A highly scalable asynchronous implementation of Balancing Domain Decomposition by Constraints

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Outline

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1. BDDC preconditioner
2. Overlapped BDDC implementation
3. Scalability analysis (overlapped)
4. Inexact BDDC
5. Scalability analysis (overlapped/inexact)
6. Conclusions and future work

Problem statement

Given a bounded domain Ω and a FE partition $\mathcal{T}$, we build a conforming (nodal) finite element (FE) space, i.e. $V_h \subset H_0^1(\Omega)$.

- **Variational problem**: find $u \in V_h$ such that

$$a(u, v) = (f, v), \qquad \text{for any } v \in V_h,$$

assuming $a(\cdot, \cdot)$ symmetric, coercive (e.g. Laplacian or linear elasticity)

- **Algebraic problem**: Equivalent to find $x \in \mathbb{R}^n$ such that

$$Ax = b,$$

where A is a large and sparse symmetric positive definite matrix

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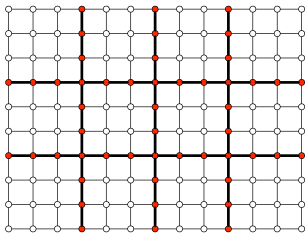
Motivation:

Efficient exploitation of distributed-memory machines for large scale FE problems $\Rightarrow$ Domain decomposition framework



$\circ$: interior DoFs (I); $\bullet$: interface dofs (Γ)

Interface (Schur complement) problem

- The domain partition induces a block structure

$$Ax = \begin{bmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} x_I \\ x_\Gamma \end{bmatrix} = \begin{bmatrix} b_I \\ b_\Gamma \end{bmatrix} = b,$$

where

$$A_{II} = diag\left(A_{II}^{(1)}, A_{II}^{(2)}, \ldots, A_{II}^{(P)}\right)$$

- After the interior correction $[A_{II}^{-1} b_I, 0]$, a reduced system for x_Γ is obtained

$$Sx_\Gamma = g,$$

where $S = A_{\Gamma\Gamma} - A_{\Gamma I} A_{II}^{-1} A_{I\Gamma}$ is the interface *Schur complement*

- **Approach:** Consider a Krylov subspace solver for $Sx_\Gamma = g$
  $\rightarrow$ Preconditioning plays a major role for optimality and scalability
- Alternatively, the preconditioner can be extended to $Ax = f$ (equivalent as soon as A_{II}^{-1} *exactly*)

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[Dohrmann, Mandel, Cros, Fragakis, Papadrakakis, Le Tallec, Vidrascu, ...]

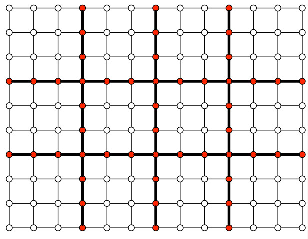
Idea: Solve global problem w/ reduced continuity

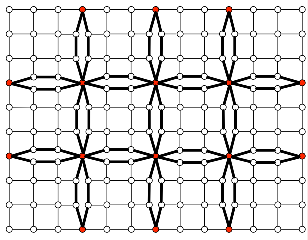
- Replace V_h by $\tilde{V}_h$ (reduced continuity)
- Define the injection $I : \tilde{V}_h \longrightarrow V_h$ weight, comm and add
- Find $\tilde{x}_h \in \tilde{V}_h$ such that:

$$a(\tilde{x}_h, \tilde{v}_h) = \langle I^t r_h, \tilde{v}_h \rangle, \quad \forall \tilde{v}_h \in \tilde{V}_h$$

and obtain $z_h = \mathcal{E} I \tilde{x}_h$, where $z_h = M_{BDDC}^{-1} r_h$

V_h





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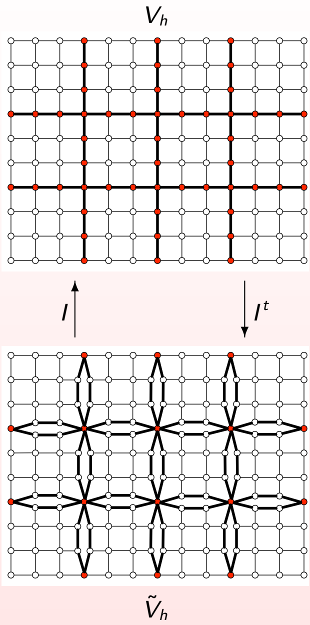

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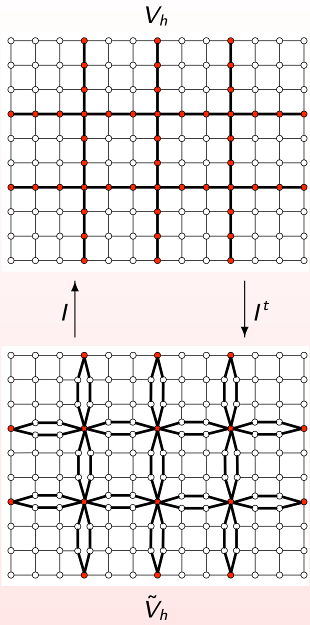
- Last correction: $\mathcal{E}$ is the harmonic extension of the boundary values, which implies local Dirichlet solvers



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Idea: Solve global problem w/ reduced continuity

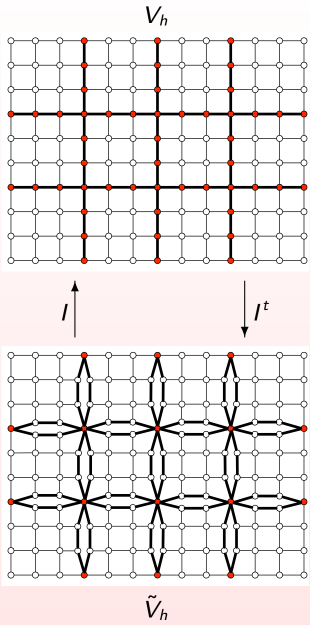
- Alternatively,

  Find $\tilde{x} \in \mathbb{R}^{\tilde{n}}$ such that:

$$\tilde{A}\tilde{x} = I^t r$$

  and obtain $z = \mathcal{E} I \tilde{x}$, where $z = M_{BDDC}^{-1} r$

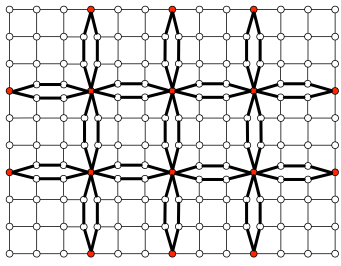
- $\tilde{A}$ is a sub-assembled global matrix (only assembled the red corners in the figure)
- $\mathcal{E} = \begin{bmatrix} 0 & -A_{II}^{-1} A_{I\Gamma} \\ 0 & I_\Gamma \end{bmatrix}$



