Spectral Analysis for Computational Fluid Dynamics

A. Dauptain

April 21, 2011

1 Overall Context

Spectral analysis is a widely used method coming from the signal processing science. However, misuses and misunderstandings of the method are quite common, to say the least. In addition, most of the signal processing tutorial/books/courses consider the time-dependent signals a particular case of the theory, making their material very hard to link with Fluid Dynamic applications.

Contents

2 Amplitude distributions

2.1 Spectral amplitude function

Fourier function allows to represent a signal in terms of its spectral components in the frequency domain.

The Fourier transform is a complex exponential transform which is related to the Laplace transform. The Fourier transform is also referred to as a trigonometric transformation since the complex exponential function can be represented in terms of trigonometric functions. Specifically,

$$\exp(i\omega t) = \cos\omega t + i\sin\omega t,\ \exp(-i\omega t) = \cos\omega t - i\sin\omega t \tag{1}$$

where $i^2 = -1$.

The Fourier transform $X(f)$ for a continuous time series x(t) is defined as:

$$X(f) = \int_{-\infty}^{+\infty} x(t)\exp(-i2\pi ft)dt \tag{2}$$

where $-\infty < f < +\infty$. Thus, the Fourier transform is continuous over an infinite frequency range. The inverse transform is:

$$x(t) = \int_{-\infty}^{+\infty} X(f)\exp(i2\pi ft)df \tag{3}$$

where $-\infty < t < +\infty$. Also note that $X(f)$ is a complex function. It may be represented in terms of real and imaginary components, or in terms of magnitude and phase. The conversion is made as follows for a complex variable V. Note that $X(f)$ has dimensions of [amplitude.time].

$$V = a + ib \tag{4}$$

$$Magnitude(V) = \sqrt{a^2 + b^2} \tag{5}$$

$$Phase(V) = \arg V \neq \arctan(b/a) \tag{6}$$

As an example, consider a sine wave

$$x(t) = A\sin(2\pi f_0 t) \tag{7}$$

where $-\infty < t < +\infty$. The Fourier transform of the sine wave is

$$X(f) = \left(\frac{iA}{2}\right)(-\delta(f - f_0) + \delta(-f - f_0)) \tag{8}$$

where δ is the Dirac delta function. As illustrated on Fig 1a, The transform of a sine wave is purely imaginary. On the other hand, the Fourier transform of a cosine wave is

$$X(f) = \left(\frac{A}{2}\right)(\delta(f - f_0) + \delta(-f - f_0)) \tag{9}$$

As illustrated on Fig 1b, The transform of a cosine wave is purely real. The two results depicted in Fig 1 demonstrates two characteristics of the fourier transforms of real time history functions:

1. The real Fourier transform is symmetric about the $f = 0$ line.
2. The imaginary Fourier transform is antisymmetric about the f = 0 line.

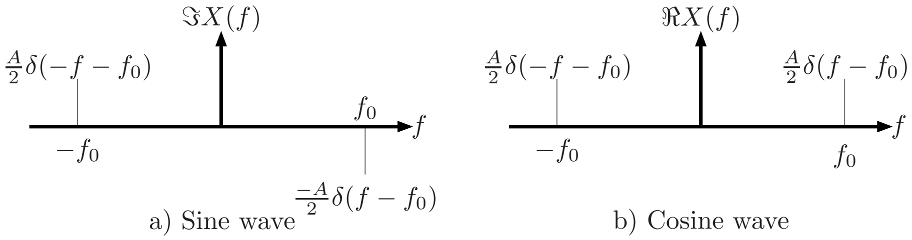


Figure 1: Fourier transform of sine and cosine waves

2.2 Dimensions and discretization

The dirac delta function is a distribution : while the peak has no width neither amplitude in the continuous sense, its integral on a frequency domain is equal to one. Therefore, the discrete version of the sine fourier transform is:

$$X_n = \left(\frac{iA}{2\Delta f}\right)(-\delta_{n,n_0} + \delta_{n,-n_0}) \tag{10}$$

With Δf standing for the frequency resolution, and $i_0\Delta f = f_0$. The term Δf is necessary to keep the discrete integral consistent with the continuous one:

$$\sum_{n=-\infty}^{n=+\infty} \frac{A\delta_{n,n_0}}{\Delta f}\Delta f = A = \int_{f=-\infty}^{f=+\infty} A\delta(f_0)df \tag{11}$$

If you stare blanky at this equality you will note that LHS clearly have the dimension [amplitude]. RHS must the same dimension, illustrating the strange dimensional property of the continuous dirac delta function.

- In the frequency domain, $\delta(f)$, the dirac delta function, is of dimension [frequency^{-1}].
- In the discrete formalism, one must use $\delta_{n,n_0}/\Delta f$ to be consistent. Here δ_{n,n_0} is the Kroenecker symbol, without dimension.

These problems are usually not treated in signal processing books, and arise here because CFD signals are time-dependent. Fig. 2 shows a discrete version of Fig. 1 consistent with the integral properties of the Fourier transform. Note that the estimator drawn strictly from mathematics is dependent upon the duration of the signal. The same estimator multiplied by Δf is far more interesting as it gives directly the amplitude of the harmonic components.

2.3 Discrete form of Fourier transform

The discrete Fourier transform F_k for a discrete time series x_n is

$$F_k = \frac{1}{N}\sum_{n=0}^{N-1}\left(x_n \exp\left(-i\frac{2\pi}{N}nk\right)\right), k = 0, 1, ..., N-1 \tag{12}$$

where N is the even[1] number of time domain samples, n is the time domain sample index, k is the frequency domain index. This is the form recommended

[1]The case of odd number of time domain samples is not discussed here. *Hic sunt leones.*

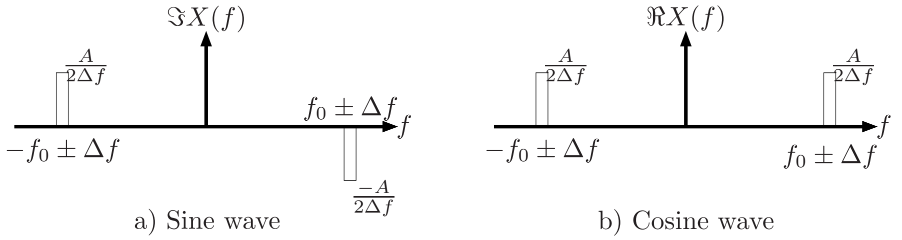


Figure 2: Discrete fourier transform of sine and cosine waves

to get directly the amplitude of the signal, independently from the frequency resolution[2]. Note that the frequency domain increment Δf is drawn from the time domain period T with

$$\Delta f = \frac{1}{T} \tag{13}$$

The frequency f_k is obtained from the index parameter k as follows

$$f_k = k\Delta f \tag{14}$$

Note that F_k has dimensions of [amplitude].

The corresponding inverse transform is

$$x_n = \sum_{k=0}^{N-1} \left(F_k \exp\left(+i\frac{2\pi}{N}nk \right) \right), n = 0, 1, ..., N-1 \tag{15}$$

A characteristic of the discrete Fourier transform is that the frequency domain is taken from 0 to $(N-1)\Delta f$. The line of symmetry is at a frequency of $N/2\Delta f$ which marks the Nyquist frequency (one-half of the sampling rate). Shannons sampling theorem states that a sampled time signal must not contain components at frequencies above the Nyquist frequency. This point is under focus in Exercise 1.

