Chapter 5

Flood waves

O. Thual, September 20, 2010

Contents

Introduction

Open channel flows often come in the investigation and intervention scope of the environment engineer. Very often, the water layer depth is weak compared to the horizontal extension of the observed phenomena. This is the case for flows in rivers or channels and of the ground runoff.

One restrains here to channels which section is a rectangle of top width L large compared to the depth, but the generalization to channels with variable sections can be done without difficulties.

The transition from turbulent Navier-Stokes equations is tackled by using the Leibnitz formula. A series of approximations allowing to neglect one or several terms of the Saint-Venant equations is presented.

Figure **5**.1: *The Garonne in floods. Photo CNRS*

The kinematic flood wave approximation is then used for its hydraulic relevance and its simplicity in the illustration of the method of characteristics. The exemple of the expansion wave eases the understanding of the notion of invariant carried by the characteristic velocity. The computation of the propagation velocity of a shock from the global formulation of the model is explicited.

1 Navier-Stokes equations with a free surface

We present here the two-dimensional incompressible and turbulent Navier-Stokes equations with a bottom and a free surface. The choice of a set of units representative of the pertinent orders of magnitudes enables to write this model under a dimensionless form.

1.1 Two-dimensional Navier-Stokes equations

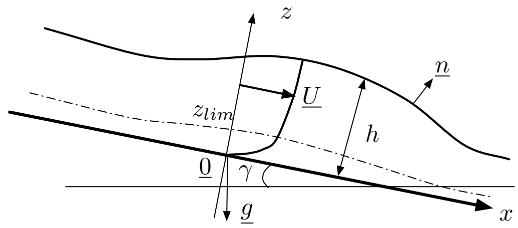

Figure **5**.2: *Tilted channel with a flat bottom.*

Let consider a two-dimensional flow on a tilted plane making an angle γ with the horizontal. By choosing the Ox axe to be parallel to the tilted plane, one denotes by $(\underline{e}_x, \underline{e}_z)$ the unit vectors of the (x, z) plan. The gravity vector then reads $\underline{g} = g \sin\gamma\, \underline{e}_x - g \cos\gamma\, \underline{e}_z$. One considers that the fluid is incompressible and denotes by ρ its constant mass density. One chooses the axes origin at the bottom so the $z = 0$ be the tilted plane equation.

When the flow is turbulent, the model of the incompressible Navier-Stokes equations with turbulent viscosity reads, in Reynolds average,

$$\begin{aligned} \operatorname{div} \underline{U} &= 0 \\ \frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \underline{\operatorname{grad}}\, \underline{U} &= -\underline{\operatorname{grad}} \left(\frac{p}{\rho} + \frac{2}{3}\, k \right) + \underline{g} + \underline{\operatorname{div}} \left[(\nu + \nu_t) \underline{\underline{d}} \right] , \end{aligned} \qquad (\mathbf{5}.1)$$

where ν is the kinematic molecular viscosity, ν_t the turbulent viscosity, $\underline{\underline{d}}$ the average strain tensor, $k = \frac{1}{2}\overline{\underline{U}'^2}$ the turbulent kinetic energy where $\underline{U}'$ is the turbulent fluctuation of the velocity field around its average. For simplicity, the notations $\underline{U}$, p and $\underline{\underline{d}}$ are used for the Reynolds means rather than $\overline{\underline{U}}$, $\overline{p}$ or $\overline{\underline{\underline{d}}}$. In the two-dimensional case which is studied here, one denotes by $\underline{U}(x, z, t) = u(x, z, t)\ \underline{e}_x + w(x, z, t)\ \underline{e}_z$ the velocity field and $p(x, z, t)$ the pressure field.

One assumes that the turbulent viscosity ν_t is constant for $z \geq z_{lim}(x, t)$ where z_{lim} is the top of a bottom boundary layer which empirical adjustment is part of the present model. The consideration of a more complex turbulent model would lead to the same Saint-Venant equations than those which are about to be derived, but the presentation of this derivation would have been heavier. One supposes than ν is negigible in front of ν_t in the upper layer, which allows to write there the Navier-Stokes equations under the form

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0 \\ \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p_t}{\partial x} + g\ \sin\gamma + \nu_t\ \Delta u \\ \frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + w\,\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p_t}{\partial z} - g\ \cos\gamma + \nu_t\ \Delta w\ , \end{aligned} \qquad (5.2)$$

where $p_t = p + \frac{3}{2}\,\rho\,k$ is the "turbulent pressure".

