Chapter 4

Open channel flows

O. Thual, September 20, 2010

Contents

Introduction

The objective of this chapter is to present the basic models which describe open channel flows. Such flows are “gradually varied” the size of these sections are small compared to the chararcteristic scales of variation of their slope. We only consider here stationary flows under the “gradual variation” hypothesis.

Figure 4.1: *The Amazon seen from space. Photo NASA.*

The concept of “hydraulic head” for open channel flow is presented on the Bernoulli equation derived from the Navier-Stokes equations. The equation for the hydraulic head along the channel in the stationary case is written and applied to the case of a flow over an obstacle when the friction can be neglected. The concepts of “super-critical” and “sub-critical” flows, depending on the Froude number, are presented of this example.

Considerations on boundary conditions explain the advent of hydraulic jumps. The discontinuity equations for these hydraulic jumps are presented in a simple manner. The turbulent friction of the bottom on the channel is modelled through a Manning-Strickler parametrization, which is very common for open channel flow engineer approach. The description of the “back water curves” is given in the case of this parametrization and the notions of “normal height” and “critical height” are used to classify these curves.

1 Hydraulic head

The Bernoulli equation leads naturally to the notion of hydraulic head. One cas then average it on the section of a channel and define the notion of lineic head loss along its axis.

1.1 Bernoulli equation

We start by considering the incompressible and turbulent Navier-Stokes equations

$$\operatorname{div} \underline{U} = 0 \quad , \quad \frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \underline{\operatorname{grad}}\, \underline{U} = \underline{F} - \frac{1}{\rho}\, \underline{\operatorname{grad}}\, p + \nu\, \Delta \underline{U} - \underline{\operatorname{div}}\, \underline{\underline{R}} \,, \qquad (\mathbf{4}.1)$$

in which ρ is the constant mass density and $\underline{\underline{R}}$ is the Reynolds stress tensor, defined by $R_{ij} = \overline{U_i' U_j'}$ where the components U_i' are the velocity turbulent fluctuations. The fields $\overline{\underline{U}}$ and $\overline{p}$ of these equations are the "Reynolds averaged" velocity and pressure that we omit to denote by $\overline{\underline{U}}$ and $\overline{p}$ for simplicity. We assume that the volume forces $\underline{F} = -g\, \underline{e}_z = -\underline{\operatorname{grad}}\, (g\, z)$ are due to gravity.

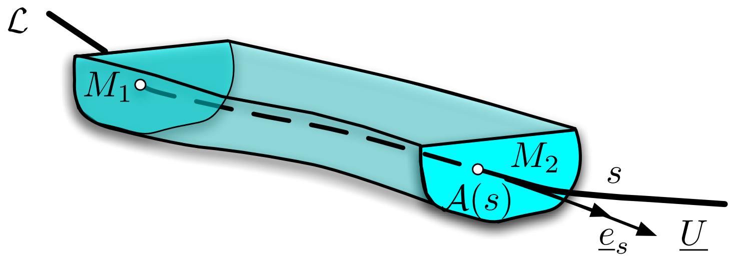

Figure **4**.2: *Streamline $\mathcal{L}$ in an open channel flow.*

Let us consider a stream line $\mathcal{L}$ going from a point M_1 to a point M_2. By using the relation

$$\underline{U} \cdot \underline{\operatorname{grad}}\, \underline{U} = \frac{1}{2}\, \underline{\operatorname{grad}}\, \underline{U}^2 + \underline{\operatorname{rot}}\, \underline{U} \wedge \underline{U} \,, \qquad (\mathbf{4}.2)$$

one can derive the "Bernoulli equation"

$$\int_{\mathcal{L}} \underline{\operatorname{grad}}\, H \cdot \underline{dM} = \frac{1}{g} \int_{\mathcal{L}} \left(-\frac{\partial \underline{U}}{\partial t} + \nu\, \Delta \underline{U} - \underline{\operatorname{div}}\, \underline{\underline{R}} \right) \cdot \underline{dM} \,, \qquad (\mathbf{4}.3)$$

where H is the "hydraulic head" defined by the relation

$$H = \frac{p}{\rho\, g} + z + \frac{1}{2\, g}\, \underline{U}^2 \,. \qquad (\mathbf{4}.4)$$

By integrating from M_1 to M_2, the Bernoulli equation reads

$$H(M_2) = H(M_1) - \int_{\mathcal{L}} \left(\frac{1}{g} \frac{\partial \underline{U}}{\partial t} + \underline{J} \right) \cdot \underline{dM} \;, \quad \underline{J} = \frac{1}{g}(-\nu \, \Delta \underline{U} + \underline{\mathrm{div}} \, \underline{\underline{R}}) \;. \quad (\mathbf{4}.5)$$

The term $\frac{1}{g}\frac{\partial \underline{U}}{\partial t}$ is the lineic head loss which vanishes for stationary regimes that are considered here. The term $\underline{J}$ is the lineic head loss due to viscous and turbulent frictions.

