Sparse Days
June 5–6, 2014
CERFACS, Toulouse

Preconditioning for various Cahn–Hilliard systems

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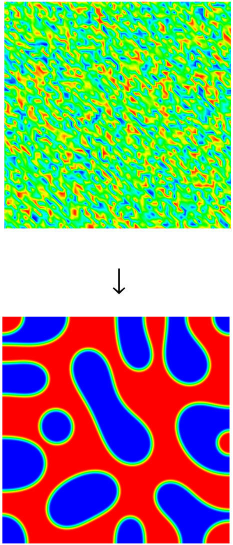
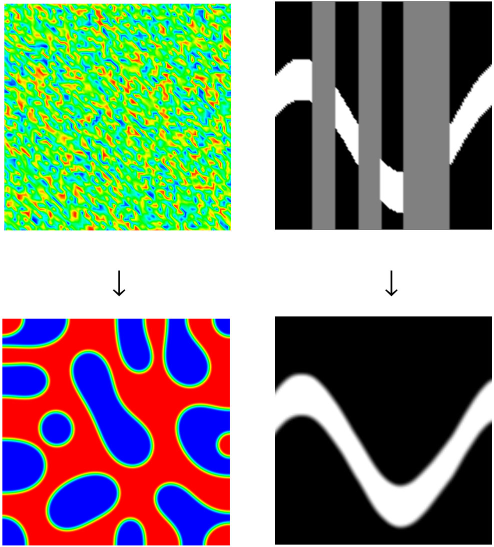
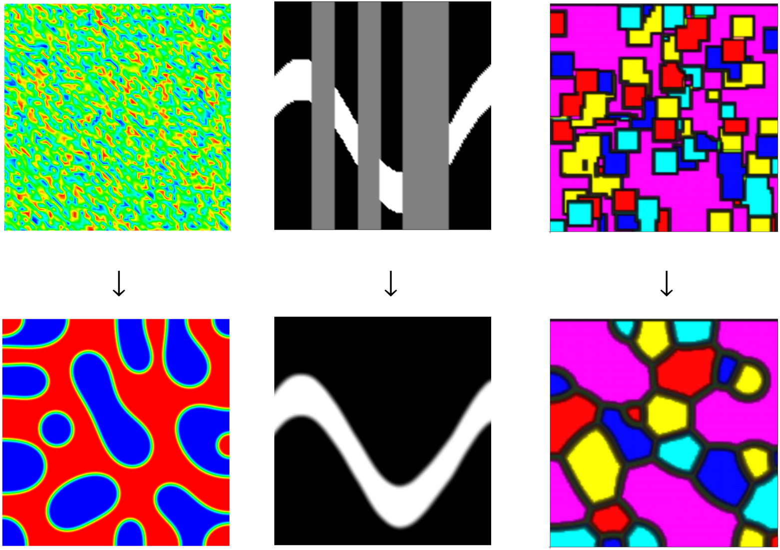
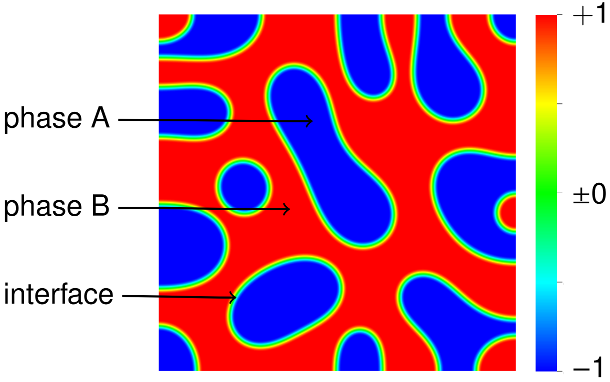






Two Phases

Two-Phase Structure



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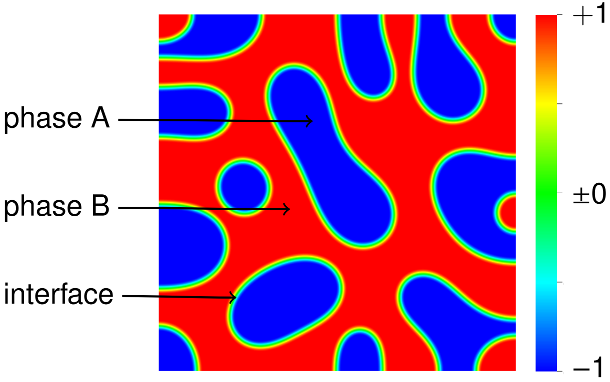