1.2 Flow boundary conditions

One assumes that the free surface equation reads $F(x,z,t) = z - h(x,t) = 0$, which exclude deformations of the breaking type. The normal $\underline{n} = \underline{\text{grad}}\ F/\|\underline{\text{grad}}\ \underline{F}\|$ to the surface is proportional to the vector $\underline{\text{grad}}\ F = -\frac{\partial h}{\partial x}\,\underline{e}_x + \underline{e}_z$. The kinematic boundary condition $\frac{dF}{dt} = 0$ then reads

$$\frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial x} = w \qquad \text{for} \qquad z = h(x,t)\ . \qquad (5.3)$$

The dynamic boundary condition on the free surface expresses the continuity of the surface forces. If one assumes that the fluid (water) is in touch with a perfect fluid (air) of constant pressure p_a, it reads

$$\underline{\underline{\sigma}}_t \cdot \underline{n} = -p_a\ \underline{n} \qquad \text{for} \qquad z = h(x,t)\ , \qquad (5.4)$$

where $\underline{\underline{\sigma}}_t(x,z,t)$ is the "turbulent stress tensor"

$$\begin{aligned} \underline{\underline{\sigma}}_t &= -p_t\ \underline{\underline{I}} + 2\,\rho\,\nu_t\ \underline{\underline{d}}\ , \\ \text{with} \quad \underline{\underline{d}}(x,z,t) &= \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{\partial w}{\partial z} \end{pmatrix} . \end{aligned} \qquad (5.5)$$

The settlement of the boundary conditions at $z = z_{lim}$ necessitates a matching with the bottom boundary layer. A very simple model consists in considering that z_{lim} is very small and to impose the boundary conditions:

$$w = 0 \qquad \text{et} \qquad \underline{e}_x \cdot \underline{\underline{\sigma}}_t \cdot \underline{e}_z = \tau_* \qquad \text{pour} \qquad z = 0\ , \qquad (5.6)$$

where τ_* is the shear stress applied by the fluid on the wall. The stress τ_* must then be modelled as a function of the velocity, of the height as well as other parameters such as the molecular viscosity ν of the fluid or the characteristic height k_s of the bottom roughnesses.

1.3 Model dimensionless form

One chooses to put in a dimensionless form the Navier-Stokes equations with the following unit system:

$$x = L_0\, x^+, \qquad z = h_0\, z^+, \qquad t = \frac{L_0}{U_0}\, t^+,$$
$$u = U_0\, u^+, \qquad w = U_0 \frac{h_0}{L_0} w^+ \quad \text{and} \quad p_t = \rho\, g'\, h_0\, p_t^+ \;, \tag{5.7}$$

where L_0 is a horizontal length scale, h_0 is a vertical length scale, U_0 a longitudinal velocity scale and $g' = g\, \cos\gamma$ a constant close to g when γ is small. The dimensionless fields depend on dimensional variables through relations of the type $u(x,z,t) = U_0\; u^+(x^+, z^+, t^+)$. A simple calculus enables to then obtain a dimensionless model which reads

$$\begin{aligned}
\frac{\partial u^+}{\partial x^+} + \frac{\partial w^+}{\partial z^+} &= 0 \\
\frac{\partial u^+}{\partial t^+} + u^+ \frac{\partial u^+}{\partial x^+} + w^+ \frac{\partial u^+}{\partial z^+} &= -\frac{1}{Fr^2} \frac{\partial p_t^+}{\partial x^+} + \frac{\tan\gamma}{\epsilon\, Fr^2} + \frac{1}{\epsilon\, Rt} \Delta^+ u^+ \\
\epsilon^2 \left(\frac{\partial w^+}{\partial t^+} + u^+ \frac{\partial w^+}{\partial x^+} + w^+ \frac{\partial w^+}{\partial z^+} \right) &= -\frac{1}{Fr^2} \frac{\partial p_t^+}{\partial z^+} - \frac{1}{Fr^2} + \frac{\epsilon}{Rt} \Delta^+ w^+ \\
\text{with} \qquad \Delta^+ &= \epsilon^2\, \frac{\partial^2}{\partial x^{+2}} + \frac{\partial^2}{\partial z^{+2}}
\end{aligned} \tag{5.8}$$

and where the four dimensionless number contributing in these equations are

$$\epsilon = \frac{h_0}{L_0}, \qquad Fr = \frac{U_0}{\sqrt{g'\, h_0}}\;, \qquad Rt = \frac{h_0\, U_0}{\nu_t} \quad \text{and} \quad \tan\gamma\;. \tag{5.9}$$

The Froude number Fr is the ratio between the characteristic velocity U_0 of the flow and a velocity $c_0 = \sqrt{g'\, h_0}$ which happens to be the propagation velocity of waves in a shallow medium (see below). The "Friction Reynolds number" $Rt = h_0\, U_0/\nu_t$, which is frankly smaller than the molecular Reynolds numbers $h_0\, U_0/\nu$ or $L_0\, U_0/\nu$, measures the magnitude of the turbulent friction compared to the other forces. At last, ϵ is small for flows which are shallow compared to the horizontal considered scales. One sees that the continuity equation (mass conservation) does not exhibit any of these dimensionless numbers.