1.2 Geometric parameters

Given an open channel flow, on can consider the family of sections $\mathcal{A}(s)$ which follow a streamline $\mathcal{L}$, parametrised by the curvilinear coordinate s (see Figure **4**.3). We denote by $\mathcal{P}(s)$ the part of the boundary of the section $\mathcal{A}(s)$ which is in contact with the inner surface of the channel of width $L(s)$. For each section $\mathcal{A}(s)$, one can define its area $A(s)$, the perimeter $P(s)$ of $\mathcal{P}(s)$, that one denotes by "wet perimeter" and the width $L(s)$, that one denotes by "top width".

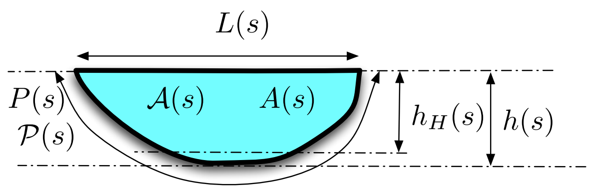

Figure **4**.3: *Section $A(s)$, wet perimeter $P(s)$, top width $L(s)$, maximal depth $h(s)$ and hydraulic depth $h_H(s)$.*

One then define the "hydraulic radius" $R_H(s)$, the "hydraulic diameter" $D_H(s)$ and the "hydraulic height" $h_H(s)$ by the relations

$$R_H(s) = \frac{A(s)}{P(s)} \quad , \quad D_H(s) = 4\, R_H(s) \quad , \quad h_H(s) = \frac{A(s)}{L(s)} \;. \qquad (\mathbf{4}.6)$$

The hydraulic height $h_H(s)$ is not necessarily equal to the true height $h(s)$ which is defined as being the greatest distance between the bottom and the free surface. But in most practical applications, the approximation $h \sim h_H$ is assumed.

When the section of the channel is a rectangle of height h and of length L with $L \gg h$, one has $R_H = h$ and $h_H = h$. Numerous typical sections, such as trapezoids, can be considered to describe open channels.

1.3 Averaged head

In numerous applications, open channel flows can be described by quantities averaged on the section $\mathcal{A}(s)$ of area $A(s)$.

For a given abscissa s, the volumetric flux $Q(s)$ and the averaged velocity $U(s)$ are defined by the relation

$$Q(s) = \iint_{\mathcal{A}} \underline{U} \cdot \underline{e}_s \, dS = A(s)\, U(s) \implies U(s) = \frac{1}{A(s)} \iint_{\mathcal{A}} \underline{U} \cdot \underline{e}_s \, dS \ . \qquad (4.7)$$

The averaged hydraulic head is defined by

$$H(s) = \frac{1}{A(s)} \iint_{\mathcal{A}} \left(\frac{p}{\rho\, g} + z + \frac{\underline{U}^2}{2g} \right) dS = \frac{P_*(s)}{\rho\, g} + \alpha(s) \frac{U^2(s)}{2g} \ , \qquad (4.8)$$

where P_* and α are defined by

$$P_*(s) = \frac{1}{A(s)} \iint_{\mathcal{A}} (p + \rho\, g\, z)\, dS \quad , \quad \alpha(s) = \frac{1}{U^2(s)} \frac{1}{A(s)} \iint_{\mathcal{A}} \underline{U}^2 \, dS \ . \qquad (4.9)$$

One has $\alpha = 1$ when the velocity field $\underline{U}$ is constant on the section $\mathcal{A}$. For turbulent flows, the velocity profile is flat and α is nearly close to one.

The quantity $P_*(s)$ can be called the "averaged piezometric pressure". When the local pressure p can be considered as hydrostatic, which is often the case for open channel flows, the local piezometric pressure $p_* = p + \rho\, g\, z$ is everywhere equal to the averaged piezometric pressure P_*. This is the case when the flow is "gradually varied", that is when the variation scale of the fields in the s direction is large compared to the scale of the sections. We make this hypothesis for the flows of this chapter.