- $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$

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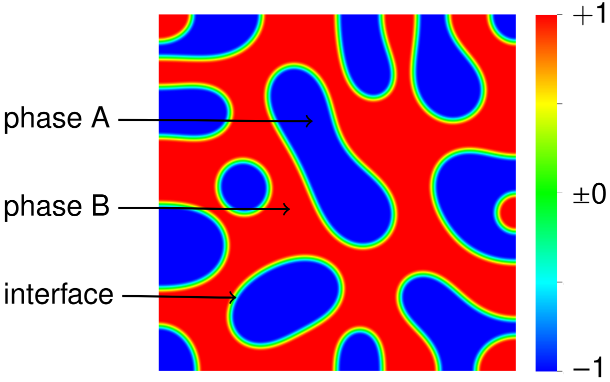

- $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$
- $u \colon \Omega \times (0, T) \to \mathbb{R}$ concentration
- $u \in [-1, 1]$

Two Phases

Energy Functional

$$\mathcal{E}(u) = \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} \psi(u) \, \mathrm{d}\mathbf{x}$$

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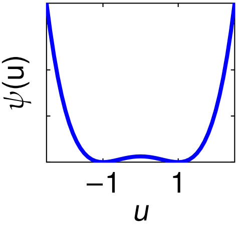

Smooth potential

$$\psi(u) = \frac{1}{4}(u^2 - 1)^2$$

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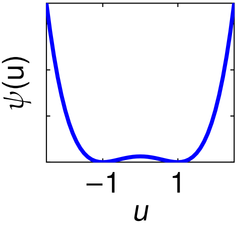

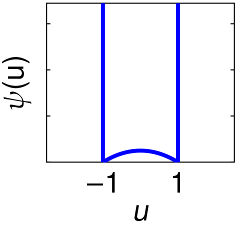

Smooth potential

$$\psi(u) = \frac{1}{4}(u^2-1)^2$$

Nonsmooth potential

$$\psi(u) = \begin{cases} \frac{1}{2}(1-u^2) & \text{if } |u| \leq 1, \\ \infty & \text{otherwise} \end{cases}$$

$$= \psi_0(u) + I_{[-1,1]}(u)$$

Two Phases

Moreau–Yosida Regularization

$$\mathcal{E}(u) = \int_\Omega \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} \left(\psi_0(u) + I_{[-1,1]}(u) \right) \mathrm{d}\mathbf{x}$$

Two Phases

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$$\downarrow$$

$$\vartheta_\nu(u_\nu) := \frac{1}{2\nu}\left(|\max(0, u_\nu - 1)|^2 + |\min(0, u_\nu + 1)|^2\right)$$

Two Phases

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$$\downarrow$$

$$\mathcal{E}_\nu(u_\nu) = \int_\Omega \frac{\varepsilon}{2} |\nabla u_\nu|^2 + \frac{1}{\varepsilon} \psi_0(u_\nu) + \vartheta_\nu(u_\nu) \, \mathrm{d}\mathbf{x}$$

Two Phases

Regularized Subproblem

$$\partial_t u(t) = -\mathrm{grad}_{H^{-1}} \mathcal{E}(u(t))$$

Two Phases

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System of two coupled PDEs

$$\begin{aligned} \partial_t u_\nu &= \Delta w_\nu \\ w_\nu &= -\varepsilon \Delta u_\nu + \frac{1}{\varepsilon}\psi_0'(u_\nu) + \theta_\nu(u_\nu) \\ \nabla u_\nu \cdot \mathbf{n} = \nabla w_\nu \cdot \mathbf{n} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

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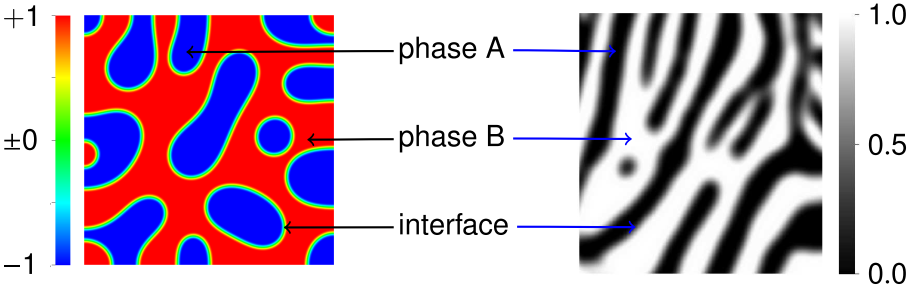
$$\theta_\nu(u_\nu) := \frac{1}{\nu}\left(\max(0, u_\nu - 1) + \min(0, u_\nu + 1)\right)$$

Black-White Inpainting

Idea



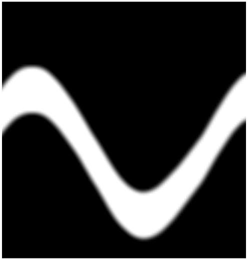
Original Cahn–Hilliard.

Inpainting version.

Black-White Inpainting

Idea

Original image f with inpainting domain D.



Inpainted image.