The dimensionless boundary conditions on the free surface read

$$\frac{1}{Rt} \frac{\partial u^+}{\partial z^+} + \frac{\epsilon}{Fr^2} \left(p_t^+ - p_a^+ \right) \frac{\partial h^+}{\partial x^+} + \frac{\epsilon^2}{Rt} \left(\frac{\partial w^+}{\partial x^+} - 2\, \frac{\partial u^+}{\partial x^+} \frac{\partial h^+}{\partial x^+} \right) = 0\;,$$
$$-\frac{1}{Fr^2} (p_t^+ - p_a^+) + \frac{\epsilon^2}{Rt} \left(-\frac{\partial u^+}{\partial z^+} \frac{\partial h^+}{\partial x^+} + 2 \frac{\partial w^+}{\partial z^+} \right) - \frac{\epsilon^3}{Rt} \frac{\partial w^+}{\partial x^+} \frac{\partial h^+}{\partial x^+} = 0$$

$$\text{and} \qquad \frac{\partial h^+}{\partial t^+} + u^+ \frac{\partial h^+}{\partial x^+} = w^+ \qquad \text{for} \qquad z^+ = h^+(x^+, t^+) \ . \qquad (\mathbf{5}.10)$$

The dimensionless boundary conditions at the bottom read

$$\frac{\partial u^+}{\partial z^+} + \frac{\partial w^+}{\partial x^+} = \frac{Rt \ \tan\gamma}{Fr^2} \, \tau_*^+ \qquad \text{et} \qquad w^+ = 0 \qquad \text{for} \qquad z^+ = 0 \ , \qquad (\mathbf{5}.11)$$

where one has chosen to put the shear stress τ_* in the dimensionless form

$$\tau_*(x,t) = \rho \, g' \, h_0 \ \tan\gamma \ \tau_*^+(x^+, t^+) \ . \qquad (\mathbf{5}.12)$$

2 Derivation of the Saint-Venant equations

The Saint-Venant equations are obtained by integrating in the vertical direction the turbulent Navier-Stokes equations, assuming that the depth is weak in front of the variation scale of the phenomena in the direction of the flow. Complementary approximations can be discussed by studying the order of magnitude of other parameters such as the bottom slope, the mean velocity or still the turbulence intensity.

2.1 Classification of the approximations

One is interested in the case of shallow layers of fluid, which is traduced by $\epsilon \ll 1$. Table **5**.1 lists the various approximations which lead to non trivial solutions, assuming that all the dimensionless fields u^+, w^+, h^+, p_t^+ and τ_*^+ remain of order 1 when all or part of the dimensionless parameters ϵ, $\tan\gamma$, $1/Rt$ or Fr go to zero. This hypothesis is not neutral and the performed unit choices, for example for τ_*, are justified afterwards by studying the models obtained by such or such approximation.

All these approximations lead to

$$\frac{\partial p_t^+}{\partial z^+} = -1 \qquad \text{with} \qquad p_t^+ = p_a^+ \qquad \text{for } z^+ = h^+(x^+, t^+). \qquad (\mathbf{5}.13)$$

The turbulent pressure is thus hydrostatic. Coming back to field with dimensions, one can then write

$$p_t(x,z,t) = p_a - \rho \, g' \, [z - h(x,t)] \ . \qquad (\mathbf{5}.14)$$

One will henceforth replace p_t by its value as a function of h in the equations. The remaining boundary conditions read, at the dominant order of all the approximations,

$$\frac{\partial h}{\partial t} + u \, \frac{\partial h}{\partial x} = w \qquad \text{and} \qquad \frac{\partial u}{\partial z} = 0 \qquad \text{for } z = h \ ,$$

	$\tan\gamma \ll \epsilon$	$\tan\gamma = O(\epsilon)$	$\tan\gamma = O(1)$
$Fr^2 = O(1)$			
$\frac{1}{Rt} \ll \epsilon$	Saint-Venant no slope no friction	Saint-Venant no friction	
$\frac{1}{Rt} = O(\epsilon)$	Saint-Venant no slope	**Saint-Venant equations** full terms	
$\frac{1}{Rt} = O(1)$			**Kinematic flood waves**
$Fr^2 = O(\epsilon)$			
$\frac{1}{Rt} = O(1)$		**Diffusive kinematic flood waves**	

Table **5**.1: Approximations in the $\epsilon \ll 1$ case.

$$w = 0 \qquad \text{and} \qquad \rho\,\nu_t\,\frac{\partial u}{\partial z} = \tau_* \qquad \text{for } z = 0\ . \tag{5.15}$$

The approximations of Table **5**.1 differentiate from each others in the projection of the momentum equation on the $\underline{e}_x$ direction. For the "full Saint-Venant Equations approximation" obtained for $\epsilon \ll 1$, $Fr^2 = O(1)$, $\tan\gamma = O(\epsilon)$ and $1/Rt = O(\epsilon)$, all the terms of the momentum equation are of the same order of magnitude and one has

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + w\,\frac{\partial u}{\partial z} = -g'\frac{\partial h}{\partial x} + g\;I + \nu_t\;\frac{\partial^2 u}{\partial z^2}\ , \tag{5.16}$$

with $g' = g\,\cos\gamma$ and $I = \sin\gamma$. For other approximations, the friction term $\nu_t\,\frac{\partial^2 u}{\partial z^2}$ or the gravity forcing $g\,I$ can be negligible. For the diffusive kinematic flood waves approximation, the first member, traducing the acceleration, is negligible in front of the other terms. These latter are all three of the same order for the "diffusive kinematic flood wave" approximation, the pressure term $-g'\frac{\partial h}{\partial x}$ being negligible for the "kinematic flood wave" approximation.