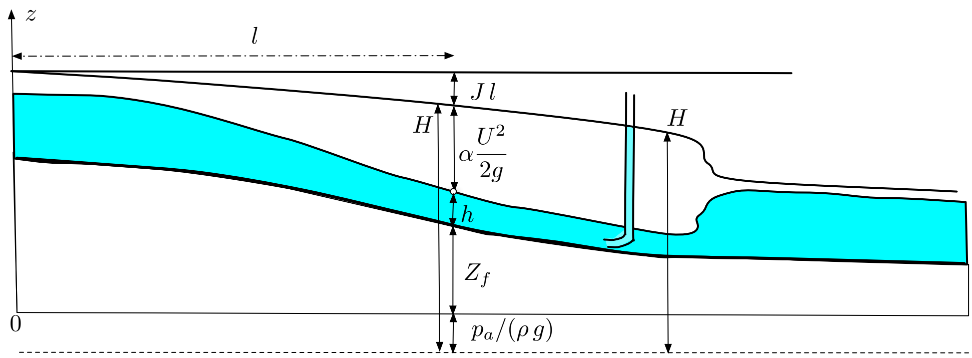

Figure 4.4: *Averaged head* $H = \frac{p_a}{\rho\, g} + Z_f + h + \alpha \frac{U^2}{2\, g}$ *and lineic head loss* J.

We denote by $z = Z_f(s)$ the equation of the deepest point (figure **4**.4) and we denote $I(s) = -Z_f'(x)$, with an abuse of language, the "slope of the channel". If $I(s)$ is small, this definition meets the notion of slope with respect to the horizontal and on can, furthemore, consider than $h(s)$ is close to the depth of the channel measured on a vertical axis. We make this hypothesis for the channels of this chapter. With all these hypotheses ($\alpha \sim 1$, $I \ll 1$, gradually varied), the averaged head reads

$$H(s) = \frac{p_a}{\rho\, g} + Z_f(s) + h(s) + \alpha(s) \frac{U^2(s)}{2\, g} \; . \tag{4.10}$$

When averaging the Bernoulli equation in the stationary case $\frac{\partial U}{\partial t} = 0$, one obtains

$$H(s_2) - H(s_1) = -\int_{s_1}^{s_2} J(s)\, ds \quad , \quad J(s) = \frac{1}{A(s)} \iint_{\mathcal{A}} \underline{J} \cdot \underline{e}_s \, dS \; . \tag{4.11}$$

By deriving with respect to s_2, the lineic head loss $J(s)$ satisfies the relation

$$\frac{dH}{ds} = -J \; . \tag{4.12}$$

2 Specific head and impulsion

The stationary equations are reduced to a single ordinary differential equation for the water height. In the absence of friction, le specific head is conserved through an obstacle. Le jump conditions through a hydraulic jump are explicited.

2.1 Stationary equations

We consider an open channel flow which is stationary ($\frac{\partial}{\partial t} = 0$), turbulent ($\alpha \sim 1$), with a small slope ($I \ll 1$) and gradually varied. The equilibrium equation reads

$$\frac{dH}{ds} = -J \quad , \quad H = \frac{p_a}{\rho\, g} + h + Z_f + \frac{U^2}{2g} \; , \tag{4.13}$$

where J is the lineic head loss which must be parameterized as a function of h and U with the help of a turbulence model.

To simplify the following presentation, we assume that the mirror width L is constant and the hydraulic height h_H can be replaced byt the depth h. One then has $h_H = h$, $A = L\, h_H = L\, h$ and $Q = A\, U = L\, h\, U$ where Q is the discharge rate. One then define the lineic discharge rate $q = Q/L = h\,U$ which is then constant for the stationnary flows considered here.

With these hypotheses, the expression of the hydraulic head read

$$H(s) = h(s) + \frac{q^2}{2\,g\,h^2(s)} + Z_f(s) + \frac{p_a}{\rho\,g}\ . \qquad (\mathbf{4}.14)$$

By denoting $I = -Z_f'(s)$ the slope of the bottom, the equilibrium equation (**4**.13) reads

$$\frac{dh}{ds} = \frac{I - J}{1 - Fr^2} \quad , \quad Fr = \frac{U}{\sqrt{g\,h}} = \frac{q}{g^{1/2}\,h^{3/2}}\ , \qquad (\mathbf{4}.15)$$

where the dimensionless number Fr, called the "Froude number", appears. On thus define the "critical height" h_c by the relation par la relation

$$h_c = \left(\frac{q^2}{g}\right)^{1/3} \quad \Longrightarrow \quad Fr = \left(\frac{h_c}{h}\right)^{3/2}\ . \qquad (\mathbf{4}.16)$$

2.2 Flow over an obstacle

As a first example of application of the equilibrium equation (**4**.15), we consider a bottom of equation $z = Z_f(s)$ such that Z_f is a constant excepted on a obstacle of finite extend and of small slope. In this case, s can be chosen as the horizontal x coordinate. For instance, one can consider a Gaussian shaped obstacle by choosing $Z_f(s) = a\ \exp\left(-\frac{s^2}{2\,\sigma^2}\right)$ where a is small in front of σ.

