nearly constant on a section $\mathcal{A}(s)$. The "gradually varying flow" hypothesis implies, in particular, that $\frac{\partial}{\partial s}(\underline{e}_s \cdot \underline{\underline{\overline{\tau}}} \cdot \underline{e}_s)$ is neglectable. The application of the divergence theorem on a small portion of the pipe the leads to the important relation

$$\tau_*(s) = \rho\, g\, R_H(s)\, J(s) \; . \qquad (\mathbf{3}.42)$$

The parametrization of the mean friction $\tau_*(s)$ as a function of $U(s)$, $D_H(s)$, $Re(s) = U(s) D_H(s)/\nu$ et $Ru(s) = k_s/D_H(s)$ is equivalent to the parametrization of the averaged lineic head loss $J(s)$, which is summed up by the relations

For stationnary ($\frac{\partial}{\partial t} = 0$), turbulent ($\alpha \sim 1$) and gradually varying ($\tau_s = \rho\, g\, R_H\, J$) flows, one can then write

$$H = \frac{p_f}{\rho\, g} + Z_f + \frac{U^2}{2g} \quad , \qquad \frac{dH}{ds} = -J \quad , \qquad J = \lambda(Re, Ru)\, \frac{U^2}{2\, g\, D_H} \ . \quad (3.43)$$

These relations, where λ is coming out the Moody diagram is a first step in the computation of a pipe network.

3.3 Singular head loss

To compute the head losses in a pipe network, one must also take into account, in addition to the slowly variable in space duct parts, singularities such as the sharp widenings, the bends and other singularities of the network. The head then undergoes a sharp decrease that one models by a signular head loss that must be parametrized for each geometries. One often expresses the singular head loss under form

$$\Delta H = K_g\, \frac{U^2}{2\, g} = K_g\, \frac{Q^2}{2\, g\, A^2} \ , \quad (3.44)$$

where U and A are respectively the velocity and the section upstream of the singularity, K_g the singular head loss coefficient and $Q = U\, A$ the flow discharge.

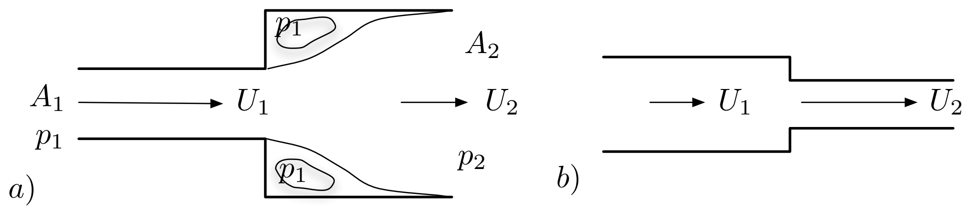

Figure 3.12: *a) Singular head loss un a sharp widening. b) No singular head loss in a sharp narrowing.*

In the case of a sharp widening, one can give an estimation of the singular head loss by assuming that the fluid pressure in the recirculation zone is approximatively equal to the entrance pression p_1. In this cas, a global momentum budget (Euler theorem) on a domain including the singularity leads to

$$\rho\, U_1^2\, A_1 + p_1\, A_2 = \rho\, U_2^2\, A_2 + p_2\, A_2 \ . \quad (3.45)$$

One deduces of this the singular head loss

$$\Delta H = \frac{(U_2 - U_1)^2}{2\, g} = K_g\, \frac{U_1^2}{2\, g} \qquad \text{avec} \qquad K_g = \left(1 - \frac{A_1}{A_2}\right)^2 \ . \quad (3.46)$$

In the case of a section sharpening, one can consider that the singular head loss is not neglectable.

In the case of a bend of radius of curvature ρ_c and of deviation φ, one can use the formula $K_g = \frac{\varphi}{\pi/2}\left[0.131 + 1.847\left(\frac{2\rho_c}{D_H}\right)^{-3.5}\right]$ for the singular head loss coefficient.

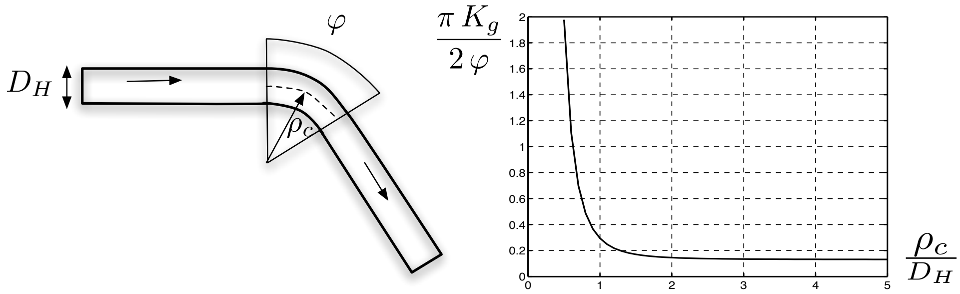

Figure **3**.13: *Singular head loss in a bend.*

FORMULAS

Turbulence modelling

Mean and fluctuations:

$$B = \overline{B} + B' \ , \qquad \underline{U} = \overline{\underline{U}} + \underline{U}' \ .$$