Black-White Inpainting

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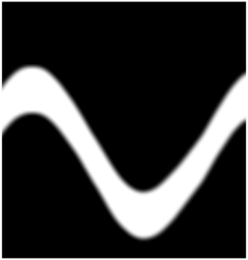
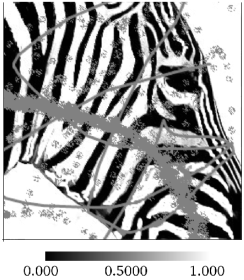


Inpainted image.

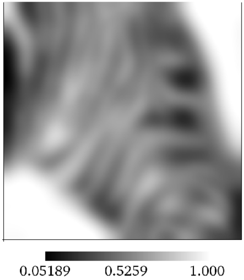
$$\omega(x) = \begin{cases} 0 & \text{if } x \in D, \\ \omega_0 & \text{if } x \in \Omega \setminus D \end{cases}$$

Black-White Inpainting

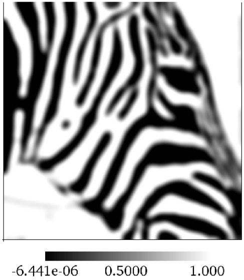
Damaged Zebra Image



$n = 0$



$n = 57$

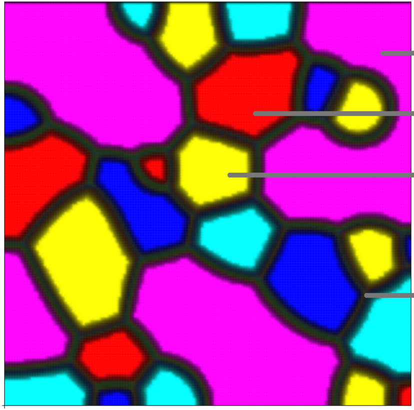


$n = 758$

Multi Phases

Multi-Phase Structure

$\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$



phase A	$u_1 : \Omega \times (0, T) \to \mathbb{R}$
phase B	$u_2 : \Omega \times (0, T) \to \mathbb{R}$
phase C	$u_3 : \Omega \times (0, T) \to \mathbb{R}$
...	...
interface	mixture of phases

Multi Phases

Modeling of Multi-Phase Systems

- number of phases N

Multi Phases

Modeling of Multi-Phase Systems

- number of phases N
- vector-valued order parameter

$$\mathbf{u} = (u_1, \ldots, u_N)^T : \Omega \times (0, T) \to \mathbb{R}^N$$

Multi Phases

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- $u_i \in \begin{cases} \{0\} & \text{if phase } i \text{ is absent,} \\ (0,1) & \text{if phase } i \text{ is present,} \\ \{1\} & \text{if only phase } i \text{ is present} \end{cases}$

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- admissible states belong to

$$\mathcal{G}^N := \left\{ \mathbf{v} \in \mathbb{R}^N \,\middle|\, \sum_{i=1}^{N} v_i = 1,\ v_i \geq 0 \ \ i = 1, \ldots, N \right\}$$

Multi Phases

Regularized Subproblems

Systems of two coupled PDEs

$$\partial_t u_{\nu,i} = (L\Delta \mathbf{w}_\nu)_i$$

$$w_{\nu,i} = -\varepsilon^2 \Delta u_{\nu,i} + \frac{\partial \psi_0(\mathbf{u}_\nu)}{\partial u_{\nu,i}} + \theta_\nu(u_{\nu,i}) - \frac{1}{N}\sum_{j=1}^{N}\left(\theta_\nu(u_{\nu,j}) + \frac{\partial \psi_0(\mathbf{u}_\nu)}{\partial u_{\nu,j}}\right)$$

$$\nabla u_{\nu,i} \cdot \mathbf{n} = (L\nabla \mathbf{w}_\nu)_i \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega$$

$$i = 1, \dots, N$$

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$$i = 1, \ldots, N$$

Mobility: $L = I - \frac{1}{N}\mathbf{1}\mathbf{1}^T \quad (L = I)$

Linear Systems and Preconditioning

Final Steps to the Linear Systems

- Time discretization:

Linear Systems and Preconditioning

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Linear Systems and Preconditioning

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Linear Systems and Preconditioning

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- Nonlinear systems: (Semismooth) Newton method
- Space discretization: Finite element method

Linear Systems and Preconditioning

Optimal Preconditioners

$$\mathcal{K} = \begin{pmatrix} A & -B \\ C & D \end{pmatrix}$$

- A nonsingular
- Schur complement $S = D + CA^{-1}B$

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$$\mathcal{P} = \begin{pmatrix} A & 0 \\ C & -\hat{S} \end{pmatrix}$$

- $\hat{S} = S$:
  $\Lambda(\mathcal{P}^{-1}\mathcal{K}) = \{1, -1\}$

[Murphy/Golub/Wathen '00]

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[Murphy/Golub/Wathen '00]

Goal

Good and easy to compute approximation $\hat{\mathcal{S}}$ of the Schur complement S.