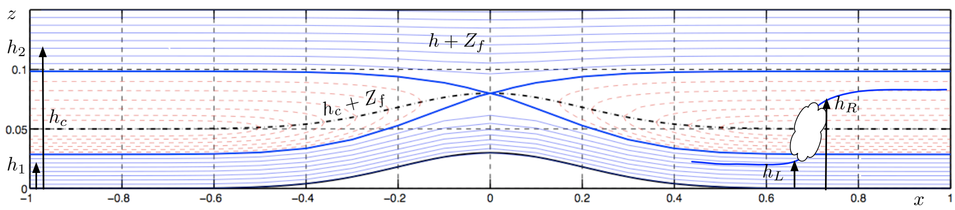

Figure **4**.5: *Curves* $z = h(x) + Z_f(x)$.

If the horizontal extension of the obstacle is not too big, one can consider that the lineic head loss $J \sim 0$ is negligible compared to I. The equilibrium equation (**4**.15) reads

$$\frac{dh}{ds} = \frac{I(s)}{1 - (h/h_c)^{-3}} \quad \Longrightarrow \quad \left[1 - (h/h_c)^{-3}\right]\frac{dh}{ds} = I = -\frac{dZ_f}{ds}\ . \qquad (\mathbf{4}.17)$$

Since h_c is considered as constant on the studied part of the channel, this differential equation can be integrated into $h(s) + \frac{1}{2}h_c^3/h^2(s) = H_0 - Z_f(s)$

where H_0 is an integration constant. One finds back the head conservation

$$H = h(s) + \frac{U(s)^2}{2\,g} + \frac{p_a}{\rho\,g} + Z_f(s) \quad \Longrightarrow \quad H_0 = H - \frac{p_a}{\rho\,g}\,. \tag{4.18}$$

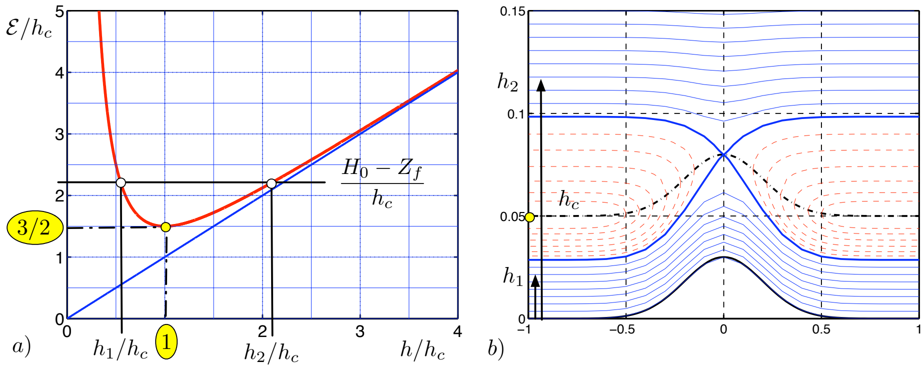

Figure 4.6: *a) Dimensionless specific head* $\frac{\mathcal{E}(q,h)}{h_c} = \frac{h}{h_c} + \frac{1}{2}\left(\frac{h}{h_c}\right)^{-2}$ *with* $h_c = q^{2/3}\,g^{-1/3}$. *b) Possible free surface profiles.*

One then defines the "specifici head" by the relations

$$\mathcal{E}(q,h) = h + \frac{U^2}{2\,g} = h + \frac{q^2}{2\,g\,h^2} = h + \frac{1}{2}\frac{h_c^3}{h^2}\,. \tag{4.19}$$

where H_0 is an integration constant. The specific head $\mathcal{E}(h)$, is drawn on Figure 4.6. For a given flow rate q, it is minimum for h_c with $\mathcal{E}(q,h_c) = \frac{3}{2}\,h_c$. For $H_0 - Z_f \geq E_c$, there exists two solutions h_1 and h_2 to the equation $\mathcal{E}(q,h) = H_0 - Z_f$. These two solutions are called "conjugate heights for the specific head".

At the point s where $h < h_c$, and thus $Fr > 1$, the flow is said to be "super-critical". When $h > h_c$, and thus $Fr < 1$, the flow is say to be "sub-critical". It is "critical" at the point where $h = h_c$ and thus $Fr = 1$.

Figure 4.6b shows all the curves $z = Z_f(s) + h(s)$ where $h(s)$ are all the solutions of the equation $\mathcal{E}[q, h(s)] = H_0 - Z_f(s)$ for fixed q with all the possible values of H_0. The dashed lines corresponds to non physical curves, since they cannot cross the obstacle.