2.2 Models integrated on the fluid layer

One defines the fluid layer longitudinal velocity $U(x,t)$ by the relation

$$U(x,t) = \frac{1}{h(x,t)}\int_0^{h(x,t)} u(x,z,t)\;dz\ . \tag{5.17}$$

To integrate from 0 to $h(x,t)$ the model equations, it is necessary to use the Leibnitz formula

$$\frac{d}{ds}\int_{a(s)}^{b(s)} f(s,z)\;dz = \int_{a(s)}^{b(s)} \frac{\partial f}{\partial s}(s,z)\;dz + \frac{db}{ds}(s)\;f[s,b(s)] - \frac{da}{ds}(s)\;f[s,a(s)]$$

valid for all integrable and derivable function $f(s,z)$ and all interval $[a(s),b(s)]$ which boundaries vary with s.

The integration on the vertical of the continuity equation leads to

$$\int_0^{h(x,t)} \frac{\partial u}{\partial x}(x,z,t)\,dz + \int_0^{h(x,t)} \frac{\partial w}{\partial z}(x,z,t)\,dz = 0\;. \qquad (\mathbf{5}.18)$$

By applying the Leibnitz formula and by integrating $\frac{\partial w}{\partial z}$, on deduces from it

$$\frac{\partial}{\partial x}\int_0^{h(x,t)} u(x,z,t)\,dz - u[x,h(x,t),t]\frac{\partial h}{\partial x}(x,t) + w[x,h(x,t),t] - w(x,0,t) = 0\;.$$

By using the boundary conditions $w=0$ for $z=0$ and $\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = w$ for $z = h(x,t)$ and by using the definition of $U(x,t)$, the mass conservation equation integrated on the section finally reads

$$\frac{\partial}{\partial x}\,(h\,U) + \frac{\partial h}{\partial t} = 0\;. \qquad (\mathbf{5}.19)$$

By using the relation $\underline{U}\cdot\underline{\text{grad}}\;u = \text{div}\;(\underline{U}\,u)$ in the momentum equation, one obtains

$$\frac{\partial u}{\partial t} + \frac{\partial (u\,u)}{\partial x} + \frac{\partial (w\,u)}{\partial z} = -g'\,\frac{\partial h}{\partial x} + g\;I + \nu_t\,\frac{\partial^2 u}{\partial z^2}\;. \qquad (\mathbf{5}.20)$$

By using the Leibnitz formula, one then obtains

$$\frac{\partial}{\partial t}\int_0^h u\;dz - \frac{\partial h}{\partial t}\;u|_{z=h} + \frac{\partial}{\partial x}\int_0^h u^2\;dz - \frac{\partial h}{\partial x}\;u^2|_{z=h} + \Big[wu\Big]_{z=0}^{z=h} =$$
$$-g'\,h\,\frac{\partial h}{\partial x} + g\,I\,h + \nu_t\left[\frac{\partial u}{\partial z}\right]_{z=0}^{z=h}\;. \qquad (\mathbf{5}.21)$$

By using the the boundary conditions, among which $\rho\,\nu_t\,\frac{\partial u}{\partial z} = \tau_*$ for $z=0$, and the definition of U, one obtains

$$\frac{\partial (U\;h)}{\partial t} + \frac{\partial}{\partial x}\int_0^h u^2\;dz + g'\,h\,\frac{\partial h}{\partial x} = gh\,\sin\gamma - \frac{\tau_*}{\rho}\;. \qquad (\mathbf{5}.22)$$

2.3 Saint-Venant equations

One seeks to obtain a model which only involves the $U(x,t)$ and $h(x,t)$ fields. There thus only remains to express $\int_0^h u^2\,dz$ and τ_* as functions of these fields. For this, one resorts to experimental observations, the empirical modelling effort being, on more time, necessary. A first modelling consists in writing, through a dimensional analysis, that

$$\int_0^h u^2\,dz = \alpha\,U^2\,h\,. \qquad (\mathbf{5}.23)$$

Since the flow is turbulent, $u(z)$ is nearly constant on a big part of the layer and one can assume $\alpha = 1$.

For the modelling of the bottom shear τ_*, one defines the dimensionless quantity $C_f(h,U)$ by the dimensional relation

$$\tau_* = \frac{1}{2}\,C_f(h,U)\,\rho\,U\,|U|\,. \qquad (\mathbf{5}.24)$$

A crude model consists in considering that C_f, called the "Chezy coefficient", is constant. A model very often used in open chanel hydraulics is the Manning-Strickler formula which reads under one of these forms

$$C_f(h) = \frac{2\,g}{K_s^2\,h^{1/3}} \quad \text{or} \quad C_f(h) = \frac{\Phi_{MS}}{4}\left(\frac{k_s}{4\,h}\right)^{1/3}, \qquad (\mathbf{5}.25)$$

where K_s is the "Strickler number" (in $\mathrm{m}^{1/3}\ \mathrm{s}^{-1}$), k_s is the bottom rugosity and Φ_{MS} a dimensionless number which can be chosen of the ordre $\Phi_{MS} = 0.2$. One deduces from this that $K_s = g^{1/2}\,\phi_{MS}^{-1/2}\,2^{11/6}\,k_s^{-1/6}$.