Linear Systems and Preconditioning

Linear Cahn–Hilliard Systems

$$\mathcal{K} = \begin{pmatrix} I_N \otimes M & -\mathcal{B} \\ \alpha L_N \otimes K & I_N \otimes (\beta M + \gamma K) \end{pmatrix} \qquad \alpha, \beta > 0, \gamma \geq 0$$

Linear Systems and Preconditioning

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Two Phases

- $N = 1$
- $\mathcal{B}$ indefinite
- $\gamma = 0$

Inpainting

- $N = 1$
- $\mathcal{B}$ symmetric positive (semi-)definite

Linear Systems and Preconditioning

Schur Complement Approximation – Scalar Problems

$$\mathcal{K} = \begin{pmatrix} M & -\mathcal{B} \\ \alpha K & \beta M + \gamma K \end{pmatrix} \qquad \alpha, \beta > 0, \gamma \geq 0$$

Linear Systems and Preconditioning

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$$\mathcal{S} = \beta M + \alpha K M^{-1} \mathcal{B} + \gamma K$$

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$$\hat{\mathcal{S}} = \left(\sqrt{\beta} M + \sqrt{\alpha} K\right) M^{-1} \left(\sqrt{\beta} M + \sqrt{\alpha} \mathcal{B}\right)$$

Linear Systems and Preconditioning

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Linear Systems and Preconditioning

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Linear Systems and Preconditioning

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Linear Systems and Preconditioning

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The shift $\sqrt{\beta} M$ makes $\boxed{\left(\sqrt{\beta} M + \sqrt{\alpha} \mathcal{B}\right)}$ positive definite.

Linear Systems and Preconditioning

Linear Multi-Phase Cahn–Hilliard Systems

For e.g. $N = 3$ phases, $\mathcal{K}$ is given as

$$\mathcal{K} = \left(\begin{array}{ccc|ccc} M & 0 & 0 & -B_{11} & -B_2 & -B_3 \\ 0 & M & 0 & -B_1 & -B_{22} & -B_3 \\ 0 & 0 & M & -B_1 & -B_2 & -B_{33} \\ \hline \tau L_{1,1} K & \tau L_{1,2} K & \tau L_{1,3} K & M & 0 & 0 \\ \tau L_{1,2} K & \tau L_{2,2} K & \tau L_{2,3} K & 0 & M & 0 \\ \tau L_{1,3} K & \tau L_{2,3} K & \tau L_{3,3} K & 0 & 0 & M \end{array}\right).$$

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We need a Schur complement approximation of

$$\mathcal{S} = I \otimes M + \tau (L \otimes K)(I \otimes M)^{-1} \mathcal{B}.$$

Linear Systems and Preconditioning

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1. Smooth potential $\rightsquigarrow$ 'simple'

Linear Systems and Preconditioning

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1. Smooth potential $\rightsquigarrow$ 'simple'
2. Nonsmooth potential $\rightsquigarrow$ penalty terms in every block of $\mathcal{B}$

Linear Systems and Preconditioning

Smooth Potential – Approximation of the Block $\mathcal{B}$

$$\mathcal{B} = \begin{pmatrix} B_{11} & B_2 & B_3 \\ B_1 & B_{22} & B_3 \\ B_1 & B_2 & B_{33} \end{pmatrix} \qquad (N = 3)$$

Linear Systems and Preconditioning

Smooth Potential – Approximation of the Block $\mathcal{B}$

$$\mathcal{B} = \begin{pmatrix} B_{11} & B_2 & B_3 \\ B_1 & B_{22} & B_3 \\ B_1 & B_2 & B_{33} \end{pmatrix} \qquad (N = 3)$$

The block $\mathcal{B}$ is given for $i = 1, \dots, N$ as

$$F_i = \operatorname{diag}\left(3\left(u_i^{(k)}(\mathbf{x}_h)\right)^2 - 3u_i^{(k)}(\mathbf{x}_h) + \frac{1}{2}\right)$$

Linear Systems and Preconditioning

Smooth Potential – Approximation of the Block $\mathcal{B}$

$$\mathcal{B} = \begin{pmatrix} B_{11} & B_2 & B_3 \\ B_1 & B_{22} & B_3 \\ B_1 & B_2 & B_{33} \end{pmatrix} \qquad (N = 3)$$

The block $\mathcal{B}$ is given for $i = 1, \ldots, N$ as

$$F_i = \operatorname{diag}\left(3\left(u_i^{(k)}(\mathbf{x}_h)\right)^2 - 3u_i^{(k)}(\mathbf{x}_h) + \frac{1}{2}\right)$$