When the flow is everywhere sub-critical ($h > h_c$), the free surface height decreases when passing over the obstacle. When the the flow is everywhere super-critical, the free surface varies the other way. There is only one curve which goes from sub-critical to super-critical. For this curve the flow is critical at the summit of the obstacle.

2.3 Hydraulic jumps

One observes, in natural flows, that the free surface height can jump abruptly from $h_L < h_c$ to $h_R > h_c$ through stationary "hydraulic jump". Such an hydraulic jump allows the matching with a downstream boundary condition.

Figure **4**.7: *Stationary hydraulic jump.*

The discontinuity relation for the hydraulic jumps are obtained from a mass and momentum budget. Let us denote (h_L, U_L) the height and velocity at the left of the hydraulic jump and (h_R, U_R) the corresponding quantities at the right.

For the stationary case that is considered here, the mass conservation law says that the lineic discharge flux $q = U\,h$ is constant, which reads

$$U_L\; h_L = U_R\, h_R = q\;. \qquad (\mathbf{4}.20)$$

Since the pressure is supposed hydrostatic on both side of the jump, a momentum budget on a small domain including the discontinuity leads to

$$h_L\, U_L^2 + \frac{1}{2}\; g\, h_L^2 = h_R\, U_R^2 + \frac{1}{2}\; g\, h_R^2\;. \qquad (\mathbf{4}.21)$$

By eliminating U_L and U_R, one writes this relation under the form

$$\mathcal{I}(q, h_L) = \mathcal{I}(q, h_R)\;, \qquad (\mathbf{4}.22)$$

where the "impulse function" $\mathcal{I}(q, h)$ is defined by

$$\mathcal{I}(q, h) = h\, U^2 + \frac{1}{2}\, g\, h^2 = \frac{q^2}{h} + \frac{1}{2}\; g\, h^2 = \frac{g}{h}\left(h_c^3 + \frac{1}{2} h^3\right)\;. \qquad (\mathbf{4}.23)$$

The two heights h_L and h_R are called "conjugated heigths for the impulsion".

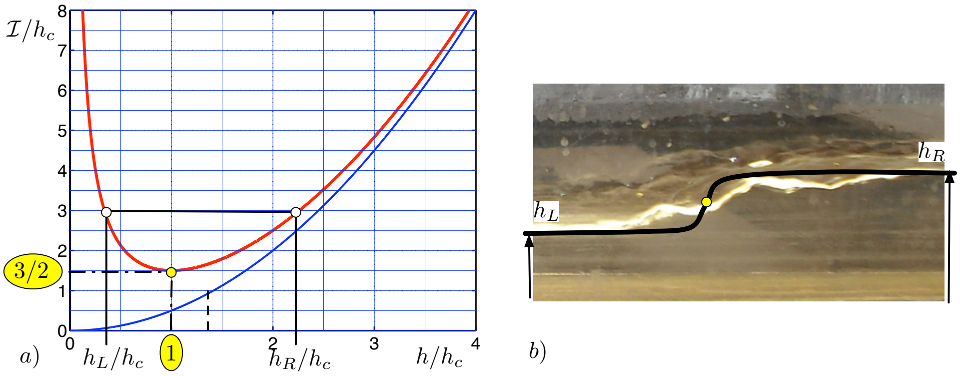

Figure **4**.8: *a) Dimensionless impulse function* $\frac{\mathcal{I}(q,h)}{h_c} = \left(\frac{h}{h_c}\right)^{-1} + \frac{1}{2}\left(\frac{h}{h_c}\right)^2$ *with* $h_c = q^{2/3}\, g^{-1/3}$. *b) Hydraulic jump.*

For q fixed, the minimum of $\mathcal{I}$ is reached for $h = h_c$. Since $F_r = (h_c/h)^{3/2}$, one sees then(Figure **4**.8) that the flow is and supercritical ($Fr > 1$) on one side of the jump subcritical ($Fr < 1$) on the other. Consideration on the energy dissipation implies that the Froude number decreased when following the flow direction.

3 Backwater curves

One shows that the mean friction is proportional to the lineic head loss. Both quantities are then jointly modelled, for example with the Manning-Strickler parametrization. The classification of the backwater curves is useful to describe the stationary solutions.

3.1 Friction and head loss

The lineic head loss due to the friction of the flow on the bottom is defined with

$$J(s) = \frac{1}{A(s)} \iint_{\mathcal{A}(s)} \underline{J} \cdot \underline{e}_s \, dS \; . \qquad (\mathbf{4}.24)$$

Coming back the turbulent Navier-Stokes equations (**4**.1), on can write

$$\underline{J} = \frac{1}{g}(-\nu\, \Delta \underline{U} + \underline{\mathrm{div}}\, \underline{\underline{R}}) = -\frac{1}{\rho\, g}\, \underline{\mathrm{div}}\, (\underline{\underline{\tau}}) \quad , \quad \underline{\underline{\tau}} = \rho \left(2\, \nu\, \underline{\underline{d}} - \underline{\underline{R}}\right) \; , \qquad (\mathbf{4}.25)$$

where $\underline{\underline{\tau}}$ is the stress tensor of the viscous and turbulent forces and $\underline{\underline{d}}$ the deformation rate tensor.