Eventually, the Saint-Venant equations read

$$\begin{aligned} \frac{\partial h}{\partial t} + U\,\frac{\partial h}{\partial x} &= -h\,\frac{\partial U}{\partial x}\,, \\ \frac{\partial U}{\partial t} + U\,\frac{\partial U}{\partial x} + g'\,\frac{\partial h}{\partial x} &= g\,I - \frac{C_f}{2}\,\frac{U|U|}{h}\,, \end{aligned} \qquad (\mathbf{5}.26)$$

where $C_f(h,U)$ models the wall friction, $g'\frac{\partial h}{\partial x}$ represents the pressure gradient and $g\,I$ is the projection of the gravity in the x direction.

In the case of the Manning-Strickler parameterization that we will use from now on, the momentum equation is put under the form

$$\frac{\partial U}{\partial t} + U\,\frac{\partial U}{\partial x} + g'\,\frac{\partial h}{\partial x} = g\left(I - \frac{U|U|}{K_s^2\,h^{4/3}}\right)\,. \qquad (\mathbf{5}.27)$$

3 Flood waves dynamics

The notion of wave is often linked to the linear dynamics of a medium described by a small perturbations models around an equilibrium state. But the notion of wave can also be extended to the nonlinear case by denoting by this name any recognizable signal which moves with an identifiable velocity. The theory of characteristics provides a rigourous framework to define the trajectories along which the information propagates. One settles here in the framework of the diffusive or non diffusive kinematic flood wave approximation (voir tableau **5**.1).

3.1 Linear dynamics

One considers first of all the "kinematic flood wave approximation" obtained for $\epsilon \ll 1$, $\frac{1}{Rt} = O(1)$,$Fr^2 = O(1)$ and $\tan\gamma = O(1)$. It reads

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(U\,h) = 0 \qquad \text{and} \qquad 0 = g\,I - \frac{1}{2}C_f\,\frac{U\,|U|}{h}\,. \tag{5.28}$$

One chooses the Manning-Strickler model and is interested in regimes such that $U \geq 0$. One then obtains the "Strickler relation"

$$U(h) = K_s\,I^{1/2}\,h^{2/3}\,. \tag{5.29}$$

By carrying over in the continuity equation, the kinematic flood wave approximation reads

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\left(K_s\,I^{1/2}\,h^{5/3}\right) = \frac{\partial h}{\partial t} + \frac{5\,U(h)}{3}\,\frac{\partial h}{\partial x} = 0\,. \tag{5.30}$$

One then considers an equilibrium state (h_n, U_n) where h_n and U_n are constant satifying $U_n = K_s\,I^{1/2}\,h_n^{2/3}$. One sets

$$h = h_n + \widetilde{h} \qquad \text{and} \qquad U = U_n + \widetilde{U} \tag{5.31}$$

and one assumes that $\widetilde{h}$ and $\widetilde{U}$ are small perturbations of the equilibrium. By carrying over in the model and neglecting the order two terms, the linearized model reads

$$\frac{\partial \widetilde{h}}{\partial t} + \frac{5\,U_n}{3}\,\frac{\partial \widetilde{h}}{\partial x} = 0\,. \tag{5.32}$$

A small "bulge" $\widetilde{h}(x,t)$ is thus convected with a velocity equal to 5/3 of the one of the mean flow (see figure **5**.3a). This bulge is, for example, a weak flood on a river which slope is not negligible.

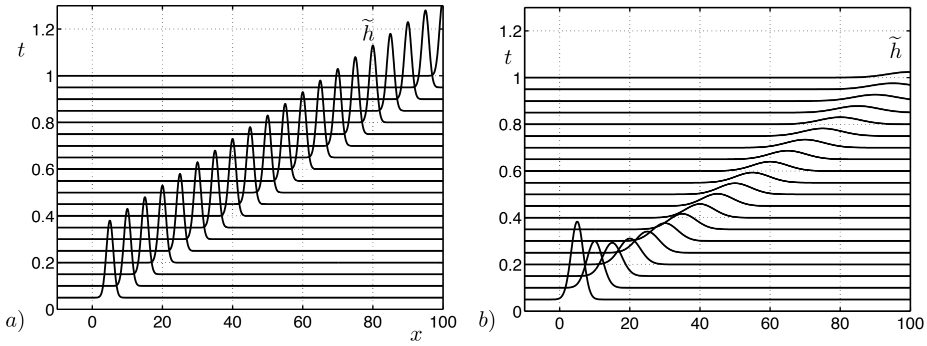

Figure **5**.3: *Solution* $\widetilde{h}(x,t)$ *a) of the nondiffusive kinematic flood wave* ($k_n = 0$)*, b) of the kinematic flood wave* ($k_n \neq 0$)*.*

When the river slope as well as its velocity become weaker, one can consider the "diffusive flood wave" model obtained for $\epsilon \ll 1, \frac{1}{Rt} = O(1)$, $Fr^2 = O(\epsilon)$ and $\tan\gamma = O(\epsilon)$. This model reads