Turbulent diffusion:

$$\frac{\partial \overline{B}}{\partial t} + \overline{\underline{U}} \cdot \underline{\mathrm{grad}}\ \overline{B} = k_B\, \Delta \overline{B} - \mathrm{div}\ (\overline{\underline{U}'\, B'})\ , \quad \underline{F}_{Bt} = \overline{\underline{U}'\, B'} = -k_{Bt}\ \underline{\mathrm{grad}}\ \overline{B}\ .$$

Reynolds tensor:

$$\mathrm{div}\ \overline{\underline{U}} = 0\ , \quad \frac{\partial}{\partial t}\overline{\underline{U}} + \overline{\underline{U}} \cdot \underline{\mathrm{grad}}\ \overline{\underline{U}} = -\frac{1}{\rho}\underline{\mathrm{grad}}\ \overline{p} - \underline{\mathrm{grad}}\ (g\, z) + \nu\, \Delta \overline{\underline{U}} - \underline{\mathrm{div}}\ \underline{\underline{R}}\ .$$

Turbulent viscosity:

$$R_{ij} = \overline{U_i'\,U_j'} = -2\,\nu_t\,\overline{d}_{ij} + \frac{2}{3}\,k\,\delta_{ij}\;, \qquad k = \frac{1}{2}\mathrm{tr}\;\underline{\underline{R}} = \frac{1}{2}\left(\overline{U_1'^2} + \overline{U_2'^2} + \overline{U_3^2}\right)\;.$$

Velocity profiles

Mixing length:

$$\nu_t = l_m^2\,\sqrt{2\,\overline{\underline{\underline{d}}} : \overline{\underline{\underline{d}}}}\;.$$

Parallel flow:

$$\overline{u'w'} = -\nu_t\,\frac{\partial\overline{u}}{\partial z}\;, \quad \nu_t = l_m^2\,\left|\frac{\partial\overline{u}}{\partial z}\right|\;.$$

Von Karman law:

$$l_m = \kappa\,z \qquad , \qquad \kappa = 0.41\;.$$

Flat bottom:

$$\tau(z) = \rho\,\left(\nu + \kappa^2\,z^2\left|\frac{\partial\overline{u}}{\partial z}\right|\right)\frac{\partial\overline{u}}{\partial z} = \tau_* - G\,z \sim \tau_*\;, \qquad \tau_* = \rho\,u_*^2\;.$$

Smooth of rough:

$$\frac{u_*\,k_s}{\nu} < 5\;, \quad z_{vis} = 11\,\frac{\nu}{u_*}\;, \qquad \frac{u_*\,k_s}{\nu} > 70\;, \quad z_0 = k_s/33\;.$$

$$\frac{\overline{u}_{sth}}{u_*} = \frac{1}{\kappa}\;\ln\left(\frac{u_*\,z}{\nu}\right) + 5.2\;, \qquad \frac{\overline{u}_{rgh}}{u_*} = \frac{1}{\kappa}\;\ln\left(\frac{z}{k_s}\right) + 8.5\;.$$

Moody diagram

Friction coefficients:

$$\tau_* = \frac{1}{2}\,C_f\,\rho\,U^2 = \frac{1}{8}\,\lambda\,\rho\,U^2\;.$$

Smooth bottom:

$$\frac{1}{\sqrt{\lambda_{sth}}} = -2.0 \log_{10}\left(\frac{\beta_f}{Re\sqrt{\lambda}}\right) \quad , \quad Re = \frac{U\,D_H}{\nu} \ .$$

Rough bottom:

$$\frac{1}{\sqrt{\lambda_{rgh}}} = -2.0 \log_{10}\left(\frac{Ru}{\alpha_f}\right) \quad , \quad Ru = \frac{k_s}{D_H} \ .$$

Colebrook formula:

$$\frac{1}{\sqrt{\lambda}} = -2.0 \log_{10}\left(\frac{Ru}{\alpha_f} + \frac{\beta_f}{Re\sqrt{\lambda}}\right) \ .$$

Confined hydraulics

Averaged hydraulic load:

$$H(s) = \frac{p_f(s)}{\rho\, g} + Z_f(s) + \alpha(s)\,\frac{U^2(s)}{2\,g} \ .$$

Stationary flow:

$$\frac{dH}{ds} = -J \quad \text{avec} \quad H = \frac{p_f}{\rho\, g} + Z_f + \frac{U^2}{2g} \quad , \quad J = \lambda(Re, Ru)\,\frac{U^2}{2\,g\,D_H} \ .$$

Gradually varying flow:

$$\tau_*(s) = \rho\, g\, R_H(s)\, J(s) \ .$$

Singular head loss:

$$\Delta H = K_g\,\frac{U^2}{2\,g} = K_g\,\frac{Q^2}{2\,g\,A^2} \ .$$

EXERCISES

See exercices in French language in the book:

O. THUAL, Hydrodynamique de l'Environnement, Éditions de l'École Polytechnique, 2010.

or at `http://thual.perso.enseeiht.fr/xsee`