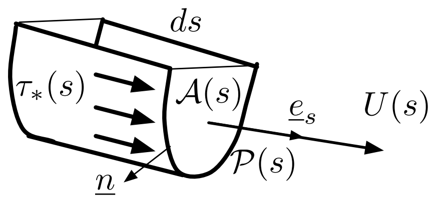

Figure **4**.9: *Shear stress τ_* applied by the flow on the walls of the wet perimeter P.*

One considers an open channel flow of wet perimeter P. We denote by τ_* the mean shear stress applied by the fluid on the wall and defined by

$$\frac{1}{P(s)} \int_{\mathcal{P}(s)} \underline{e}_s \cdot \underline{\underline{\tau}} \cdot \underline{n} \, dl = -\tau_*(s) \; , \tag{4.26}$$

where $\underline{n}$ is the unit vector normal the bottom (Figure **4**.9). When the flow is gradually varying, one can assume that

$$\iint_{\mathcal{A}} \frac{\partial}{\partial s} \left(\underline{e}_s \cdot \underline{\underline{\tau}} \cdot \underline{e}_s \right) \, dS \sim 0 \; . \tag{4.27}$$

The application of the divergence theorem to the tensor $\underline{\underline{\tau}}$, integrated on a small channel volume of infinitesimal length ds, then leads, by using the definitions (**4**.24) and (**4**.25) of J and $\underline{J}$, to the important equality

$$\tau_* = \rho \, g \, R_H \, J \qquad , \qquad R_H = \frac{A}{P} \; . \tag{4.28}$$

Thanks to this relation, a parametrization of the shear stress τ_* gives a parametrization of the lineic head loss J.

3.2 Strickler coefficient

The dimensional analysis expressing the tangential shear stress τ_* as a function of the mean velocity U, the hydraulic diameter D_H, the kinematic viscosity ν, the characteristic size of the roughness k_s, leads to

$$\tau_* = \frac{1}{8} \, \lambda(Re, Ru) \, \rho \, U^2 \; , \tag{4.29}$$

where the Reynolds number Re and Ru are defined by

$$Re = \frac{U \, D_H}{\nu} \qquad , \qquad Ru = \frac{k_s}{D_H} \; . \tag{4.30}$$

The relations $\tau_* = \rho\, g\, R_H\, J$ between J and τ_* and $D_H = 4\, R_H$ between the hydraulic diameter D_H and the hydraulic radius R_H then implies

$$J = \lambda\,(Re, Ru)\ \frac{U^2}{2\, g\, D_H}\ . \tag{4.31}$$

For turbulent open flow channel the Reynolds number is high for pratical applications and the bottom can thus be considered as rough. In this case, the friction parameter λ can be parametrized by the Manning-Strickler formula which reads

$$\lambda = \phi_{MS}\, Ru^{1/3}\ . \tag{4.32}$$

For practical studies in open channel flows, the value $\phi_{MS} = 0.1$ is relevant when choosing $k_s = d_{50}$, where d_{50} is the median of the distribution spectrum of the sediment sizes. Another approach, very common in hydraulics, consists in considering directly the Strickler number K_s which links J to U and R_H through the formula

$$U = K_s\, R_H^{\frac{2}{3}}\, J^{\frac{1}{2}}\ . \tag{4.33}$$

When comparing the definition of K_s with the definition of ϕ_{MS}, one obtains the relation $K_s = g^{1/2}\, \phi_{MS}^{-1/2}\, 2^{11/6}\, k_s^{-1/6}$.

For instance, one will choose $K_s = 75$ m$^{1/3}$s^{-1} for a concrete made open channel and $K_s = 30$ m$^{1/3}$s^{-1} for a river with irregular bottom.

3.3 Backwater curves

We now consider the stationary flow momentum equation

$$\frac{dh}{ds} = \frac{I-J}{1-Fr^2} \qquad , \qquad I = -\frac{dZ_f}{ds}\ , \tag{4.34}$$

when $I > 0$. Contrarily to the case of the localized obstacle J is not negligible on the channel length under study. The solutions $h(s)$ of this equation are the "backwater curves" which give the response of the free surface water to the bottom shape $Z_f(s)$ and to a downstream boundary condition.