Spectrum analyzer devices typically represent the Fourier transform in terms of magnitude and phase rather than real and imaginary components. Furthermore, spectrum analyzers typically only show only half the total frequency band due to the symmetry relationship. The spectrum analyzer amplitude may either represent the half-amplitude or the full- amplitude of the spectral components. The one-sided, full-amplitude Fourier transform magnitude would be calculated as

$$\begin{aligned} F_k &= Magn\left[\frac{1}{N} \sum_{n=0}^{N-1} \left(x_n \exp\left(-i\frac{2\pi}{N}nk \right) \right) \right], k = 0 & (16) \\ &= 2Magn\left[\frac{1}{N} \sum_{n=0}^{N-1} \left(x_n \exp\left(-i\frac{2\pi}{N}nk \right) \right) \right], k = 1, ..., \frac{N}{2} - 1 & (17) \end{aligned}$$

with N as an even integer. Note that k = 0 is a special case. The Fourier transform at this frequency is already at full- amplitude.

[2] An alternate form which has dimensions of[amplitude.time] is :
$F_k = \Delta t \sum_{n=0}^{N-1} \left(x_n \exp\left(-i\frac{2\pi}{N}nk \right) \right), k = 0, 1, ..., N-1$, where $T = N\Delta t$. In other words, Eq. 12 divided by the frequency resolution Δf. This form is in strict equivalence with the original Fourier transform of Eq. 2.

2.4 Exercise 1 : numerical application of Amplitude spectra

Let us consider a signal defined on 8 instants on a duration $T = 1s$, i.e. $0, 1/8s, 2/8s, 3/8s, 4/8s, 5/8s, 6/8s, 7/8s$. The sampling frequency is $f_s = 8Hz$. The Real and Imaginary terms of the Fourier series are detailed in the tables 1: The sum of Fourier terms is done following the columns of these tables. Each

$\cos(2\pi\frac{nk}{8})$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=0$	1	1	1	1	1	1	1	1
$n=1$	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$
$n=2$	1	0	-1	0	1	0	-1	0
$n=3$	1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	-1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$
$n=4$	1	-1	1	-1	1	-1	1	-1
$n=5$	1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	-1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$
$n=6$	1	0	-1	0	1	0	-1	0
$n=7$	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$

$\sin(2\pi\frac{nk}{8})$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=0$	0	0	0	0	0	0	0	0
$n=1$	0	$\sqrt{2}/2$	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$
$n=2$	0	1	0	-1	0	1	0	-1
$n=3$	0	$\sqrt{2}/2$	-1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	1	$-\sqrt{2}/2$
$n=4$	0	0	0	0	0	0	0	0
$n=5$	0	$-\sqrt{2}/2$	1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	-1	$\sqrt{2}/2$
$n=6$	0	-1	0	1	0	-1	0	1
$n=7$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$	0	$\sqrt{2}/2$	1	$\sqrt{2}/2$

Table 1: Real and Imaginary coefficients of the discrete Fourier transform for any signal on the time domain $0, 1/8s, 2/8s, 3/8s, 4/8s, 5/8s, 6/8s, 7/8s$

column is contributing to a Fourier coefficient (or frequency). One can observe the symmetry of columns $k = 1, 2, 3$ and $k = 5, 6, 7$ with respect to the "Nyquist" column $k = 4$.

- **Question** : With an Excel spreadsheet, compute the coefficients for a 1Hz sine wave.
- **Answer** : The signal and coefficients are numerically given by Tab. 2.4 and Fig. 3. This result is in agreement with Eq. 8, except that the positive dirac at $-f_0$ is lying here at $k = 7$, or $f_s - f_0$. The repetition of the pattern around f_s and its multiples is due to the finite duration of the signal.
- **Question** : Now compute with the same spreadsheet the coefficients of a signal summing four cosine waves at $1Hz, 2Hz, 3Hz, 4Hz$. What would happen with a sine wave at the Nyquist frequency $sin(2\pi t \times 4)$? What whould happen with frequencies greater than the Nyquist frequency?
- **Answer** : The signal and coefficients are numerically given by Tab. 2.4 and Fig. 4. The results are again consistent with Eq. 9. However, the

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
0	$\sqrt{2}/2$	1	$\sqrt{2}/2$	0	$-\sqrt{2}/2$	-1	$-\sqrt{2}/2$

X_0	X_1	X_2	X_3	X_4	X_5	X_6	X_7
0	$-0,5i$	0	0	0	0	0	$+0,5i$

Table 2: Coefficients of the function $sin(2\pi t)$ on the time domain $0, 1/8s, 2/8s, 3/8s, 4/8s, 5/8s, 6/8s, 7/8s$

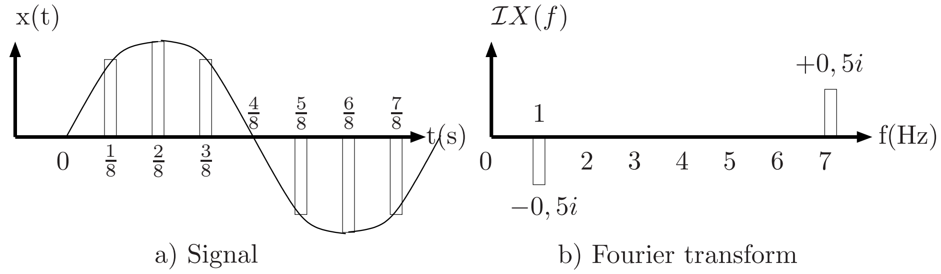


Figure 3: Discrete Fourier transform of the function $sin(2\pi t)$

Nyquist frequency coefficient receives the contribution from both sides , $X_4 = 1$.

Considering a sine wave at the same frequency $sin(2\pi t \times 4)$, $X_4 = 0i$.(Indeed, $x_{0,1,\ldots,7} = 0$ due to undersampling). The original signal should never exhibit a significant content beyond the Nyquist frequency.

x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
4	-1	0	-1	0	-1	0	-1

X_0	X_1	X_2	X_3	X_4	X_5	X_6	X_7
0	$0,5$	0,5	0,5	1	0,5	0,5	0,5

Table 3: Coefficients of the function $cos(2\pi t) + cos(2\pi t \times 2) + cos(2\pi t \times 3) + cos(2\pi t \times 4)$ on the time domain $0, 1/8s, 2/8s, 3/8s, 4/8s, 5/8s, 6/8s, 7/8s$

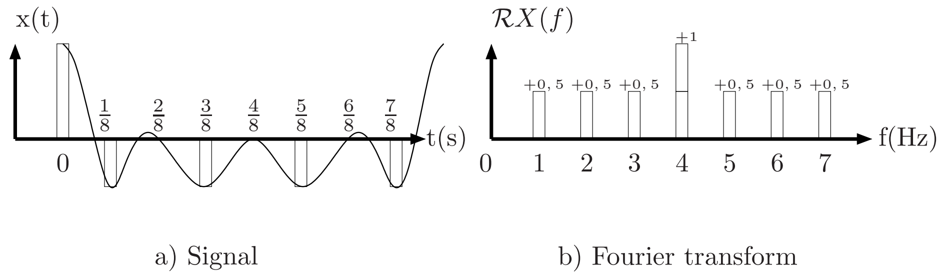


Figure 4: Discrete Fourier transform of the function $cos(2\pi t) + cos(2\pi t \times 2) + cos(2\pi t \times 3) + cos(2\pi t \times 4)$

3 Power distributions

3.1 Power spectral density function

First of all, the term "Power" refers to the instantaneous power of a signal $x(t)$, defined equal to $|x(t)|^2$. Consequently the signal "Power" is equal to its Mean Square value. From a discrete point of view, the power spectral density distribution is thus the evaluations of Mean Square (MS) values of frequency bands on the signal. Before any further reading, let us recall some statistical estimators of Tab. 4.