- Let $\tilde{V}_h = [\tilde{v}_\circ \ \tilde{v}_\bullet]$ and decompose $\tilde{V}_h$ as

$$\tilde{V}_h = \tilde{V}_F \oplus \tilde{V}_C, \ \text{with} \begin{cases} \tilde{V}_F = [\tilde{v}_\circ \ 0] \\ \tilde{V}_C \perp_{\tilde{A}} \tilde{V}_F \end{cases}$$

- Now, problem split into fine-grid ($\tilde{x}_F$) and coarse-grid ($\tilde{x}_C$) correction



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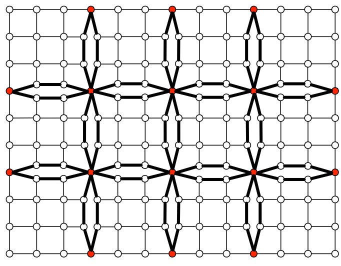

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Balancing domain decomposition by constraints (BDDC)

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Fine-grid correction $(\tilde{x}_F)$

- Find $\tilde{x}_F \in \mathbb{R}^{\tilde{n}}$ such that

$$\tilde{A}\tilde{x}_F = I^t r$$

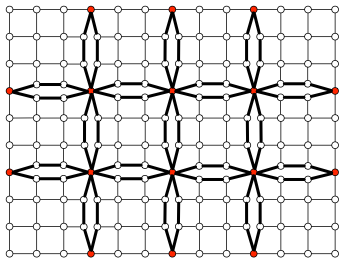
constrained to $(\tilde{x}_F)_\bullet = 0$

- Equivalent to P independent problems

Find $\tilde{x}_F^{(i)} \in \mathbb{R}^{\tilde{n}^{(i)}}$ such that

$$A^{(i)}\tilde{x}_F^{(i)} = I_i^t r$$

constrained to $(\tilde{x}_F^{(i)})_\bullet = 0$



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Coarse-grid correction ($\tilde{x}_C$)

Computation of $\tilde{V}_C = \mathrm{span}\{\Phi_1, \Phi_2, \ldots, \Phi_{n_C}\}$

- Find $\Phi \in \mathbb{R}^{\tilde{n} \times n_C}$ such that

$$\tilde{A}\tilde{\Phi} = 0$$

  constrained to $\Phi_{\bullet} = I$

- Equivalent to P independent problems

  Find $\Phi^{(i)} \in \mathbb{R}^{\tilde{n} \times n_C^{(i)}}$ such that

$$A^{(i)}\Phi^{(i)} = 0$$

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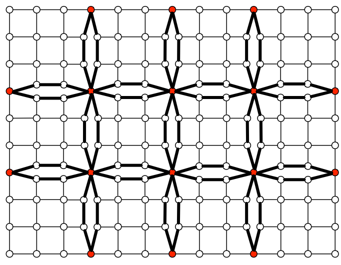

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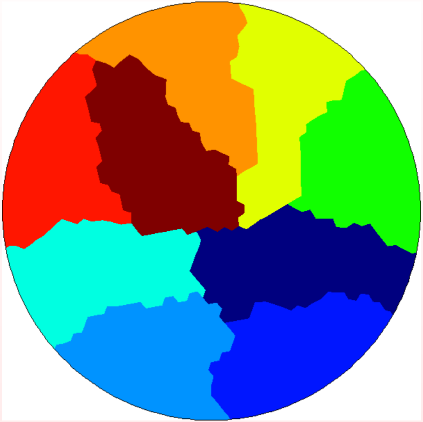
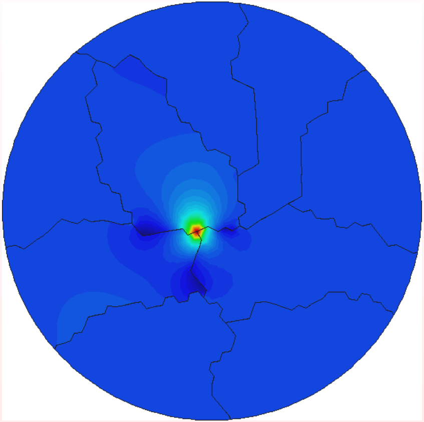


Circle domain partitioned into 9 subdomains



Φ_j ($\tilde{V}_C$'s basis vector)

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Coarse-grid correction ($\tilde{x}_C$)

Assembly and solution of coarse-grid problem

$$A_C = \text{assembly}(A_C^{(i)}) = \text{assembly}(\Phi^t A^{(i)} \Phi), \qquad \text{Solve } A_C \alpha_c = \Phi^t I^t r, \qquad \tilde{x}_C = \Phi \alpha_C$$

coarse-grid problem is

- Global, i.e. couples all subdomains
- But much smaller than S (size $\mathrm{n_C}$)
- Potential loss of parallel efficiency with P

Coarse dofs definition

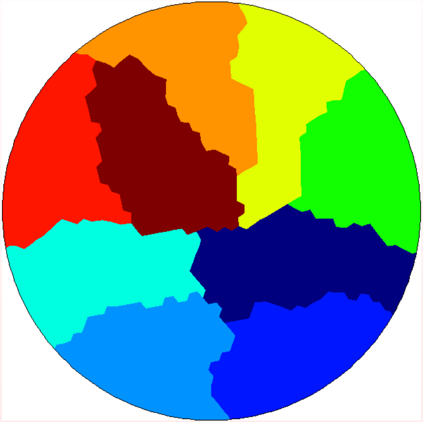
Key aspect: Selection of coarse dofs, i.e. continuity among subdomains

Properties of BDDC preconditioner

- Optimality ($\kappa(M^{-1}S)$ bounded by a constant for fixed N/P and $\uparrow P$)
- $N/P = (H/h)^d$ large in practice (e.g. $\mathcal{O}(10^4)$ for sparse direct solvers)
- In general, BDDC(ce) and BDDC(cef) require much less iterations in $3D$
- But at the expense of a more costly coarse-grid problem

Coarse dofs vs. $\kappa(M^{-1}S)$:	$d = 2$	$d = 3$
Continuity on corners	$\left[1 + d^{-1}\log^2\left(\frac{N}{P}\right)\right]$	$\frac{N}{P}\left[1 + d^{-1}\log^2\left(\frac{N}{P}\right)\right]$
Continuity of mean value on edges too	$\left[1 + d^{-1}\log^2\left(\frac{N}{P}\right)\right]$	$\left[1 + d^{-1}\log^2\left(\frac{N}{P}\right)\right]$
Continuity of mean value on faces too	-	$\left[1 + d^{-1}\log^2\left(\frac{N}{P}\right)\right]$