We consider the simple case of rectangular channel section of width L large compared to the depth h. This hypothesis implies $R_H = A/(L{+}h) \sim A/L = h$.

We suppose here that the lineic head loss J is parametrized by the Manning-Strickler formula $U = K_s\, h^{2/3}\, J^{1/2}$ which can be written in the following form

$$J = I\ \left(\frac{h}{h_n}\right)^{-10/3} \qquad , \qquad h_n = \left(\frac{q^2}{I\, K_s^2}\right)^{3/10}\ , \tag{4.35}$$

where h_n is called the "normal height". We notice that this height depends on the slope I while the "critical height" h_c, defined by

$$Fr^2 = \left(\frac{h_c}{h}\right)^3 \quad , \quad h_c = \left(\frac{q^2}{g}\right)^{1/3} \quad , \tag{4.36}$$

is independent of I. With these definitions of h_n and of h_c, the stationary momentum equation reads

$$\frac{dh}{ds} = \mathcal{F}(h) = I \, \frac{1-(h/h_n)^{-10/3}}{1-(h/h_c)^{-3}} \; . \tag{4.37}$$

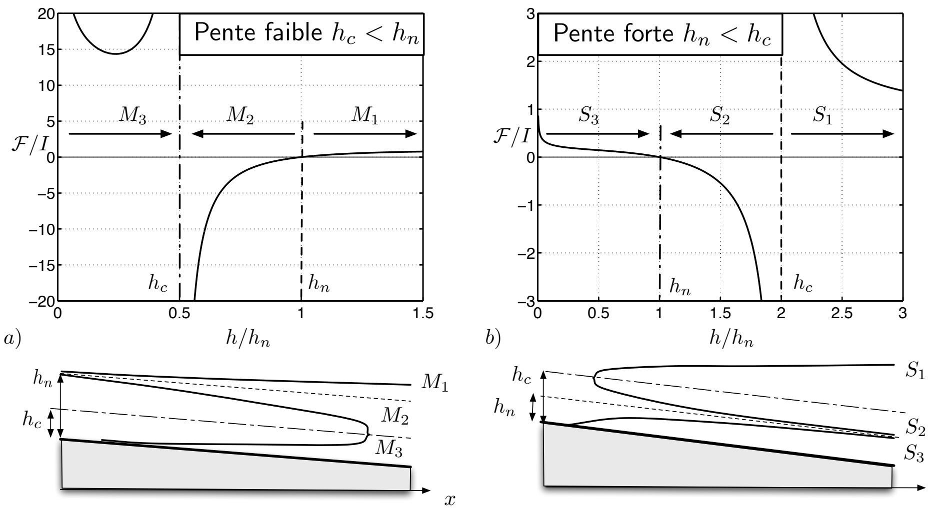

Figure 4.10: *Function $\mathcal{F}(h)$ and associated back water curves $h(s)$. a) Weak slope regime $h_c < h_n$ with type M curves. b) Strong slope regime $h_n < h_c$ with S curves.*

For a given lineic flux q, the normal height $h_n(I)$, defined by $I = J$, and the critical height h_c, defined by $Fr = 1$, are important quantities which can be compared to each other in order to characterize to flow at every location. At the points where $h_c < h_n$, the flow is in a "weak slope" regime. The "strong slope" regime is obtained for $h_n < h_c$.

The transition between a strong slope regime and a weak slope regime is obtained at a point where the slope I is equal to the "critical slope" I_c defined by the relation $h_n(I_c) = h_c$.

Only boundary condition $h(s_0) = h_0$ imposed somewhere downstream or upstream the considered reach (subdivision of the channel) is sufficient to determine the "backwater curve" $h(s)$. This name comes from the downstream conditions case by also applies for the upstream case.

Depending on the position of $h(s_0) = h_0$ compared to both h_n and h_c, one obtains increasing or decreasing backwater curves which are denoted by (M_1, M_2, M_3) in the case of a week slope regime and (S_1, S_2, S_3) in the case of a strong slope regime (figure **4**.10).

The drawing of these backwater curves show that singularities appears when Fr becomes equal to 1. On can observe the appearance of a hydraulic jump before reaching the singularity $Fr = 1$. The position of such stationary hydraulic jumps provide a new degree of freedom to the system when both an upstream and a downstream boundary conditions are imposed.