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(U\,h) = 0 \qquad \text{and} \qquad 0 = -g'\,\frac{\partial h}{\partial x} + g\,I - \frac{1}{2}C_f\,\frac{U\,|U|}{h}\ . \tag{5.33}$$

The (h_n, U_n) equilibrium satisfies the same Strickler relation than for the previous model, but a small perturbation $\widetilde{h}$ now satisfies

$$\frac{\partial \widetilde{h}}{\partial t} + \frac{5\,U_n}{3}\,\frac{\partial \widetilde{h}}{\partial x} = k_n\,\frac{\partial^2 \widetilde{h}}{\partial x^2} \qquad \text{with} \qquad k_n = \frac{U_n\,h_n}{2\,\tan\gamma}\ . \tag{5.34}$$

The supplementary term effect will be to diminish the flood amplitude by diffusing it. If the initial condition $\widetilde{h}(x,t_0)$ is a gaussian of standard deviation l_0 and of maximum amplitude $\widetilde{h}_m$, the solution is the gaussian

$$\widetilde{h}(x,t) = \widetilde{h}_m\,\frac{l_0}{l(t)}\ \exp\left[-\frac{(x - \frac{5\,U_n}{3}\,t)^2}{2\,l^2(t)}\right]\quad ,\quad l^2(t) = l_0^2 + 2\,k_n\,(t - t_0)\ , \tag{5.35}$$

of increasing standard deviation $l(t)$ and of decreasing maximum amplitude (see figure **5**.3b).

3.2 Nonlinear dynamics

One is now interested in the nonlinear dynamics of the non diffusive kinematic flood wave approximation which is written under the form

$$\frac{\partial h}{\partial t} + \lambda(h)\,\frac{\partial h}{\partial x} = 0 \tag{5.36}$$

with $\lambda(h) = \frac{5}{3}U(h)$ and $U(h) = K_s\, I^{1/2}\, h^{2/3}$. One qualifies "characteristic" a curve $\mathcal{C}$ of equation $x = x_{\mathcal{C}}(t)$ satisfying the ordinary differential equation

$$\dot{x} = \lambda[h(x,t)]\ , \qquad (5.37)$$

where $h(x,t)$ is a solution of the partial differential equation. This solution then satisfies

$$\left(\frac{dh}{dt}\right)_{\mathcal{C}} = 0 \qquad \text{with} \qquad \left(\frac{d}{dt}\right)_{\mathcal{C}} = \frac{\partial}{\partial t} + \dot{x}_{\mathcal{C}}(t)\,\frac{\partial}{\partial x}\ . \qquad (5.38)$$

Put differently, the value $h_{\mathcal{C}}(t) = h[x_{\mathcal{C}}(t),t]$ "measured along the curve $\mathcal{C}$" is constant. One says that h is a "Riemann invariant" of the system. This constant is, for instance, $h_{\mathcal{C}} = h(a,0)$ if $x_{\mathcal{C}}(0) = a$, when $\mathcal{C}$ goes through the point $(x,t) = (a,0)$, or else $h_{\mathcal{C}} = h(0,\tau)$ if $x_{\mathcal{C}}(\tau) = 0$, when $\mathcal{C}$ goes throught the point $(x,t) = (0,\tau)$. Since h is constant along these "curves" and $\lambda(h)$ only depends on h, the characteristics are straight lines.

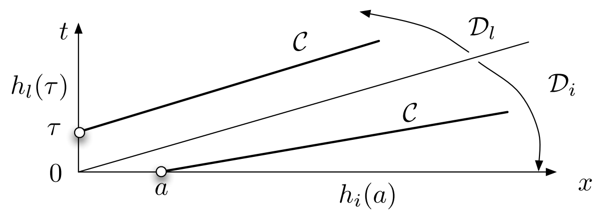

Figure 5.4: *Examples of characteristics $\mathcal{C}$ going through $(a,0)$ or $(0,\tau)$ in the (x,t) plane. Influence domains $\mathcal{D}_i$ and $\mathcal{D}_l$ of, respectively, initial and boundary conditions.*

If one is given an initial condition $h_i(a)$ such that $h(x,0) = h_i(x)$ for $x \geq 0$ and a boundary condition $h_l(\tau)$ such that $h(0,t) = h_l(t)$ for $t \geq 0$, the solution $h(x,t)$ is obtained by eliminating, respectively, a or else τ from one of the systems

$$\left\{\begin{array}{rcl} x-a &=& \lambda(h)\,t \\ h &=& h_i(a) \end{array}\right. \quad \text{or else} \quad \left\{\begin{array}{lcc} x &=& \lambda(h)\,(t-\tau) \\ h &=& h_l(\tau) \end{array}\right.\ , \qquad (5.39)$$

depending wheter one is in the initial conditions influence domain $\mathcal{D}_i$ or of the boundary conditions $\mathcal{D}_l$.

The graphical resolution of these implicit equations is done by drawing the characteristic lines $x = a + \lambda[h_i(a)]\,t$ starting from the initial conditions or else the characteristic lines $x = \lambda[h_l(\tau)](t-\tau)$ starting from the boundary conditions.