Balancing domain decomposition by constraints (BDDC)



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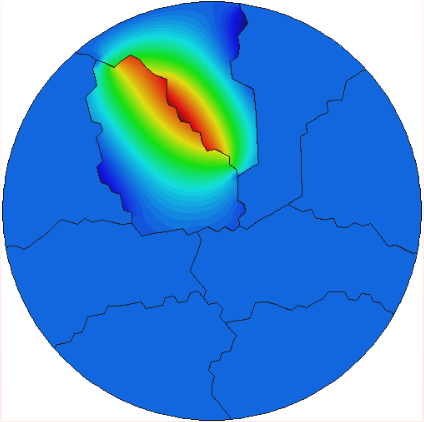

Φ_j ($\tilde{V}_C$'s basis vector)

Good features of BDDC towards extreme scales

BDDC has some salient properties that make it an excellent candidate for extreme scale solver design:

1. The method allows for a (mathematically supported) extremely aggressive coarsening ($10^5 - 10^6$ size reduction between fine/coarse level)
2. The coarse matrix has a similar sparsity as the original matrix
3. Coarse and local components can be computed in a parallel (additive) way
4. Local (constrained) Neumann and coarse solvers can be solved in an inexact way (AMG-cycle instead of sparse direct solvers)
5. A multilevel extension of the method is possible (for extreme core counts)

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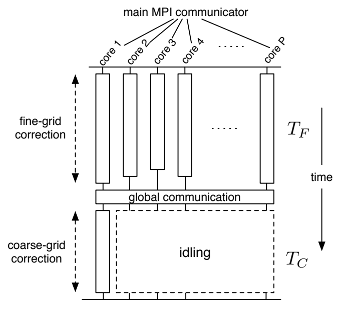
- (1)-(2) always exploited in BDDC implementations
- Let us see how to exploit (3), in order to reduce synchronization and boost scalability (overlapped implementation)

Outline

Typical parallel implementation
(e.g., PETSc. FreeFEM, BDDCML)



- All MPI tasks have f-g duties and one/several have also c-g duties
- Computation of f-g and c-g correction is serialized (but they are independent!)
- T_C grows as $O(P^2)$ and mem as $\mathcal{O}(P^{\frac{4}{3}})$
  $\rightarrow$ becomes a bottleneck with P
  $\rightarrow$ mem per core rapidly exceeded
- Parallel coarse solvers / multilevel extensions reduce this effect

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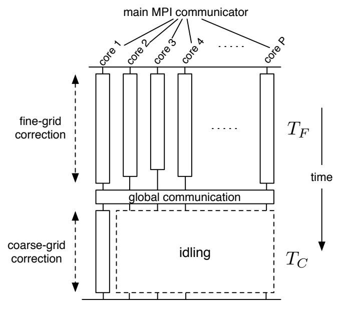

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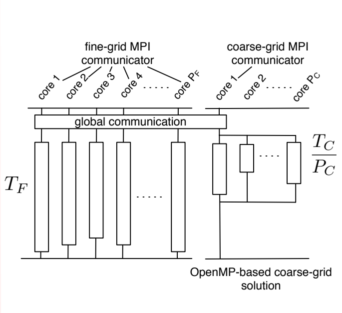
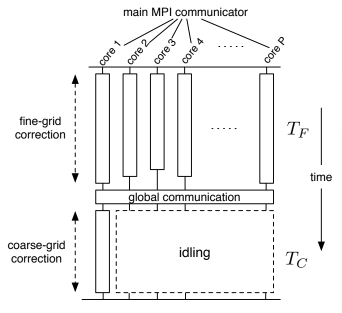
Highly-scalable parallel implementation
Overlapping of fine-grid/coarse-grid duties

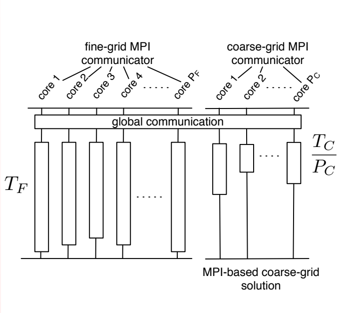


- MPI tasks have either f-g duties or c-g duties (but not both)
- Computation of f-g and c-g correction can be overlapped in time (asynchronous)
- Full node(s) resources (memory and cores) can be devoted to coarse-grid duties
- MPI-based or OpenMP-based (this work) solutions are possible for c-g correction

Typical parallel implementation (e.g., PETSc)



Highly-scalable parallel implementation Overlapping of fine-grid/coarse-grid duties



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BDDC-PCG basic building blocks

Solve $Ax = b$ via BDDC-PCG

Schur complement set-up (S)
Precond set-up (M_{BDDC})
$g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$
call PCG(S,M_{BDDC},g,x_Γ)
$x_I := A_{II}^{-1}(b_I - A_{I\Gamma} x_\Gamma)$

PCG

$r_0 := g - S x_\Gamma$
$z_0 := M_{\mathrm{BDDC}}^{-1} r_0$
$p_0 := z_0$
for $j = 0, \ldots,$ till CONV **do**
  $s_{j+1} = S p_j$
  ...
  $z_{j+1} := M_{\mathrm{BDDC}}^{-1} r_{j+1}$
  ...
end for

Fine-grid tasks	Coarse-grid task

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization($G_{A_{II}^{(i)}}$)

Schur set-up (numeric)

Numerical factorization($A_{II}^{(i)}$)

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization($G_{A_F^{(i)}}$)
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization($A_F^{(i)}$)
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C :=$ assemble($A_C^{(i)}$)
Numerical factorization(A_C)

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization($G_{A_{II}^{(i)}}$)

Schur set-up (numeric)

Numerical factorization($A_{II}^{(i)}$)

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization($G_{A_F^{(i)}}$)
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization($A_F^{(i)}$)
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C :=$ assemble($A_C^{(i)}$)
Numerical factorization(A_C)

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization($G_{A_{II}^{(i)}}$)

Schur set-up (numeric)

Numerical factorization($A_{II}^{(i)}$)

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization($G_{A_F^{(i)}}$)
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization($A_F^{(i)}$)
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C :=$ assemble($A_C^{(i)}$)
Numerical factorization(A_C)

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization($G_{A_{II}^{(i)}}$)

Schur set-up (numeric)