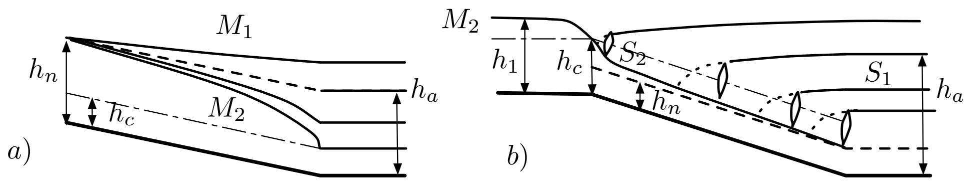

Figure **4**.11: *Slope changes and associated backwater curves. a) Weak slope regime. b) Strong slope example with hydraulic jump.*

As an example of application of this backwater concept, let us consider a flow in a weak slope regime ($h_c < h_n$) with a transition to a horizontal flat bottom (Figure **4**.11a). If the boundary condition $h = h_a$, with $h_c < h_a$, is imposed downstream of this transition point (for instance with a lake of big size), one can compute the backwater curves which are of the M_2 type for $h_c < h_a < h_n$ and of the M_1 type for $h_n < h_a$. In the cas of a strong slope (figure **4**.11b), one sees that a hydraulic jump must appear so that the free surface profile can match with the downwards boundary condition $h = h_a$ if $h_c < h_a$.

When the slopes vanishes ($I = 0$) or is negative ($I < 0$), one can define new backwater curves (C_1 and C_2 for $I = 0$, A_1 and A_2 for $I < 0$) by using an argument similar to the case of positive slope from Equation (**4**.34).

FORMULAS

Hydraulic head

Local head:

$$H = \frac{p}{\rho\, g} + z + \frac{1}{2\, g} \underline{U}^2 \; .$$

Geometric parameter:

$$R_H = \frac{A}{P} \quad , \quad D_H = 4\, R_H \quad , \quad h_H = \frac{A}{L} \; .$$

Averaged head:

$$H(s) = \frac{p_a}{\rho\, g} + h(s) + Z_f(s) + \alpha(s) \frac{U^2(s)}{2\, g} \; .$$

Specific head and impulsion

Lineic head loss:

$$\frac{dH}{ds} = -J \quad , \quad H = \frac{p_a}{\rho\, g} + h + Z_f + \frac{U^2}{2g} \; .$$

Equilibrium equation:

$$q = U\; h \; , \qquad \frac{dh}{ds} = \frac{I - J}{1 - Fr^2} \quad , \quad Fr = \frac{U}{\sqrt{g\, h}} \; .$$

Specific charge:

$$\mathcal{E}(q, h) = h + \frac{U^2}{2\, g} = h + \frac{1}{2} \frac{h_c^3}{h^2} \quad , \quad h_c = \left(\frac{q^2}{g} \right)^{1/3} \; .$$

Sationary hydraulic jump:

$$U_L\, h_L = U_R\, h_R = q \quad , \quad h_L\, U_L^2 + \frac{1}{2}\, g\, h_L^2 = h_R\, U_R^2 + \frac{1}{2}\, g\, h_R^2 \,.$$

Impulse function:

$$\mathcal{I}(q, h_L) = \mathcal{I}(q, h_R) \quad \text{avec} \quad \mathcal{I}(q, h) = h\, U^2 + \frac{1}{2}\, g\, h^2 = \frac{g}{h}\left(h_c^3 + \frac{1}{2} h^3\right) \,.$$

Backwater curves

Friction:

$$J = -\frac{1}{A} \iint_{\mathcal{A}} \frac{1}{\rho\, g} \underline{\mathrm{div}}\, (\underline{\underline{\tau}}) \cdot \underline{e}_s\, da \quad , \quad \tau_* = -\frac{1}{P} \int_{\mathcal{P}} \underline{e}_s \cdot \underline{\underline{\tau}} \cdot \underline{n}\, dl \,.$$

Coefficient λ:

$$\tau_* = \frac{1}{8}\, \lambda(Re, Ru)\, \rho\, U^2 \,, \qquad Re = \frac{U\, D_H}{\nu} \quad , \quad Ru = \frac{k_s}{D_H} \,.$$

Head loss :

$$\tau_* = \rho\, g\, R_H\, J \quad \Longrightarrow \quad J = \lambda\, (Re, Ru)\, \frac{U^2}{2\, g\, D_H} \,.$$

Strickler coefficient:

$$U = K_s\, R_H^{\frac{2}{3}}\, J^{\frac{1}{2}} \,.$$

Backwater curves:

$$\frac{dh}{ds} = \mathcal{F}(h) = I\, \frac{1 - (h/h_n)^{-10/3}}{1 - (h/h_c)^{-3}} \,,\; h_c = \left(\frac{q^2}{g}\right)^{1/3} \,,\; h_n = \left(\frac{q^2}{I\, K_s^2}\right)^{3/10} \,.$$

EXERCISES

See exercices in French language in the book:

O. THUAL, Hydrodynamique de l'Environnement, Éditions de l'École Polytechnique, 2010.

or at `http://thual.perso.enseeiht.fr/xsee`