Figure **5**.5a represents such a drawing starting form an increasing initial condition. Since h is increasing along characteristics which diverges when t increases, one sees that the horizontal extension grows, the amplitude remaining constant as shown on Figure **5**.5b.

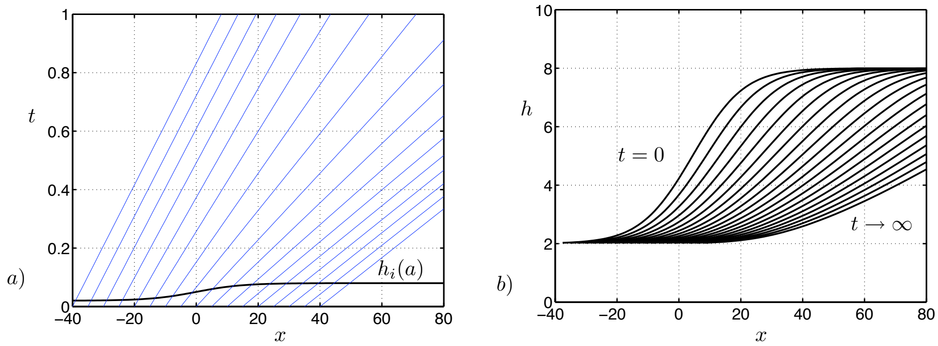

Figure **5**.5: *Expansion. a) Characteristic lines in the* (x, t) *plane and initial profile* $h_i(a) = h_n + \frac{\Delta h}{2}\left[1 + \tanh\left(\frac{a}{l_0}\right)\right]$. *b) Evolution of the profile* $h(x, t)$.

Figure **5**.6a represents such a drawing starting from a decreasing initial condition. Since h is constant along converging characteristics when t increases, one sees that the horizontal extension decreases, the amplitude remaining constant as shown on Figure **5**.6b. Beyond the first time t_c where the characteristics cross, the continuous model of the kinematic flood waves is no longer valid. Very often, this singularity traduces physical phenomena in which the fields vary on very short distances and are called "shocks" in compressible aerodynamics or "jumps" here.

3.3 Jumps of the flood wave model

One can try to model these jumps with discontinuous functions $h(x, t)$. For this, it is nessecary to formulate a model richer than the considered partial differential equation. Such a model is obtained through a global formulation of the global mass conservation which postulates that the relation

$$\frac{d}{dt}\int_{x_1}^{x_2} h\, dx + [q(h)]_{x_1}^{x_2} = 0 \qquad \text{with} \qquad q'(h) = \lambda(h) \tag{5.40}$$

is true on the whole fixed interval $[x_1, x_2]$ of the considered spatial domain.

One denotes by $x_c(t)$ the position of a moving shock and $[\![h]\!] = h_R - h_L$ the discontinuity of h through the shock where $h_R = h[x_c^+(t), t]$ is the value at its

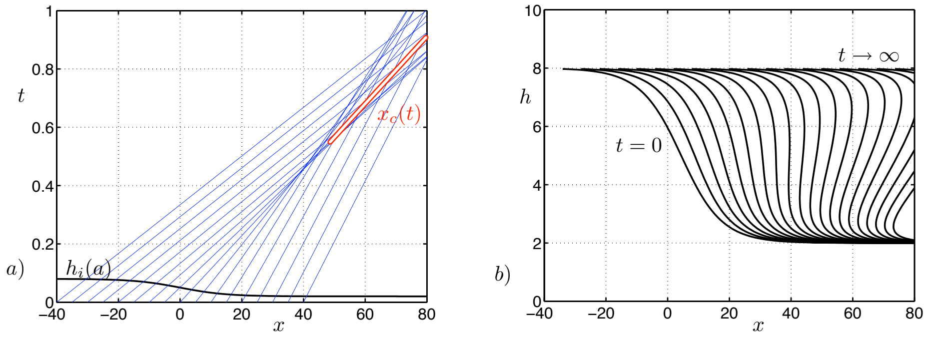

Figure 5.6: *Shock. a) Characteristic lines in the (x, t) plan and initial profile $h_i(a) = h_n + \frac{\Delta h}{2}\left[1 - \tanh\left(\frac{a}{l_0}\right)\right]$. b) Evolution of the profile $h(x, t)$.*

right and $h_L = h[x_c^-(t), t]$ is the value at its left. The Leibnitz formula enables to write

$$\begin{aligned}
\frac{d}{dt}\int_{x_1}^{x_2} h\, dx &= \frac{d}{dt}\left[\int_{x_1}^{x_c^-(t)} h\, dx + \int_{x_c^+(t)}^{x_2} h\, dx\right] \\
&= \int_{x_1}^{x_c^-(t)} \frac{\partial h}{\partial t}\, dx + \int_{x_c^+(t)}^{x_2} \frac{\partial h}{\partial t}\, dx + \dot{x}_c(t)\, h_L(t) - \dot{x}_c(t)\, h_R(t) \\
&= \int_{x_1}^{x_c^-(t)} \frac{\partial h}{\partial t}\, dx + \int_{x_c^+(t)}^{x_2} \frac{\partial h}{\partial t}\, dx - W\, [\![h]\!]\ , \qquad (5.41)
\end{aligned}$$

where $W(t) = \dot{x}_c(t)$ is the velocity of the shock. Furthermore, one can write

$$\begin{aligned}
[q]_{x_1}^{x_2} &= q[h(x_2, t)] - q[h(x_1, t)] = [q]_{x_1}^{x_c^-} + [q]_{x_c^+}^{x_2} - q[h_L(t)] + q[h_R(t)] \\
&= [q]_{x_1}^{x_c^-} + [q]_{x_c^+}^{x_2} + [\![q]\!]\ . \qquad (5.42)
\end{aligned}$$