Numerical factorization($A_{II}^{(i)}$)

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization($G_{A_F^{(i)}}$)
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization($A_F^{(i)}$)
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C :=$ assemble($A_C^{(i)}$)
Numerical factorization(A_C)

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact$(G_{A_F^{(i)}})$ $\mathcal{O}(n_i^{\frac{4}{3}})$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization$(G_{A_{II}^{(i)}})$

Schur set-up (numeric)

Numerical factorization$(A_{II}^{(i)})$

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization$(G_{A_F^{(i)}})$
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization$(A_F^{(i)})$
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C := \text{assemble}(A_C^{(i)})$
Numerical factorization(A_C)

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact($G_{A_F^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Symb fact($G_{A_{II}^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization($G_{A_{II}^{(i)}}$)

Schur set-up (numeric)

Numerical factorization($A_{II}^{(i)}$)

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization($G_{A_F^{(i)}}$)
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization($A_F^{(i)}$)
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C := \text{assemble}(A_C^{(i)})$
Numerical factorization(A_C)

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact$(G_{A_F^{(i)}})$ $\mathcal{O}(n_i^{\frac{4}{3}})$ Symb fact$(G_{A_{II}^{(i)}})$ $\mathcal{O}(n_i^{\frac{4}{3}})$ Num fact$(A_F^{(i)})$ $\mathcal{O}(n_i^2)$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization$(G_{A_{II}^{(i)}})$

Schur set-up (numeric)

Numerical factorization$(A_{II}^{(i)})$

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization$(G_{A_F^{(i)}})$
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization$(A_F^{(i)})$
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C :=$ assemble$(A_C^{(i)})$
Numerical factorization(A_C)

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact($G_{A_F^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Symb fact($G_{A_{II}^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Num fact($A_F^{(i)}$) $\mathcal{O}(n_i^2)$ Compute Φ_i $\mathcal{O}(n_i^{\frac{4}{3}})$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization($G_{A_{II}^{(i)}}$)

Schur set-up (numeric)

Numerical factorization($A_{II}^{(i)}$)

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization($G_{A_F^{(i)}}$)
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization($A_F^{(i)}$)
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C :=$ assemble($A_C^{(i)}$)
Numerical factorization(A_C)

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact($G_{A_F^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Symb fact($G_{A_{II}^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Num fact($A_F^{(i)}$) $\mathcal{O}(n_i^2)$ Compute Φ_i $\mathcal{O}(n_i^{\frac{4}{3}})$ $A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization($G_{A_{II}^{(i)}}$)

Schur set-up (numeric)

Numerical factorization($A_{II}^{(i)}$)

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization($G_{A_F^{(i)}}$)
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization($A_F^{(i)}$)
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C :=$ assemble($A_C^{(i)}$)
Numerical factorization(A_C)

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact($G_{A_F^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Symb fact($G_{A_{II}^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Num fact($A_F^{(i)}$) $\mathcal{O}(n_i^2)$ Compute Φ_i $\mathcal{O}(n_i^{\frac{4}{3}})$ $A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$
Gather $A_C^{(i)}$ GC	

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization($G_{A_{II}^{(i)}}$)

Schur set-up (numeric)

Numerical factorization($A_{II}^{(i)}$)

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization($G_{A_F^{(i)}}$)
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization($A_F^{(i)}$)
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C :=$ assemble($A_C^{(i)}$)
Numerical factorization(A_C)

Fine-grid tasks		Coarse-grid task	
Identify local coarse DoFs			
Construct G_{A_C} GC			
Symb fact($G_{A_F^{(i)}}$)	$\mathcal{O}(n_i^{\frac{4}{3}})$	Symb fact(G_{A_C})	$\mathcal{O}(P^{\frac{4}{3}})$
Symb fact($G_{A_{II}^{(i)}}$)	$\mathcal{O}(n_i^{\frac{4}{3}})$		
Num fact($A_F^{(i)}$)	$\mathcal{O}(n_i^2)$		
Compute Φ_i	$\mathcal{O}(n_i^{\frac{4}{3}})$		
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$			
Gather $A_C^{(i)}$ GC			
		$A_C :=$ assble($A_C^{(i)}$)	

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization($G_{A_{II}^{(i)}}$)

Schur set-up (numeric)

Numerical factorization($A_{II}^{(i)}$)

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization($G_{A_F^{(i)}}$)
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization($A_F^{(i)}$)
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C :=$ assemble($A_C^{(i)}$)
Numerical factorization(A_C)

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact($G_{A_F^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$
Symb fact($G_{A_{II}^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$	
Num fact($A_F^{(i)}$) $\mathcal{O}(n_i^2)$	
Compute Φ_i $\mathcal{O}(n_i^{\frac{4}{3}})$	
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$	
Gather $A_C^{(i)}$ GC	
	$A_C :=$ assble($A_C^{(i)}$)
	Num fact(A_C) $\mathcal{O}(P^2)$

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization($G_{A_{II}^{(i)}}$)

Schur set-up (numeric)

Numerical factorization($A_{II}^{(i)}$)

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization($G_{A_F^{(i)}}$)
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization($A_F^{(i)}$)
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C :=$ assemble($A_C^{(i)}$)
Numerical factorization(A_C)

Fine-grid tasks		Coarse-grid task	
Identify local coarse DoFs			
Construct G_{A_C} GC			
Symb fact($G_{A_F^{(i)}}$)	$\mathcal{O}(n_i^{\frac{4}{3}})$	Symb fact(G_{A_C})	$\mathcal{O}(P^{\frac{4}{3}})$
Symb fact($G_{A_{II}^{(i)}}$)	$\mathcal{O}(n_i^{\frac{4}{3}})$		
Num fact($A_F^{(i)}$)	$\mathcal{O}(n_i^2)$		
Compute Φ_i	$\mathcal{O}(n_i^{\frac{4}{3}})$		
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$			
Gather $A_C^{(i)}$ GC			
Num fact($A_{II}^{(i)}$)	$\mathcal{O}(n_i^2)$	$A_C :=$ assble($A_C^{(i)}$)	
		Num fact(A_C)	$\mathcal{O}(P^2)$

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

Schur set-up (symbolic)

Symbolic factorization($G_{A_{II}^{(i)}}$)

Schur set-up (numeric)

Numerical factorization($A_{II}^{(i)}$)

BDDC set-up (symbolic)

Identify local coarse DoFs
Symbolic factorization($G_{A_F^{(i)}}$)
Construct G_{A_C}
Symbolic factorization (G_{A_C})

BDDC set-up (numeric)