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Linear Systems and Preconditioning

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Linear Systems and Preconditioning

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$$\boxed{\mathcal{O}(M) = h^2}$$

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Linear Systems and Preconditioning

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$$B_{ii} = \varepsilon^2 K + \left(\frac{N-1}{N}\right) F_i M F_i \quad \longrightarrow \quad \boxed{\hat{B} = \varepsilon^2 K + \left(\frac{N-1}{N}\right) M}$$

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Linear Systems and Preconditioning

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Linear Systems and Preconditioning

Smooth Potential – Schur Complement Approximation

The Schur complement

$$\mathcal{S} = I \otimes M + \tau (L \otimes K)(I \otimes M)^{-1} \mathcal{B}$$

is then approximated by

Linear Systems and Preconditioning

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$$\hat{\mathcal{S}} = \left(\frac{N}{N-1} I \otimes M + \tau L \otimes K \right) (I \otimes M)^{-1} \left(\frac{N-1}{N} I \otimes M + \varepsilon^2 I \otimes K \right)$$

Linear Systems and Preconditioning

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Linear Systems and Preconditioning

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&= I \otimes M + \tau (L \otimes K)(I \otimes M)^{-1} \hat{\mathcal{B}} + \frac{\varepsilon^2 N}{N-1} I \otimes K.
\end{aligned}$$

Linear Systems and Preconditioning

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Linear Systems and Preconditioning

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&= \left(\frac{N}{N-1} I \otimes M + \tau L \otimes K \right) (I \otimes M)^{-1} \hat{\mathcal{B}} \\
&= I \otimes M + \tau (L \otimes K)(I \otimes M)^{-1} \hat{\mathcal{B}} + \frac{\varepsilon^2 N}{N-1} I \otimes K.
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Linear Systems and Preconditioning

FFT based Preconditioner [Stoll '13]

$$\left(\frac{N}{N-1} I \otimes M + \tau L \otimes K\right)\mathbf{y} = \mathbf{g} \tag{1}$$

Linear Systems and Preconditioning

FFT based Preconditioner [STOLL '13]

$$\left(\frac{N}{N-1} I \otimes M + \tau L \otimes K\right) \mathbf{y} = \mathbf{g} \qquad (1)$$

- $L = F \operatorname{diag}(\lambda_1, \ldots, \lambda_N)\, F^H$ [CHEN '87]

Linear Systems and Preconditioning

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- apply FFT to (1)

Linear Systems and Preconditioning

FFT based Preconditioner [Stoll '13]

$$(F^H \otimes I)\left(\frac{N}{N-1} I \otimes M + \tau L \otimes K\right)(F \otimes I)(F^H \otimes I)\mathbf{y} = (F^H \otimes I)\mathbf{g} \tag{1}$$

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Linear Systems and Preconditioning

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- $L = F \operatorname{diag}(\lambda_1, \ldots, \lambda_N)\ F^H$ [Chen '87]
- apply FFT to (1)
- equivalent block-diagonal system

$$\left(\frac{N}{N-1} I \otimes M + \tau \operatorname{diag}(\lambda_1, \ldots, \lambda_N) \otimes K\right)\tilde{\mathbf{y}} = \tilde{\mathbf{g}}.$$

Linear Systems and Preconditioning

Nonsmooth Potential – The Block $\mathcal{B}$

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$$G_i = \operatorname{diag}\begin{pmatrix} 1 & \text{if } u_i^{(k)}(\mathbf{x}_h) < 0, \\ 0 & \text{otherwise} \end{pmatrix}$$

Linear Systems and Preconditioning

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Linear Systems and Preconditioning

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$\nu \ll 1 \Rightarrow$ Cannot neglect this term!

Linear Systems and Preconditioning

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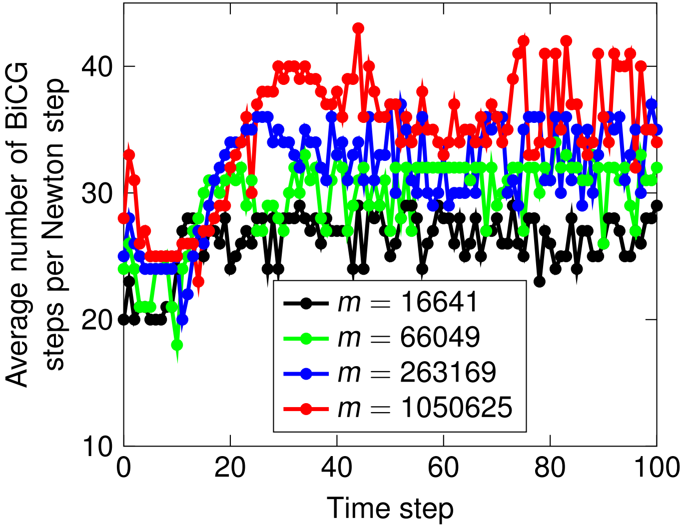
The shift with M makes the diagonal blocks of $\boxed{\left(\frac{N-1}{N} I \otimes M + \sqrt{\tau}\mathcal{B}\right)}$ positive definite.