By summing Equations (5.41) and (5.42) and applying the local budget (5.36) on the intervals $[x_1, x_c[$ and $]x_c, x_2]$, one finds that the velocity of the shock is

$$W(t) = [\![q(h)]\!]/[\![h]\!] \qquad \text{with} \qquad W(t) = \dot{x}_c(t)\ , \qquad (5.43)$$

the values on both sides of the shock being known thanks to the characteristics. In the case of the kinematic flood wave approximation, one has $q(h) = U\, h$ with $U = K_s \sqrt{I}\, h^{2/3}$. One can thus write the jump condition under the form

$$\left[\!\!\left[h(U - W) \right]\!\!\right] = 0\ . \qquad (5.44)$$

FORMULAS

Free surface Navier-Stokes equations

With turbulent viscosity:

$$\begin{aligned} \operatorname{div} \underline{U} &= 0 \\ \frac{\partial \underline{U}}{\partial t} + \underline{U} \cdot \underline{\text{grad}}\, \underline{U} &= -\frac{1}{\rho} \underline{\text{grad}}\, p_t + \underline{g} + \underline{\operatorname{div}} \left[(\nu + \nu_t) \underline{\underline{d}} \right] . \end{aligned}$$

Boundary conditions:

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = w \quad , \quad \underline{\underline{\sigma}}_t \cdot \underline{n} = -p_a \, \underline{n} \qquad / \qquad z = h(x,t) \; .$$

Dimensionless numbers:

$$\epsilon = \frac{h_0}{L_0}, \qquad Fr = \frac{U_0}{\sqrt{g' \, h_0}} \; , \qquad Rt = \frac{h_0 \, U_0}{\nu_t} \quad \text{et} \quad \tan \gamma \; .$$

Derivation of the Saint-Venant equations

Hydrostatic pressure:

$$p_t(x,z,t) = p_a - \rho \, g' \left[z - h(x,t) \right] .$$

Momentum:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} = -g' \frac{\partial h}{\partial x} + g \; I + \nu_t \; \frac{\partial^2 u}{\partial z^2} \; .$$

Bottom shear:

$$\tau_* = \frac{1}{2} \, C_f(h, U) \, \rho \, U \, |U| \; .$$

Saint-Venant equations:

$$\begin{aligned}\frac{\partial h}{\partial t} + U\,\frac{\partial h}{\partial x} &= -h\,\frac{\partial U}{\partial x}\,,\\ \frac{\partial U}{\partial t} + U\,\frac{\partial U}{\partial x} + g'\,\frac{\partial h}{\partial x} &= g\,I - \frac{C_f}{2}\,\frac{U|U|}{h}\,.\end{aligned}$$

Manning-Strickler parameterisation:

$$C_f = \frac{2\,g}{K_s^2\,h^{1/3}} \quad\Longrightarrow\quad \frac{\partial U}{\partial t} + U\,\frac{\partial U}{\partial x} + g'\,\frac{\partial h}{\partial x} = g\left(I - \frac{U|U|}{K_s^2\,h^{4/3}}\right)\,.$$

Flood wave dynamics

Diffusive flood waves:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(U\,h) = 0 \qquad \text{and} \qquad 0 = -g'\,\frac{\partial h}{\partial x} + g\,I - \frac{1}{2}C_f\,\frac{U\,|U|}{h}\,.$$

Flood waves:

$$\frac{\partial h}{\partial t} + \lambda(h)\,\frac{\partial h}{\partial x} = 0 \quad , \qquad \lambda(h) = \frac{5}{3}U(h) \quad , \qquad U(h) = K_s\,I^{1/2}\,h^{2/3}\,.$$

Characteristics:

$$\left(\frac{dh}{dt}\right)_{\mathcal{C}} = 0 \quad , \qquad \left(\frac{d}{dt}\right)_{\mathcal{C}} = \frac{\partial}{\partial t} + \dot{x}_{\mathcal{C}}(t)\,\frac{\partial}{\partial x}\,.$$

Jump relations:

$$\frac{d}{dt}\int_{x_1}^{x_2} h\,dx + [q(h)]_{x_1}^{x_2} = 0 \quad\Longrightarrow\quad \dot{x}_c(t) = W(t) = [\![q(h)]\!]/[\![h]\!]\,.$$

EXERCISES

See exercices in French language in the book:

O. THUAL, Hydrodynamique de l'Environnement, Éditions de l'École Poly-

technique, 2010.

or at http://thual.perso.enseeiht.fr/xsee