Numerical factorization($A_F^{(i)}$)
Compute Φ_i
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$
Gather $A_C^{(i)}$
$A_C :=$ assemble($A_C^{(i)}$)
Numerical factorization(A_C)

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact($G_{A_F^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$
Symb fact($G_{A_{II}^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$	
Num fact($A_F^{(i)}$) $\mathcal{O}(n_i^2)$	
Compute Φ_i $\mathcal{O}(n_i^{\frac{4}{3}})$	
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$	
Gather $A_C^{(i)}$ GC	
Num fact($A_{II}^{(i)}$) $\mathcal{O}(n_i^2)$	$A_C :=$ assble($A_C^{(i)}$)
$g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$ $\mathcal{O}(n_i^{\frac{4}{3}})$	Num fact(A_C) $\mathcal{O}(P^2)$

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

PCG

$g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$
$r_0 := g - S x_\Gamma$
$z_0 := M_{\mathrm{BDDC}}^{-1} r_0$
$p_0 := z_0$
for $j = 0, \ldots,$ till CONV **do**
$s_{j+1} = S p_j$
$\ldots$
$z_{j+1} := M_{\mathrm{BDDC}}^{-1} r_{j+1}$
$\ldots$
end for

BDDC application

$r^{(i)} := I_i^t r$
Compute $s_F^{(i)}$
$r_C^{(i)} := \Phi_i^t r^{(i)}$
Gather $r_C^{(i)}$
$r_C :=$ assemble($r_C^{(i)}$)
Solve $A_C z_C = r_C$
Scatter z_C into $z_C^{(i)}$
$s_C^{(i)} := \Phi_i z_C^{(i)}$
$z^{(i)} := I_i(s_F^{(i)} + s_C^{(i)})$

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact($G_{A_F^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Symb fact($G_{A_{II}^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Num fact($A_F^{(i)}$) $\mathcal{O}(n_i^2)$ Compute Φ_i $\mathcal{O}(n_i^{\frac{4}{3}})$ $A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$
Gather $A_C^{(i)}$ GC	
Num fact($A_{II}^{(i)}$) $\mathcal{O}(n_i^2)$ $g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$ $\mathcal{O}(n_i^{\frac{4}{3}})$ $r_0 := g - Sx_\Gamma$ $\mathcal{O}(n_i^{\frac{4}{3}})$	$A_C :=$ assble($A_C^{(i)}$) Num fact(A_C) $\mathcal{O}(P^2)$

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

PCG

$g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$
$r_0 := g - Sx_\Gamma$
$z_0 := M_{\mathrm{BDDC}}^{-1} r_0$
$p_0 := z_0$
for $j = 0, \ldots,$ till CONV **do**
  $s_{j+1} = Sp_j$
  ...
  $z_{j+1} := M_{\mathrm{BDDC}}^{-1} r_{j+1}$
  ...
end for

BDDC application

$r^{(i)} := I_i^t r$
Compute $s_F^{(i)}$
$r_C^{(i)} := \Phi_i^t r^{(i)}$
Gather $r_C^{(i)}$
$r_C :=$ assemble($r_C^{(i)}$)
Solve $A_C z_C = r_C$
Scatter z_C into $z_C^{(i)}$
$s_C^{(i)} := \Phi_i z_C^{(i)}$
$z^{(i)} := I_i(s_F^{(i)} + s_C^{(i)})$

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact($G_{A_F^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$
Symb fact($G_{A_{II}^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$	
Num fact($A_F^{(i)}$) $\mathcal{O}(n_i^2)$	
Compute Φ_i $\mathcal{O}(n_i^{\frac{4}{3}})$	
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$	
Gather $A_C^{(i)}$ GC	
Num fact($A_{II}^{(i)}$) $\mathcal{O}(n_i^2)$	$A_C :=$ assble($A_C^{(i)}$)
$g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$ $\mathcal{O}(n_i^{\frac{4}{3}})$	Num fact(A_C) $\mathcal{O}(P^2)$
$r_0 := g - Sx_\Gamma$ $\mathcal{O}(n_i^{\frac{4}{3}})$	
$r^{(i)} := I_i^t r$ LC	

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

PCG

$g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$
$r_0 := g - Sx_\Gamma$
$z_0 := M_{\mathrm{BDDC}}^{-1} r_0$
$p_0 := z_0$
for $j = 0, \ldots,$ till CONV **do**
$s_{j+1} = Sp_j$
...
$z_{j+1} := M_{\mathrm{BDDC}}^{-1} r_{j+1}$
...
end for

BDDC application

$r^{(i)} := I_i^t r$
Compute $s_F^{(i)}$
$r_C^{(i)} := \Phi_i^t r^{(i)}$
Gather $r_C^{(i)}$
$r_C :=$ assemble($r_C^{(i)}$)
Solve $A_C z_C = r_C$
Scatter z_C into $z_C^{(i)}$
$s_C^{(i)} := \Phi_i z_C^{(i)}$
$z^{(i)} := I_i(s_F^{(i)} + s_C^{(i)})$

Fine-grid tasks		Coarse-grid task	
Identify local coarse DoFs			
Construct G_{A_C} GC			
Symb fact($G_{A_F^{(i)}}$)	$\mathcal{O}(n_i^{\frac{4}{3}})$	Symb fact(G_{A_C})	$\mathcal{O}(P^{\frac{4}{3}})$
Symb fact($G_{A_{II}^{(i)}}$)	$\mathcal{O}(n_i^{\frac{4}{3}})$		
Num fact($A_F^{(i)}$)	$\mathcal{O}(n_i^2)$		
Compute Φ_i	$\mathcal{O}(n_i^{\frac{4}{3}})$		
$A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$			
Gather $A_C^{(i)}$ GC			
Num fact($A_{II}^{(i)}$)	$\mathcal{O}(n_i^2)$	$A_C :=$ assble($A_C^{(i)}$)	
$g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$	$\mathcal{O}(n_i^{\frac{4}{3}})$	Num fact(A_C)	$\mathcal{O}(P^2)$
$r_0 := g - S x_\Gamma$	$\mathcal{O}(n_i^{\frac{4}{3}})$		
$r^{(i)} := I_i^t r$	LC		
$r_C^{(i)} := \Phi_i^t r^{(i)}$			

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

PCG

$g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$
$r_0 := g - S x_\Gamma$
$z_0 := M_{\mathrm{BDDC}}^{-1} r_0$
$p_0 := z_0$
for $j = 0, \ldots,$ till CONV **do**
  $s_{j+1} = S p_j$
  ...
  $z_{j+1} := M_{\mathrm{BDDC}}^{-1} r_{j+1}$
  ...
end for