Name	Continuous form	Discrete form
Mean $\overline{X}$	$\lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt$	$\lim_{N\to\infty} \frac{1}{N}\sum_i^N x_i$
Mean Square $\overline{X^2}$	$\lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)^2dt$	$\lim_{N\to\infty} \frac{1}{N}\sum_i^N x_i^2$
Root Mean Square $\sqrt{\overline{X^2}}$	$\lim_{T\to\infty} \sqrt{\frac{1}{T}\int_0^T x(t)^2dt}$	$\lim_{N\to\infty} \sqrt{\frac{1}{N}\sum_i^N x_i^2}$
Variance σ^2	$\lim_{T\to\infty} \frac{1}{T}\int_0^T (x(t)-\overline{X})^2dt$	$\lim_{N\to\infty} \frac{1}{N}\sum_i^N (x_i-\overline{X})^2$

Table 4: Several statistical estimators

One can find three methods to compute the PSD:

1. Measuring the MS value of the amplitude in successive frequency bands, where the signal in each band has been bandpass filtered. This "brute force" method yields cumbersome computations.

2. Taking the Fourier transform of the autocorrelation function. This is the Wiener-Khintchine approach. While conceptually different (point of view of autocorrelation , *Hic sunt leones* again), this second method is mathematically the same as the third method.

3. Taking the limit of the Fourier transform $X(f)$ times its complex conjugate divided by its period T as the period approaches infinity. This last method relies on the Fourier Transform, which comes in handy since it was the topic of the previous section.

Following the third method , the Fourier transform X(f) for a continuous time series x(t)

$$X(f) = \lim_{T\to\infty} \int_{-T/2}^{+T/2} x(t)\exp\left(-i2\pi ft\right)dt \tag{18}$$

where $-\infty < f < +\infty$.

The power spectral density S(f) for a continuous Fourier transform is defined as

$$S(f) = \lim_{T\to\infty} \frac{1}{T} X(f)X^*(f) \tag{19}$$

where $-\infty < f < +\infty$ and symbol * denotes complex conjugate.

Moving to the discrete domain the single-sided power spectral density function PSD k for a discrete series is

$$PSD_k = \left[\frac{F_k F_k^*}{\Delta f}\right], k = 0 \tag{20}$$

$$= \frac{1}{2}\left[\frac{F_k F_k^*}{\Delta f}\right], k = 1, ..., \frac{N}{2} - 1 \quad (21)$$

with N an even integer and F_k the Fourier coefficients normalized[3] to be an amplitude (see Eq. 52).

The $\frac{1}{2}$ factor in equation 21 comes from the fact that the Mean Square of a sine wave is equal to half its peak value (or $\int_0^\pi \sin^2(t)dt = \pi/2$). The $k = 0$ case does not require this peak-to-MS conversions since the MS value is equal to the peak value for a signal with zero frequency. This signal is often called a DC signal. Each coefficent PSD_k is evaluated on a narrow frequency band $1/\Delta f$.

3.2 Parseval equality

The area under the power spectral density curve is equal to the mean square value. The mean square value can also be calculated directly from the time history. This equivalence can be written as an energy equivalence :

$$E = \lim_{T\to\infty} \int_{-T/2}^{+T/2} x(t)^2 dt = \lim_{T\to\infty} \int_{-\infty}^{+\infty} X(f)X^*(f)df \quad (22)$$

A similar result can be written for power equivalence:

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)^2 dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{+\infty} X(f)X^*(f)df \quad (23)$$

Almost all actions in spectral processing (normalization of spectra, zero-padding, windowing, half spectrum computation, removal of the DC term) can affect the Parseval equality. A Parseval check on the final spectrum is the proof that all these actions are correctly addressed.

3.3 Sound Pressure Level

Sound pressure level (SPL) or sound level is a logarithmic measure of the effective sound pressure of a sound relative to a reference value. It is measured in decibels (dB) above a standard reference level. The commonly used "zero" reference sound pressure in air is 20 μPa, which is usually considered the threshold of human hearing. The formula reads:

$$SPL = 10 \log\left(\frac{\sigma^2}{p_{ref}^2}\right) \quad (24)$$

where σ^2 is the variance of the signal. The variance formulas are given for continuous and digital signal respectively:

$$\sigma^2 = \lim_{T\to\infty} \left(\frac{1}{T}\int_0^T (x(t) - \overline{X})^2\right) \quad (25)$$

$$\sigma^2 = \lim_{N\to\infty} \left(\frac{1}{N}\sum_i^N (x_i - \overline{X})^2\right) \quad (26)$$

[3]With this normalization continuous coefficient $X(f)$ corresponds to discrete coefficient $T \times F_k$. This might help the reader to see the link between Eq. 19 and Eqs. 20,21

Its square root is called the standard deviation which is NOT equivalent to Root Mean Square values. Indeed, most web sources uses P_{RMS} instead of σ^2. If you compute the RMS with the DC component of the pressure, you will end up with $SPL > 180dB$, even without fluctuations).

In the case of several noises, it is possible to compute the SPL provided that signals are un correlated. In this case only, the variance of the sum is equal to the sum of variances:

$$SPL = 10\log\left(\frac{\sum_i \sigma_i^2}{p_{ref}^2}\right) \tag{27}$$

This explains why a PSD spectrum expressed in SPL cannot be directly integrated over frequencies to reach the global SPL of a signal.

3.4 Exercise 2 : statistics .vs. Parseval for PSD spectra

- **Question** : The main characteristic of the white noise is its fluctuation about the mean, described by the variance σ^2. Find the relation between Mean Square value and Variance. Consequently, what would be the mean square value of a white noise of variance $0.01Bar^2$ in the atmospheric pressure. Express this result in terms of Sound Pressure Level.

- **Answer**: The Mean Square value is the variance plus the squared mean i.e. :$\overline{X^2} = \sigma^2 + \overline{X}^2$.

$$\begin{aligned}
\sigma^2 &= \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T (x(t) - \overline{X})^2\right) && (28)\\
&= \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T x(t)^2 - 2x(t)\overline{X} + \overline{X}^2\right) && (29)\\
&= \lim_{T\to\infty}\left(\frac{1}{T}\int_0^T x(t)^2\right) - 2\overline{X}\lim_{T\to\infty}\left(\frac{1}{T}\int_0^T x(t)\right) + \overline{X}^2 && (30)\\
&= \overline{X^2} - 2\overline{X}^2 + \overline{X}^2 && (31)
\end{aligned}$$

  The mean square value of the pressure signal would be $P_{atm}^2 + \sigma^2 = 1,01Bar^2$. The variance, expressed in pascal is $1.0267\,10^6 Pa^2$. The SPL is then $20\log(1.0267\,10^6/2\,10^{-5}) = 154dB$.

- **Question** :In a more general context, what would be the Mean Square value of the signal:

$$x(t) = A + Bcos(2\pi f_0) + \mathcal{N}(C) \tag{32}$$

  with $\mathcal{N}(C)$ being a white noise of variance C. Note that the variance of a sine wave $\sigma^2\,(cos(t))$ is equal to $\frac{1}{2}$. Draw the PSD spectrum in the case of a sampled signal. Discuss the case of an increase of the sampling rate. Discuss the case of an increase of the signal duration.