Numerical Results

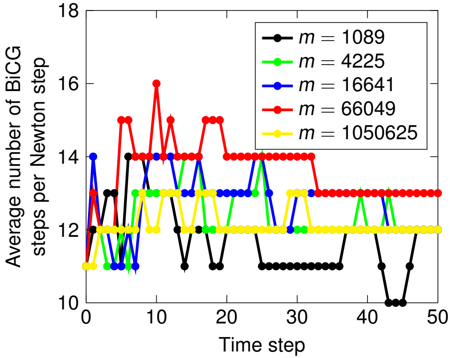
Two Phases – BiCG Iteration Numbers



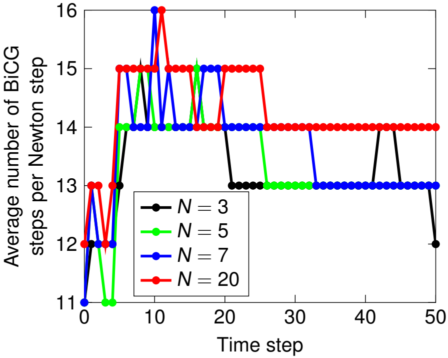
- $\varepsilon \approx \frac{9h}{\pi}$
- $\tau \approx 4\varepsilon^3$
- $\nu_{\max} = 10^{-7}$

Numerical Results

Smooth Multi-Phase Model – BiCG Iteration Numbers



(a) $N = 7$



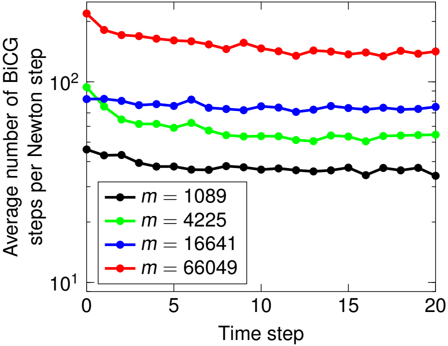
(b) $h = 2^{-8}$ $(m = 66049)$

Figure: $L = I - \frac{1}{N}\mathbf{1}\mathbf{1}^T$, $\varepsilon \approx \frac{9h}{\pi}$, $\tau \approx 4\varepsilon^2$

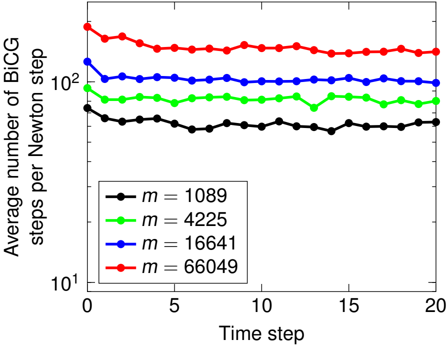
$\boxed{\text{Whole system size:} \quad 2 \cdot m \cdot N}$

Numerical Results

Nonsmooth Multi-Phase Model – BiCG Iteration Numbers



(a) $N = 5$



(b) $N = 20$

Figure: $L = I$, $\varepsilon \approx \frac{9h}{\pi}$, $\tau \approx 4\varepsilon^2$, $\nu_{\max} = 10^{-7}$

$\boxed{\text{Whole system size:} \quad 2 \cdot m \cdot N}$

Numerical Results

Results and Outlook

Results

- Smooth: Numerically mesh and phase independent preconditioners.

Numerical Results

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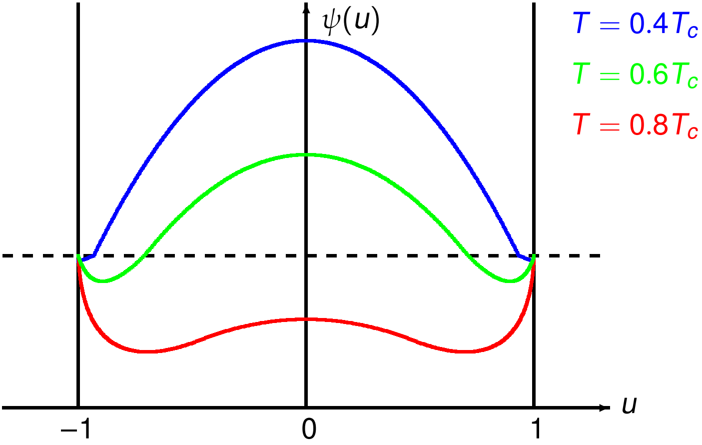
Outlook

- Proofs for the preconditioner.
- Nonsmooth: Enhanced Schur complement approximation?
- Grey/color inpainting.