BDDC application

$r^{(i)} := I_i^t r$
Compute $s_F^{(i)}$
$r_C^{(i)} := \Phi_i^t r^{(i)}$
Gather $r_C^{(i)}$
$r_C :=$ assemble($r_C^{(i)}$)
Solve $A_C z_C = r_C$
Scatter z_C into $z_C^{(i)}$
$s_C^{(i)} := \Phi_i z_C^{(i)}$
$z^{(i)} := I_i (s_F^{(i)} + s_C^{(i)})$

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact($G_{A_F^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Symb fact($G_{A_{II}^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Num fact($A_F^{(i)}$) $\mathcal{O}(n_i^2)$ Compute Φ_i $\mathcal{O}(n_i^{\frac{4}{3}})$ $A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$
Gather $A_C^{(i)}$ GC	
Num fact($A_{II}^{(i)}$) $\mathcal{O}(n_i^2)$ $g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$ $\mathcal{O}(n_i^{\frac{4}{3}})$ $r_0 := g - S x_\Gamma$ $\mathcal{O}(n_i^{\frac{4}{3}})$ $r^{(i)} := I_i^t r$ LC $r_C^{(i)} := \Phi_i^t r^{(i)}$	$A_C :=$ assble($A_C^{(i)}$) Num fact(A_C) $\mathcal{O}(P^2)$
Gather $r_C^{(i)}$ GC	

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

PCG

$g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$
$r_0 := g - S x_\Gamma$
$z_0 := M_{\mathrm{BDDC}}^{-1} r_0$
$p_0 := z_0$
for $j = 0, \ldots,$ till CONV **do**
  $s_{j+1} = S p_j$
  ...
  $z_{j+1} := M_{\mathrm{BDDC}}^{-1} r_{j+1}$
  ...
end for

BDDC application

$r^{(i)} := I_i^t r$
Compute $s_F^{(i)}$
$r_C^{(i)} := \Phi_i^t r^{(i)}$
Gather $r_C^{(i)}$
$r_C :=$ assemble($r_C^{(i)}$)
Solve $A_C z_C = r_C$
Scatter z_C into $z_C^{(i)}$
$s_C^{(i)} := \Phi_i z_C^{(i)}$
$z^{(i)} := I_i(s_F^{(i)} + s_C^{(i)})$

Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact($G_{A_F^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Symb fact($G_{A_{II}^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Num fact($A_F^{(i)}$) $\mathcal{O}(n_i^2)$ Compute Φ_i $\mathcal{O}(n_i^{\frac{4}{3}})$ $A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$
Gather $A_C^{(i)}$ GC	
Num fact($A_{II}^{(i)}$) $\mathcal{O}(n_i^2)$ $g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$ $\mathcal{O}(n_i^{\frac{4}{3}})$ $r_0 := g - S x_\Gamma$ $\mathcal{O}(n_i^{\frac{4}{3}})$ $r^{(i)} := I_i^t r$ LC $r_C^{(i)} := \Phi_i^t r^{(i)}$	$A_C :=$ assble($A_C^{(i)}$) Num fact(A_C) $\mathcal{O}(P^2)$
Gather $r_C^{(i)}$ GC	
	$r_C :=$ assble($r_C^{(i)}$)

LC: local communication (nearest neighbours)
GC: global communication (gather or scatter)

PCG

$g := b_\Gamma - A_{\Gamma I} A_{II}^{-1} b_I$
$r_0 := g - S x_\Gamma$
$z_0 := M_{\mathrm{BDDC}}^{-1} r_0$
$p_0 := z_0$
for $j = 0, \ldots,$ till CONV **do**
  $s_{j+1} = S p_j$
  ...
  $z_{j+1} := M_{\mathrm{BDDC}}^{-1} r_{j+1}$
  ...
end for

BDDC application

$r^{(i)} := I_i^t r$
Compute $s_F^{(i)}$
$r_C^{(i)} := \Phi_i^t r^{(i)}$
Gather $r_C^{(i)}$
$r_C :=$ assemble($r_C^{(i)}$)
Solve $A_C z_C = r_C$
Scatter z_C into $z_C^{(i)}$
$s_C^{(i)} := \Phi_i z_C^{(i)}$
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Fine-grid tasks	Coarse-grid task
Identify local coarse DoFs	
Construct G_{A_C} GC	
Symb fact($G_{A_F^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Symb fact($G_{A_{II}^{(i)}}$) $\mathcal{O}(n_i^{\frac{4}{3}})$ Num fact($A_F^{(i)}$) $\mathcal{O}(n_i^2)$ Compute Φ_i $\mathcal{O}(n_i^{\frac{4}{3}})$ $A_C^{(i)} := \Phi_i^t A^{(i)} \Phi_i$	Symb fact(G_{A_C}) $\mathcal{O}(P^{\frac{4}{3}})$
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Outline

1. BDDC preconditioner
2. Overlapped BDDC implementation
3. Scalability analysis (overlapped)
4. Inexact BDDC
5. Scalability analysis (overlapped/inexact)
6. Conclusions and future work

FEMPAR (in-house developed HPC software, free software GNU-GPL):

Finite Element Multiphysics PARallel software

- Massively parallel sw for the FE simulation of Multiphysics problems governed by PDEs
- Scalable preconditioning of fully coupled and implicit system via block preconditioning techniques (physics-based preconditioning)
- Scalable preconditioning for one-physics (elliptic) PDEs relies on BDDC, BNN
  $\rightarrow$ hybrid MPI/OpenMP implementation
- Relies on highly-efficient vendor implementations of the dense/sparse BLAS (Intel MKL, IBM ESSL, etc.)
- Provides interfaces to external multi-threaded sparse direct solvers (PARDISO, HSL_MA87, etc.) and serial AMG preconditioners (HSL_MI20)

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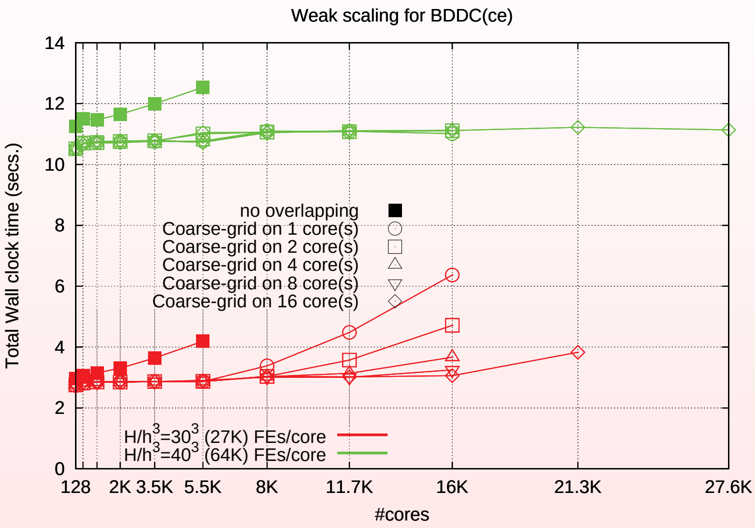
Weak scalability for 3D Poisson