- **Answer**: As the variance of the three terms are uncorrelated, it is possible to write the sum of variance and squared mean for all terms :

$$
\begin{aligned}
MS(x(t)) &= \sigma^2(A) + \overline{A}^2 & (33)\\
&+ \sigma^2(Bcos(2\pi f_0)) + \overline{Bcos(2\pi f_0)}^2 & (34)\\
&+ \sigma^2(\mathcal{N}(C)) + \overline{\mathcal{N}(C)} & (35)\\
&= 0 + A^2 & (36)\\
&+ \frac{B^2}{2} + 0 & (37)\\
&+ C + 0 & (38)
\end{aligned}
$$

The Mean Square value is therefore $A^2 + \frac{B^2}{2} + C$. The PSD spectrum is sketched in Fig. 5a. The evolution of the total energy curve is also displayed on Fig. 5b.

In the case of an infinite signal duration $\Delta f \to 0$, the contribution of the DC component A and the sine wave reduces to dirac peaks of 'infinite' amplitude, while the noise part remains at the same level.

If the sampling rate increases, $f_{Nyquist}$ is increasing. Both DC and sine peaks remain unchanged. If the noise level variance is kept constant , the noise level is decreasing as $C/f_{Nyquist}$.

The Parseval equality is satisfied taking into account all the terms of the signal (Here DC component, harmonic component, random component). As some components can be removed during the spectral analysis process (zero-padding requires the removal of the DC term, for example), the user must track these simplifications while evaluating the power distribution in the signal.

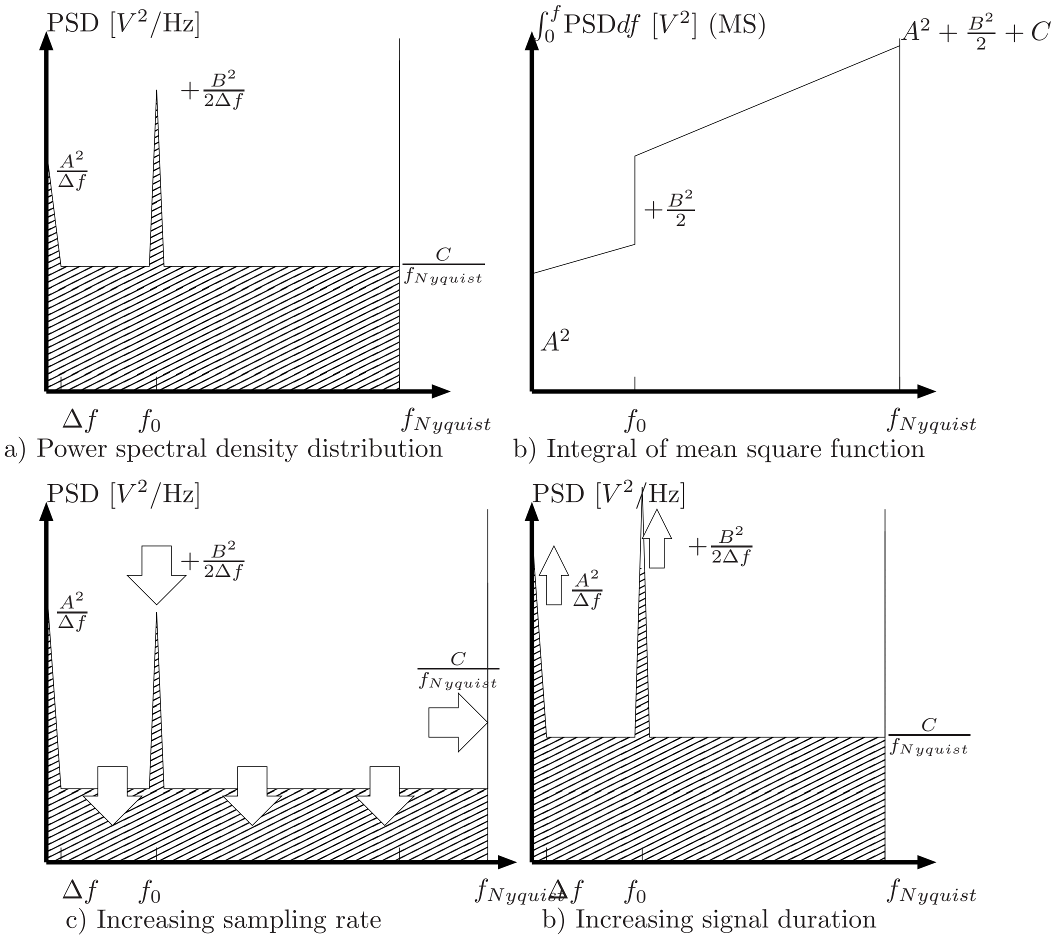


Figure 5: Illustration of different Power Spectral Density . The noise level of fig. a),c),d) is an ideal estimation,

4 Finite duration, sampled signals

4.1 Impact of finite duration signals

The case of pure continuous sine waves has been addressed in the previous section. We will investigate now the effect of finite duration signals through some mathematical considerations.

First, we consider the following infinite signal $x(t) = cos(2\pi f_0 t + \phi_0)[V]$, with $f_0[Hz]$ a strictly positive frequency and $\phi_0[rad]$ a phase in the range $[0, 2\pi][rad]$. Note : the delta function is the Fourier transform of 1: $\delta(f) = \int_{-\infty}^{+\infty} e^{-i2\pi ft}dt$.

$$\begin{aligned}
X(f) &= \int_{-\infty}^{+\infty} cos(2\pi f_0 t + \phi_0)e^{-i2\pi ft}dt && (39)\\
&= \int_{-\infty}^{+\infty} \left(\frac{e^{i(2\pi f_0 t+\phi_0)} + e^{-i(2\pi f_0 t+\phi_0)}}{2}\right) e^{-i2\pi ft}dt && (40)\\
&= \int_{-\infty}^{+\infty} \left(\frac{e^{i\phi_0}e^{i2\pi f_0 t} + e^{-i\phi_0}e^{-i2\pi f_0 t}}{2}\right) e^{-i2\pi ft}dt && (41)\\
&= \int_{-\infty}^{+\infty} \frac{1}{2}\left(e^{i\phi_0}e^{-i(2\pi(f-f_0)t)} + e^{-i\phi_0}e^{-i(2\pi(f+f_0)t)}\right) dt && (42)\\
&= \frac{1}{2}\left(e^{i\phi_0}\delta(f-f_0) + e^{-i\phi_0}\delta(f+f_0)\right) && (43)
\end{aligned}$$

This last equation is illustrated in Fig. 6. Note in particular that for $\phi_0 = \pi/2$, the result of the Fourier transform is consistent with $\cos(2\pi f_0 t + \pi/2) = \sin(2\pi f_0 t)$.