Two Phases

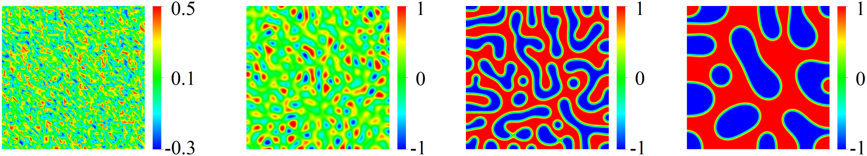
Different Potentials



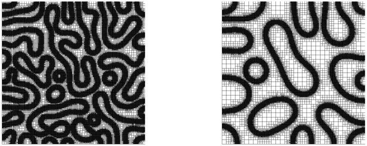
[Blowey/Elliott '91]

Two Phases

Phase Separation in 2D

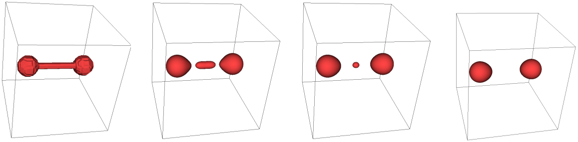


$n = 0$ $n = 5$ $n = 50$ $n = 500$

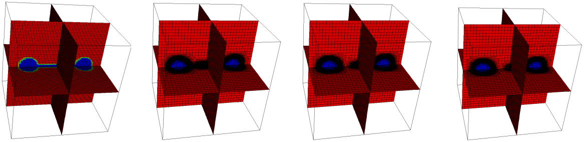


Two Phases

Dumbbell in 3D



(a) $n = 0$ (b) $n = 4$ (c) $n = 7$ (d) $n = 8$



- $\Omega = (-1, 1)^3$
- $\varepsilon = 0.03$
- $\tau = 5 \cdot 10^{-5}$
- $c_{\max} = 10^{-5}$

- $h_0 = 2^{-5}$
- $h_{\min} = \frac{\varepsilon\pi}{9}$
- $h_{\max} = 10 \cdot h_{\min}$

Two Phases

Moreau–Yosida Semismooth Newton (SSN) Solver in 1D

h	$c_{\max}$	$\|\|u_{c_{\max},h} - u_{\mathrm{ex},h}\|\|_2$	max. SSN\BiCG iterations	CPU time (s)
$\frac{1}{2^6}$	10^{-3}	$2.46337 \cdot 10^{-2}$	2\14	0.34
	10^{-6}	$3.05118 \cdot 10^{-3}$	2\14	0.41
	10^{-9}	$3.06076 \cdot 10^{-3}$	2\14	0.50
$\frac{1}{2^7}$	10^{-3}	$3.58987 \cdot 10^{-2}$	3\15	0.61
	10^{-6}	$9.32589 \cdot 10^{-4}$	3\16	0.87
	10^{-9}	$9.31919 \cdot 10^{-4}$	3\16	1.00
$\frac{1}{2^8}$	10^{-3}	$5.04977 \cdot 10^{-2}$	3\15	1.17
	10^{-6}	$6.92492 \cdot 10^{-4}$	3\16	1.80
	10^{-9}	$6.71485 \cdot 10^{-4}$	3\17	2.12
$\frac{1}{2^9}$	10^{-3}	$7.10723 \cdot 10^{-2}$	3\15	2.17
	10^{-6}	$2.63167 \cdot 10^{-4}$	3\16	3.55
	10^{-9}	$1.99167 \cdot 10^{-4}$	3\17	4.07

Two Phases

Semi-Implicit Time Discretization and Large Time Steps

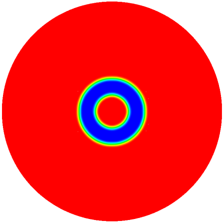


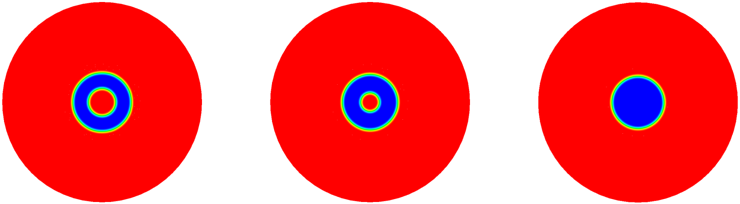
Figure: Initial state.

Exact solution

Small red circle vanishes at time $t = 1.85 \cdot 10^{-3}$.

Two Phases

Semi-Implicit Time Discretization and Large Time Steps



$\tau = 5 \cdot 10^{-4}$ $\tau = 1 \cdot 10^{-4}$ $\tau = 3 \cdot 10^{-5}$

Figure: Semi-implicit Cahn–Hilliard evolution with different time steps τ. The figure shows the solutions at time $t = 3 \cdot 10^{-3}$.