Target machine: HELIOS@IFERC-CSC
4,410 bullx B510 compute blades (2 Intel Xeon E5-2680 8-core CPUs; 64GB)

- Target problem: $-\Delta u = f$ on $\overline{\Omega} = [0,2] \times [0,1] \times [0,1]$
- Uniform global mesh of hexahedral Q1 finite elements
- Uniform partition into rectangular grids of $4m \times 2m \times 2m$ cubic local meshes
- $m = 2^3, 3^3 \ldots, 12^3$ blades $(8, 432, \ldots, 27648$ cores) devoted to fine-grid duties
- Entire 16-core blade devoted to coarse-grid duties (multi-threaded PARDISO)
- Direct solution (PARDISO) of Dirichlet, Neumann, and coarse-grid corrections
- Gradually larger fixed local problem sizes $\frac{H}{h} = 30^3, 40^3$ FEs/core

BDDC(corners+edges) :: Poisson problem

Weak scaling for BDDC(corners+edges) :: Poisson problem



Outline

Good features of BDDC towards extreme scales

BDDC has some salient properties that make it an excellent candidate for extreme scale solver design:

1. The method allows for a (mathematically supported) extremely aggressive coarsening
2. The coarse matrix has a similar sparsity as the original matrix
3. Coarse and local components can be computed in a parallel (additive) way
4. Local (constrained) Neumann and coarse solvers can be solved in an inexact way
5. A multilevel extension of the method is possible (for extreme core counts)

- (1)-(2)-(3) always exploited in our overlapped BDDC implementations
- Let us see how to exploit (4), in order to boost scalability further and reduce memory requirements (overlapped/inexact implementation)

- The exact (using direct solvers) BDDC preconditioner leads to the most effective preconditioner
- However, also to the most computationally and memory demanding one
- In order to reduce these demands, one may solve only approximately some (or even all) of the internal problems using, e.g., AMG-based solvers
- Numerical analysis says that inexact BDDC preconditioners are also algoritmically scalable [Dohrmann, 2007]
- Benefit has to be viewed in light of future parallel architectures: the most scalable architectures (e.g., IBM BG) will have more limited memory per core
- Further, the coarse solver time increases as P instead of P^2, much less degradation for high core counts

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4 different solvers in BDDC:

1. Dirichlet problem: approximate $(A_{II}^{(i)})^{-1}$ in $\mathcal{E} = \begin{bmatrix} 0 & -A_{II}^{-1}A_{I\Gamma} \\ 0 & I_\Gamma \end{bmatrix}$
2. Local Neumann problem: approximate $(A_c^{(i)})^{-1}$, where $A_c^{(i)}$ is the (sub-assembled) local matrix $A^{(i)}$ after eliminating the coarse corner rows/columns
3. Coarse problem: approximate $(A_C)^{-1}$
4. Computation of Φ: approximate $(A_c^{(i)})^{-1}$

From numerical analysis [Dohrmann, 2007]:

- (2)-(3) can be replaced by optimal preconditioners, e.g., AMG-cycle
- (1)-(4) more delicate, additional *null space* preservation required (not true in general)

Key question to be experimentally assessed

Sensivity of the algorithm to every inexact solver?

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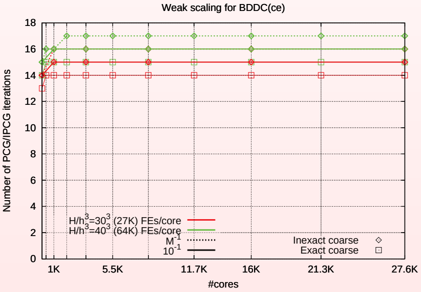
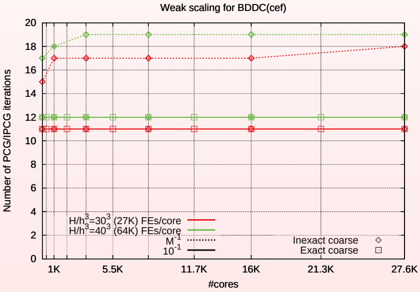
Outline

The effect of approximately solving the internal problems

Coarse-grid problem

- The problem that can harm scalability (couples ALL subdomain)
- Fortunately, it can be highly perturbed without impact in the scalability (AMG-cycle suffices)

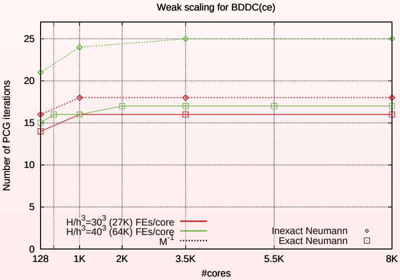


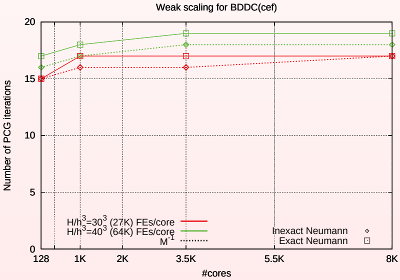


The effect of approximately solving the internal problems

Neumann Problem

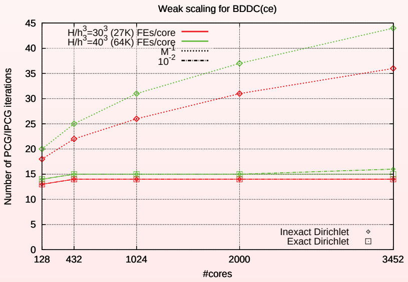
Neumann problem can be highly perturbed without impact in the scalability (AMG-cycle suffices)

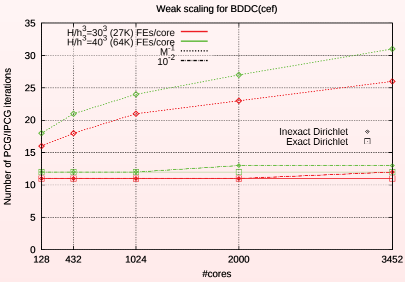




Dirichlet problem

- AMG-cycle wo/ null space preservation (deflation) not algorithmically scalable
- But with loose tolerance enough to make it scalable