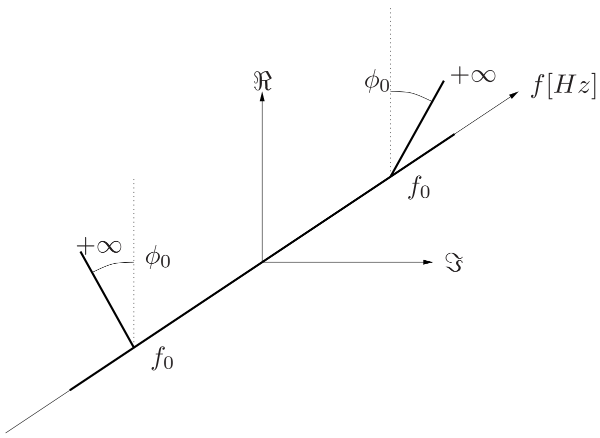


Figure 6: Fourier transform of the cosine signal, frequency f_0, phase ϕ_0

Then, let us consider the signal $x(t) = w_{rect}^T(t) \times cos(2\pi f_0 t + \phi_0)[Hz]$, with w_{rect}^T a rectangular window of duration T. Note : the Fourier transform of the rectangular function is:
$W_{rect}^T(f) = \int_{-\infty}^{+\infty} W_{rect}^T(x)e^{-i2\pi fx}dx = Tsinc(\pi fT) = \frac{sin(\pi fT)}{\pi f}$, which is illustrated on Fig. 7.

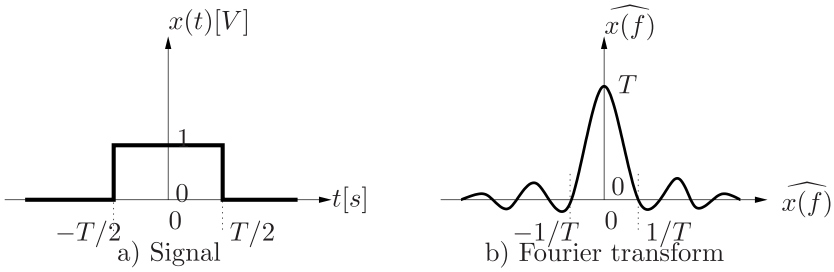


Figure 7: The Fourier transform of a rectangular window is a Sync function.

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{+\infty} w_{rect}^{T}(t) cos(2\pi f_0 t + \phi_0) e^{-i2\pi f t} dt && (44)\\
&= \int_{-\infty}^{+\infty} \frac{1}{2} w_{rect}^{T}(t) \left(e^{i\phi_0} e^{-i(2\pi(f-f_0)t)} + e^{-i\phi_0} e^{-i(2\pi(f+f_0)t)} \right) dt && (45)\\
&= \frac{T}{2} \left(e^{i\phi_0} sinc(\pi(f-f_0)T) + e^{-i\phi_0} sinc(\pi(f+f_0)T) \right) && (46)
\end{aligned}
$$

This curve degenerates to the Fourier transform of the infinite signal with $T \to +\infty$. A non-intuitive observation is the fact that the frequency support is infinite $-\infty < f < \infty[Hz]$ while the signal duration is finite. These results are illustrated on Fig 8 and Fig 9. Phase modification phase rotates the curve along the frequency axis : Fig 9 a) .vs. b). Target frequency modification shifts the *sinc*'s along the frequency axis without other alterations : Fig 9 a) .vs. c). Signal duration modification compresses the *sinc*'s around their respective centers, and the amplitude is increased : Fig 9 a) .vs. d).

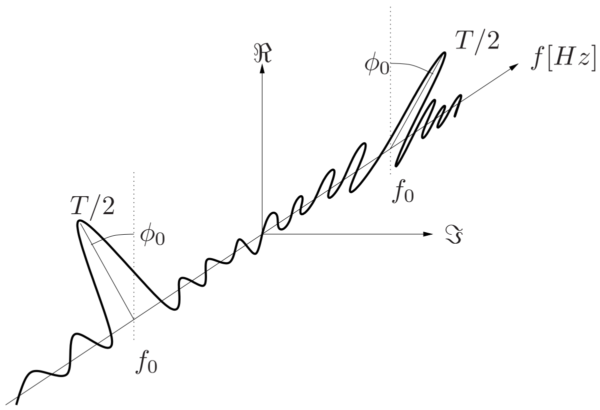
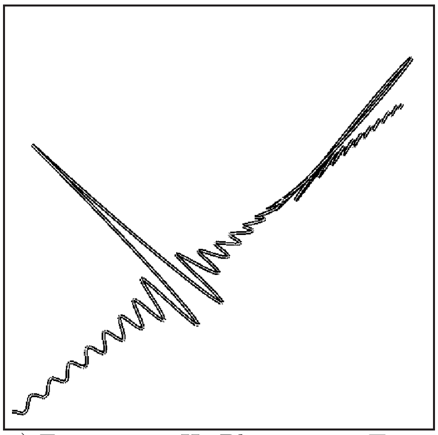
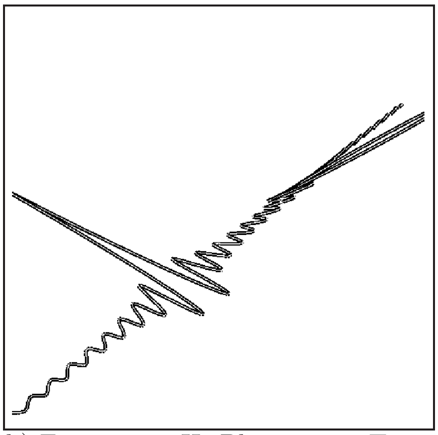
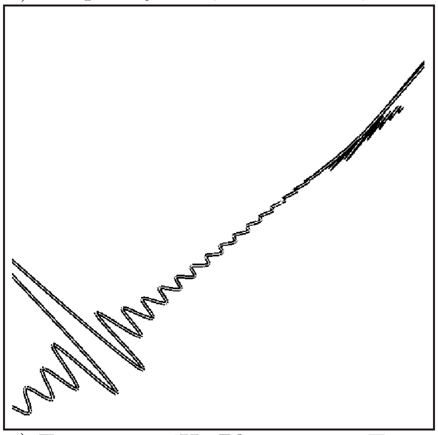
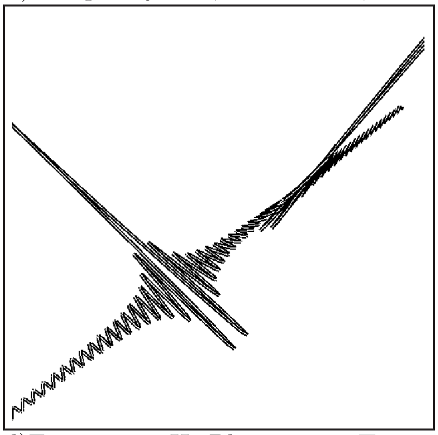


Figure 8: Fourier transform of the cosine signal, frequency f_0, phase ϕ_0

To conclude, all signals coming from Computational Fluid Dynamics problems are of finite duration. Therefore, all the estimators will describe the spectral content with wavy *sinc* functions, and not series of clean localized dirac delta functions.

a) Frequency 2Hz,Phase 0.25π,T=5s b) Frequency 2Hz,Phase 0.35π,T=5s

c) Frequency 4Hz,Phase 0.25π,T=5s d)Frequency 2Hz,Phase 0.25π,T=10s

Figure 9: Fourier transform of a cosine of finite duration : plot of the analytical solution.

4.2 Aliasing

First, let us recall some mathematical properties of the Dirac comb, with an exerpt of Wikipedia on "the dirac comb":

In mathematics, a Dirac comb, also known as an impulse train and sampling function in electrical engineering, is a periodic Schwartz distribution constructed from Dirac delta functions :

$$\delta_\tau(t) = \sum_{k=-\infty}^{+\infty} \delta(t - k\tau) \tag{47}$$

for some given period τ. Some authors, notably Bracewell as well as some textbook authors in electrical engineering and circuit theory, refer to it as the Shah function (possibly because its graph resembles the shape of the Cyrillic letter sha). Because the Dirac comb function is periodic, it can be represented *as a Fourier series :*

$$\Delta_\tau(f) = \frac{1}{\tau}\sum_{n=-\infty}^{+\infty} e^{i2\pi nt/\tau} \quad (48)$$

The Fourier transform of a Dirac comb is also a Dirac comb.

$$\Delta_\tau(f) = \frac{1}{\tau}\delta_{1/\tau}(f) \quad (49)$$

The Fourier transform of a signal $x(t)(\tau\delta_\tau(t))$, i.e. the signal $x(t)$ sampled on a period τ, is the convolution of $X(f)$ and $\Delta_\tau(f)$.

$$\begin{aligned} X(f) * \Delta_\tau(f) &= X(f) * \sum_{n=-\infty}^{+\infty} \delta(f - \frac{n}{T}) && (50)\\ &= \sum_{n=-\infty}^{+\infty} X(f - n/\tau) && (51)\end{aligned}$$

This demonstrates that the computation of the transform from a τ-sampled signal $x(t)(\tau\delta_\tau(t))$ is equal to the sum of :

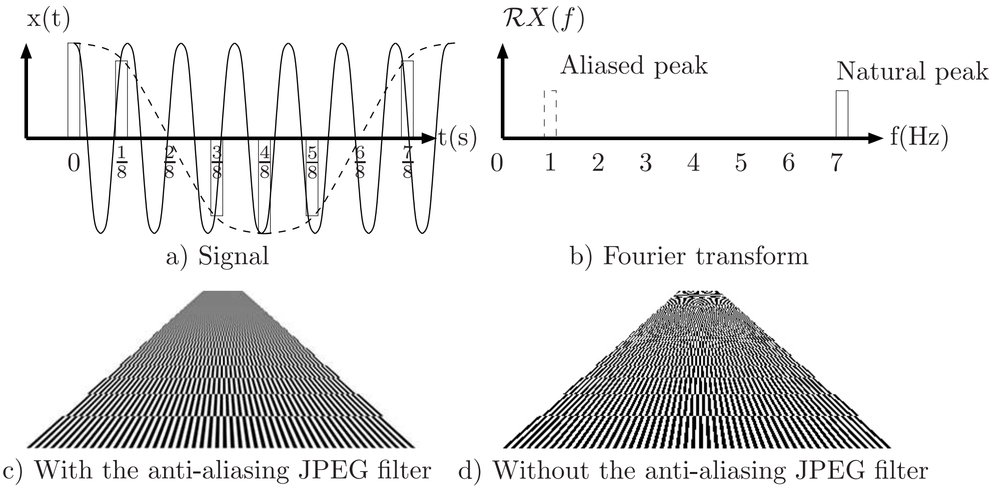
- the transform from the continuous signal $X(f)$
- of an infinity of copies of this transform, shifted by the frequency gaps n/τ.

Figure 10 illustrates the aliasing effect. In the temporal domain, the undersampling of a $7Hz$ cosine yields a serie of value equal to the $1Hz$ cosine (Fig. 10a). In the frequency domain, the natural peak is present at $7Hz$, but the only visible peak comes from the copy of the signal $X(f - 1/\tau)$, at $1Hz$ (Fig. 10b). The very same process is visible in two dimensions through the following example. A highly resolved checkboard pattern is displayed on Fig. 10c thanks to the anti aliasing JPEG filter : small black and white regions are shown as grey. Without the anti-aliasing pattern (Fig. 10d), several wavelenghts are visible, biasing the actual checkboard pattern.

4.3 Exercise 3: Filtering to prevent aliasing

- **Question** : A low pass filter rejecting the high frequencies is the common solution to prevent aliasing. Explain when to use this filter on a signal created by a CFD solver, (timestep $1\,10^{-7}s$, duration $10ms$) to design a fast signal processing tool focusing on the frequency range $0 - 10kHz$.
- **Answer** : According to the signal, the frequency resolution is 100Hz, the maximum Nyquist frequency available is $5MHz$, cf. Fig. 11a. Taking one point over 500 in the original signal yields a timestep of $5\,10^{-5}s$ and a Nyquist frequency reduced to $10kHz$ as requested. However, the direct subsampling do not prevent the aliasing, see Fig. 11b. Therefore, the low pass filter, rejecting all frequencies beyond $10kHz$ must be applied like in Fig. 11c before the downsampling of one point over 500. The aliased third peak of Fig. 11d is then rejected , as in Fig. 11e.

  In other words, applying a low-pass filter on a signal right before spectral processing shows no interest : the frequencies higher than the sampling



a) Signal
b) Fourier transform
c) With the anti-aliasing JPEG filter
d) Without the anti-aliasing JPEG filter

Figure 10: Aliasing effect : a) & b) on a $7Hz$ cosine sampled at $8Hz$; c) & d) in two dimensions on a checkboard pattern.(warning, do not read this figure on screen , print it !)

frequency have already become low frequencies by sampling effect, and will not be rejected by the filter.