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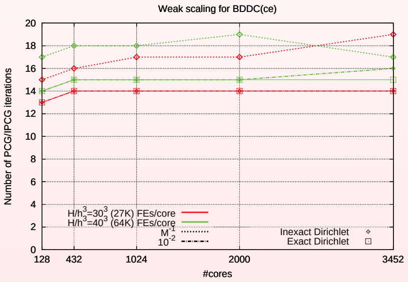
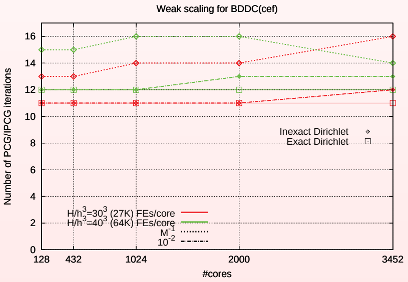
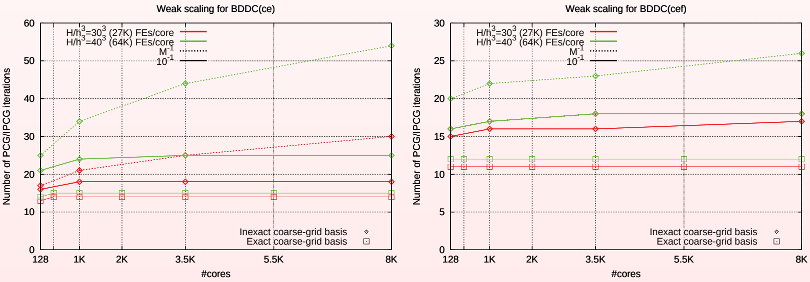
- AMG-cycle w/ null space preservation (deflation) not algorithmically scalable
- But with loose tolerance enough to make it scalable





Coarse-grid basis vectors

- AMG-cycle wo/ null space preservation not algorithmically scalable
- But with loose tolerance enough to make it scalable



A set of strategically selected inexact variants

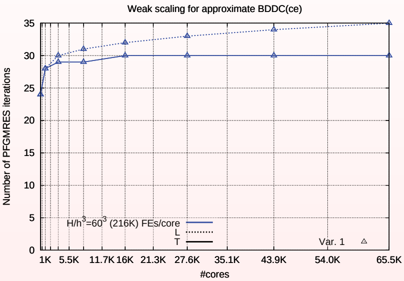
Target machine: JUQUEEN@JSC
28,672 compute nodes (16-core, 64-way threaded IBM PPC A2; 16 GB)

- Target problem: $-\Delta u = f$ on $\overline{\Omega} = [0,2] \times [0,1] \times [0,1]$
- Uniform global mesh of hexahedral Q1 finite elements
- Uniform partition into rectangular grids of $4m \times 2m \times 2m$ cubic local meshes
- $m = 2^3, 3^3 \ldots, 16^3$ nodes $(8, 432, \ldots, 65535$ cores) devoted to fine-grid duties
- Entire node devoted to coarse-grid duties (restricted to only 1 core/GB)
- Gradually larger fixed local problem sizes $\frac{H}{h} = 60^3$ FEs/core

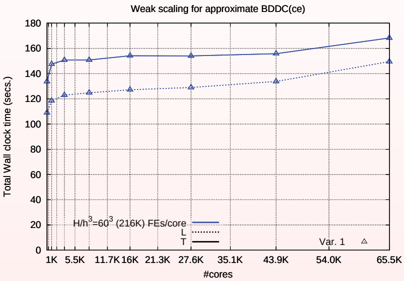
Inexact BDDC(corners+edges) :: Poisson problem

Weak scaling for inexact BDDC(corners+edges) :: Poisson problem

$$\frac{H}{h} = 60 \text{ (216K FEs/core)}$$



# of outer solver iterations



Total time (secs.)

	Outer solver	Φ	Dirichlet	Neumann	Coarse
Var. 1T	FGMRES	PCG-AMG(10^{-1})	PCG-AMG(10^{-4})	AMG(1)	AMG(1)
Var. 1L	FGMRES	PCG-AMG(10^{-1})	PCG-AMG(10^{-2})	AMG(1)	AMG(1)

Memory usage:

- Fine proc's: 538.6MB ($<$ 1GB)
- Coarse proc's (65.5K cores): 392.7MB ($<$ 1GB)

Outline

Conclusions:

- Highly scalable asynchronous implementation of BDDC
- Overlapping of fine-grid and coarse-grid computations
- OpenMP parallelization for coarse-grid problem in the exact case
- Exploitation of AMG-based solvers in the inexact case
- Weakly scalable for many ranges of interest
- Memory limitations clearly improved
- High scalability in a memory constrained environment (JUQUEEN)

Conclusions and future work

Conclusions:

- Highly scalable asynchronous implementation of BDDC
- Overlapping of fine-grid and coarse-grid computations
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Good features of BDDC towards extreme scales

BDDC has some salient properties that make it an excellent candidate for extreme scale solver design:

1. The method allows for a (mathematically supported) extremely aggressive coarsening
2. The coarse matrix has a similar sparsity as the original matrix
3. Coarse and local components can be computed in a parallel (additive) way
4. Local (constrained) Neumann and coarse solvers can be solved in an inexact way
5. A multilevel extension of the method is possible (for extreme core counts)

- (1)-(2)-(3)-(4) exploited in our inexact/overlapped BDDC implementations
- Next step: Exploit (5), for to boost scalability even further (overlapped/inexact/multilevel implementation)

- S. Badia, A. F. Martín and J. Principe. Enhanced balancing Neumann-Neumann preconditioning in computational fluid and solid mechanics. *International Journal for Numerical Methods in Engineering*. Vol. 96(4), pp. 203-230, 2013.
- S. Badia, A. F. Martín and J. Principe. Implementation and scalability analysis of balancing domain decomposition methods. *Archives of Computational Methods in Engineering*. Vol. 20(3), pp. 239-262, 2013.
- S. Badia, A. F. Martín and J. Principe. A highly scalable parallel implementation of balancing domain decomposition by constraints. *SIAM Journal on Scientific Computing*. Vol. 36(2), pp. C190-C218, 2014.
- S. Badia, A. F. Martín and J. Principe. On the scalability of inexact balancing domain decomposition by constraints with overlapped coarse/fine corrections. In preparation, 2014.

- Preprints available at `http://badia.rmee.upc.edu/sbadia_ar.html`
- COMFUS team: `https://web.cimne.upc.edu/groups/comfus/`

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