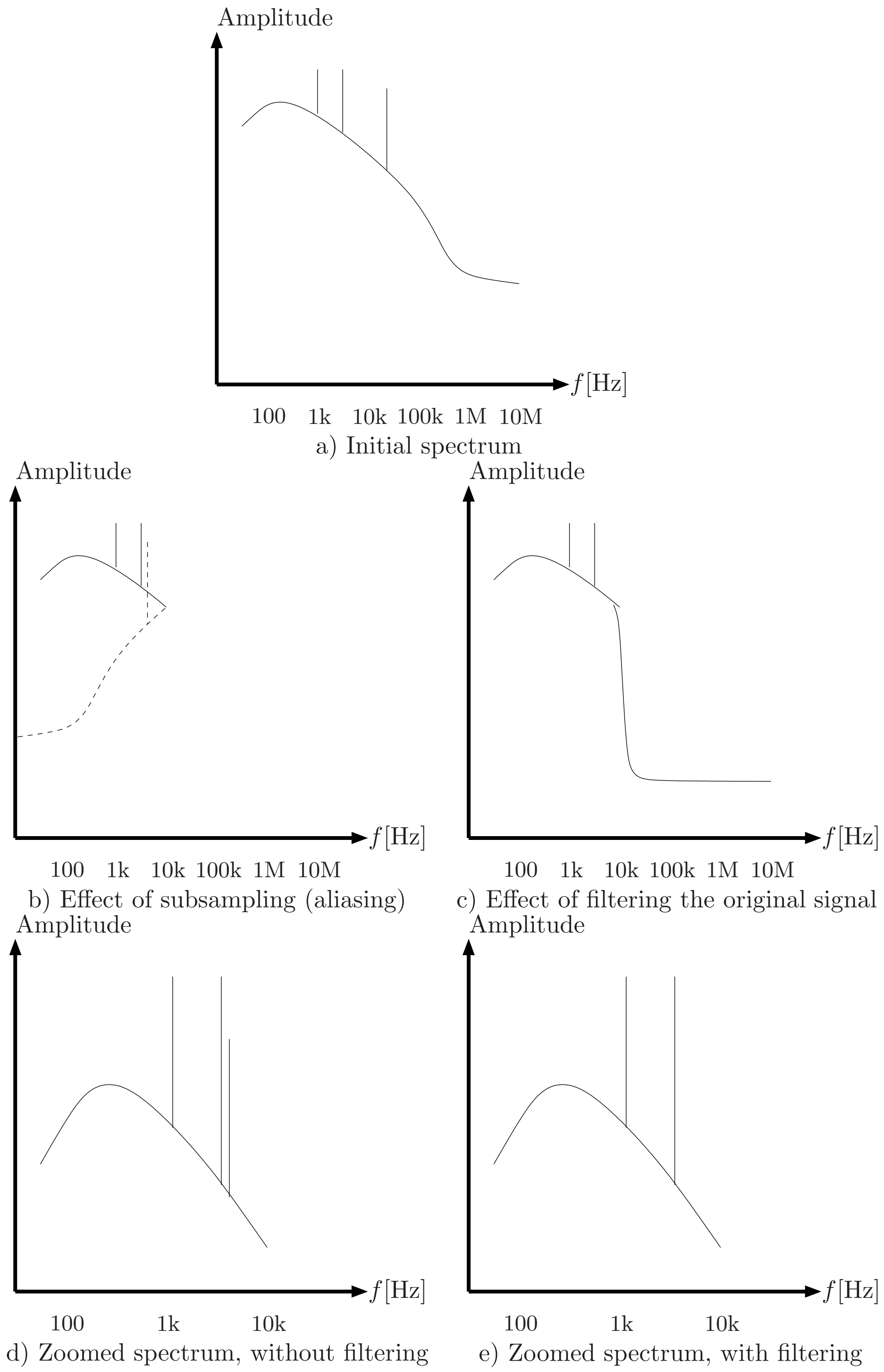


Figure 11: Ex3 : Filtering to prevent aliasing

5 Improvement of spectra

5.1 Zeropadding

Zeropadding is a numerical recipe allowing to increase the representation of the Fourier transform. As seen in the previous sections, the Fourier transform of signals are continuous on the frequency domain $-\infty < f < \infty$, even for sampled signals and finite duration signals. The discrete Fourier transform F_k for a discrete time series x_n was defined as:

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} \left(x_n \exp\left(-i \frac{2\pi}{N} nk \right) \right), k = 0, 1, ..., N-1 \tag{52}$$

However, one can arbitrarily increase the number of coefficients with $N_{zp} > N$.

$$F_k^{zp} = \frac{1}{N_{zp}} \sum_{n=0}^{N_{zp}-1} \left(x_n \exp\left(-i \frac{2\pi}{N_{zp}} nk \right) \right), k = 0, 1, ..., N_{zp}-1 \tag{53}$$

with $x_n = 0\ \forall n > N-1$, which simplifies into :

$$F_k^{zp} = \frac{1}{N_{zp}} \sum_{n=0}^{N-1} \left(x_n \exp\left(-i \frac{2\pi}{N_{zp}} nk \right) \right), k = 0, 1, ..., N-1 \tag{54}$$

This last expression gives N_{zp} coefficients F_k, built upon the sum of N terms.

The zeropadding method is thus a way to increase the number of Fourier coefficients which describes the spectrum of a finite duration signal. It is usefull to get a better graphical description of the *sinc* main peak, i.e. a better estimation of a specific harmonic amplitude. The frequency resolution (or the width of the associated *sinc*) is not improved through zeropadding.

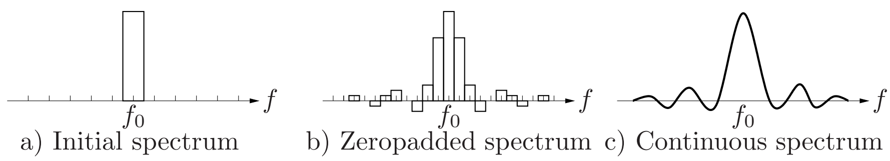
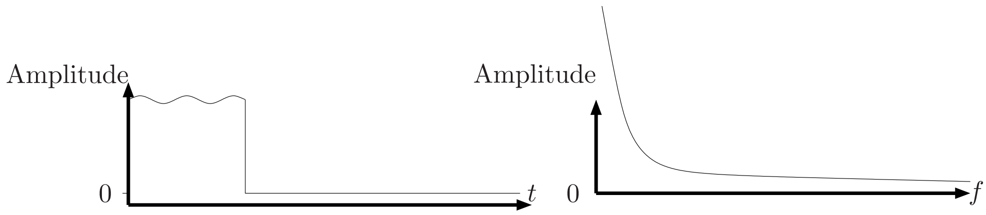


Figure 12: Zeropadding effect on the spectrum of a cosine wave of frequency f_0.

A major drawback from the zeropadding comes from the DC component of the signal. In the non-zeropadded case, the DC component is limted only to the first fourier coefficient F_0. In the zeropadded case, the signal takes the shape of an Heaviside function $h(T-x)$ like in Fig. 13a, especially if the amplitude of the DC component is big with respect to the signal fluctuation (e.g. pressure signals). Let us just note that the Fourier transform of the Heaviside function is :

$$\begin{aligned} H(f) &= \int_{-\infty}^{+\infty} e^{-2\pi i f t} h_t \qquad (55) \\ &= \frac{1}{2} \left[\delta(f) - \frac{i}{\pi f} \right] \qquad (56) \end{aligned}$$

This component can easily hide the actual spectrum with the ”spectral leak” of the imaginary term, as illustrated by Fig. 13b. For this reason, it is compulsory to remove the DC component from the signal when using the zeropadding technique.



a) Heaviside shape due to zeropadding   b) Amplitude spectrum

Figure 13: Zeropadding effect on the spectrum of a cosine wave of frequency with a strong DC component.

5.2 Multi-Windowing/Averaging

The topic of single-windowing is not discussed in the present material. The reader shall find by himself a proper description of the different windows available in his spectral processing language.

The multi-windowing method is used for a single purpose : reduce the variance of the Power Spectral Density estimator. Its use on Amplitude is pointless. Indeed, the Fourier transform of random signals of variance σ^2 yields a spectrum showing the same variance in the frequency domain, independently of the sampling. The clean way to ”smooth” the spectrum is to average several spectra made from independent realizations of the random signal, as illustrated in Fig 14a.

To avoid running several times the same experiment/computation, the multi-windowing strategy assumes the ergodicity of the random process : the statistical means (averages on realizations) are equal to time averaging. The signal is divided into bits showing the same lenght, mean, and variance, as shown on Fig 14b. Each bit is processed as an independent realization of the same random process. As this method supposes a 'long' signal , it can be tricky to apply on the short CFD-created signals.

A third strategy is to record the signal at several positions in the experiment at the same time like in Fig 14c, assuming that these positions are far enough to stay uncorrelated. For example, the pressure signal of eight probes located evenly on the circumference of a jet can replace eight runs with one probe. This method is particularly suited for CFD computations, with short signal but unlimited access to flow locations.

5.3 Exercise 4 : Bias of short signals

This exercise uses the OCTAVE jammer.m program.

- **Question** : The program generates a serie of N peaks of variable amplitudes, separated by variable silences. However, the global signal should exhibit a main frequency around 1Hz, with an amplitude near 0.5.

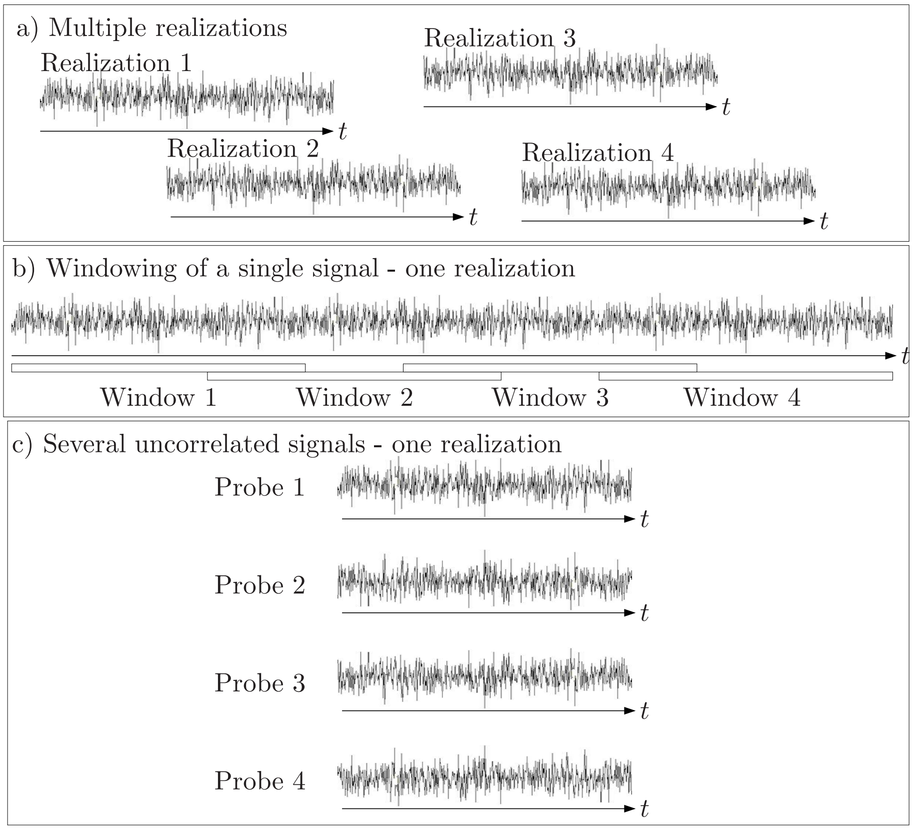


Figure 14: Three methods to reduce the variance of spectral estimators for random signals

What is the effect of time shifts and amplitude fluctuations on the spectrum (2nd and 3rd parameter)? Is this effect dependent of the signal length (1st parameter)? What can you conclude for LES signals?

- **Answer** : Pure amplitude shifts do not have strong effects on a Fourier spectrum. On the other hand, pure time shifts can quickly lead to totally biased spectra. Moreover, the biased spectra are totally case-dependant, with strong variations of the output from a realization to the other.

  The bias of the spectra slowly vanishes for long signals, i.e. substantially greater than 24 periods. In the case of short signals (dozen of periods or less) the bias makes the spectral analysis inconclusive. In this case, the spectra usually takes a sawtooth shape. The main tone is often visible, but can vanish on a pathologic realization, and is biased in frequency.

  LES signals are usually short with respect to the largest wavelenght. If the spectral content is close to an harmonic signal (negligible temporal lags), spectra will be easy to obtain, allowing a robust analysis of the flow. Short feedback loops like impinging jets belong to this category.

  On the other hand, if the spectral content is far from an harmonic case (strong temporal lags), the spectral analysis will be extremely hard to do. Signals coming from strongly non linear phenomenons like downstream a cylinder wake belong to this category.

6 Conclusions

6.1 List of the pitfalls

1. In a continuous frequency domain, the Dirac delta function is of dimension [frequency$^-1$]. In the discrete domain, it must be replaced by a Kroenecker function [no dimension], divided by the frequency resolution. This explains why the factor $1/\Delta f$ arises in the discrete formulas.

2. The normalization of an amplitude spectrum is done in order to ensure that the amplitude of harmonics remains unchanged whatever the signal duration/sampling is. Therefore, it is not strictly the magnitude of the Fourier transform.

3. The normalization of PSD spectrum is done in order to ensure that the power sectral density of noise remains unchanged whatever the signal duration/sampling is. Therefore, it is not strictly the magnitude of the squared Fourier transform.

4. The Power of a signal must not be confused with the physical quantity named Power. Its dimension is the signal dimension squared, do not expect Watts or Joules there.

5. The Mean Square is the variance plus the Squared Mean. The term RMS is often confused with the Variance. As Igor, Matlab, Octave or any math toolbox will follow the math definition, the function 'RMS' will often lead to something different than the user is expecting.

6. The Sound Pressure Level (SPL) uses the variance of the signal, not the RMS (see previous item). If you get 180 dB SPL even on a plain constant pressure signal, you have fallen in the trap.

7. Aliasing is created by the sampling process. Applying an anti-aliasing filter on the sampled signal is pointless (like pouring water on ashes).

8. Spectral maps computed from spatial solutions includes a temporal sampling. Thus, spectral maps are pretty good canditates for aliasing.

9. Zeropadding allows to increase the description of the spectrum (but not the resolution).

10. MultiWindowing is almost never usefull in CFD results, leave it to experiments. In CFD, it is easy to get the same improvement by multiplying the source of the signals.

11. A signal with significant fluctuations of the time lag between the 'event' is a good candidate for the "Chainsaw massacre" : sawtooth spectrum , extremely variable from a realization to the other. Even long signals do not fix totally the problem. Usually , moving the source closer to the start of the phenomenon is a substantial improvement.

6.2 Exercise 5 : the online spectral analysis lab

This exercise focuses on the online spectral analysis lab provided on the E-learning website of cerfacs:
`http://elearning.cerfacs.fr/numerical/signal/tutorialSpec/index.php`.

- **Amplitude** : check that the amplitude of the two harmonics are well caught on the amplitude spectrum.
- **DC componen**t : Set DC component on amplitude equal to 1, first harmonic amplitude to 1, second harmonic to 2, Set the Amplitude and PSD graphs on log-linear scale. Pay a close attention on the first point of the spectrums.

  The amplitude of DC is well caught on the Amplitude graph. The ratio between the DC component (Amplitude 1) and the first harmonic (Amplitude 1) is one.
- **Amplitude vs PSD** : With the same setup on the PSD graph, the DC component and the harmonic component do not stand on the same level, illustrating how Amplitude and PSD spectra differ.
- **Aliasing** : Rise the frequency f1 beyond the maximum frequency (Nyquist frequency). Note the reflexion of the peak when it bounces over the maximum frequency.
- **Noise** : Set the PSD graph in log-log. Then cycle between the White, Pink, and Brown noise. The Spectrum slope should move respectively from $PSD = cte$ to $PSD = 1/f$ and $PSD = 1/f^2$ (definitions of white, pink and brown noise). Note how injecting noise at low frequencies changes the signal shape.
- **Parseval** : Check the fact that the observed signal variance plus the observed signal squared mean equals the observed signal mean square. Note that this last quantity is equal to the sum of the PSD terms. Note also how the cummulative PSD reaches this last value.
- **Spectral power repartition** : Check how the noise contributes to the total amount of spectral power. In the particular case of a noise concentrated on low frequencies (Pink Brown), observe how the high frequencies becomes negligible in the Spectral Power budget.

References

[1] M. Kunt. *Traitement numérique des signaux.* Presses polytechniques et universitaires romandes.