Remember: In the exact solution the small red circle vanishes at time $t = 1.85 \cdot 10^{-3}$.

Black-White Inpainting

Modified Cahn–Hilliard Model

$$\partial_t u(t) = -\mathrm{grad}_{H^{-1}}\mathcal{E}(u(t)) - \mathrm{grad}_{L^2}\mathcal{E}_2(u(t))$$

$$\mathcal{E}_2(u) = \int_\Omega \frac{\omega}{2}(f-u)^2 \,\mathrm{d}\mathbf{x}$$

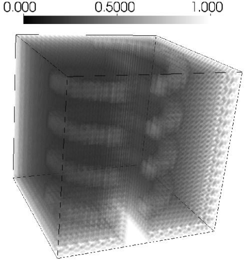
Regularized modified Cahn–Hilliard subproblem

$$\partial_t u_\nu = \Delta\Big(-\varepsilon\Delta u_\nu + \frac{1}{\varepsilon}\psi_0'(u_\nu) + \theta_\nu(u_\nu)\Big) + \omega(f-u_\nu)$$

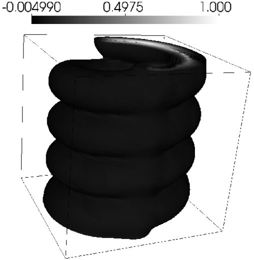
$$\nabla u_\nu \cdot \mathbf{n} = \nabla(\Delta u_\nu)\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega$$

Black-White Inpainting

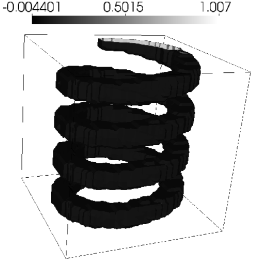
Three-Dimensional Space



(a) $n = 0$



(b) $n = 115$

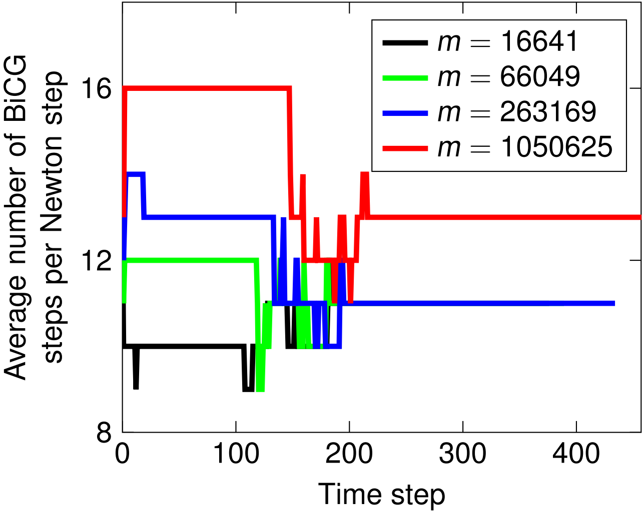


(c) $n = 900$

- $\Omega = (-1, 1)^3$
- $\varepsilon = 2 \to 0.01$
- $\tau = 1$
- $\nu_{\max} = 10^{-4}$
- $\omega_0 = 10^{-5}$
- $h = 2^{-5}$

Black-White Inpainting

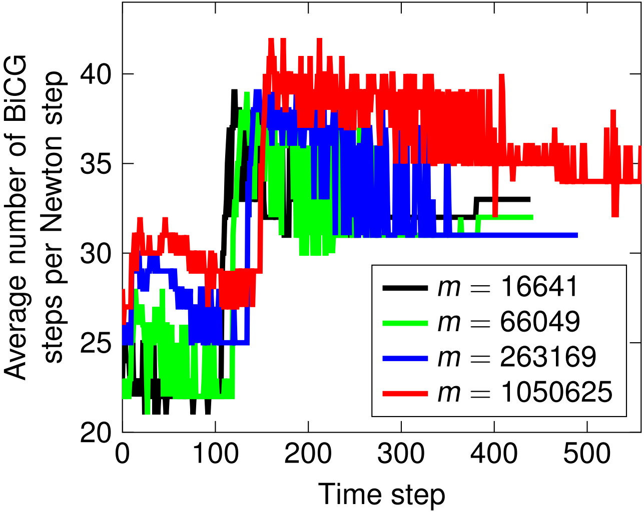
Smooth Potential – BiCG Iteration Numbers



- $\varepsilon = 0.8 \to 0.01$
- $\tau = 1$
- $\omega_0 = 10^5$

Black-White Inpainting

Nonsmooth Potential – BiCG Iteration Numbers



- $\varepsilon = 0.8 \to 0.01$
- $\tau = 1$
- $\omega_0 = 10^5$
- $\nu_{\max} = 10^{-7}$

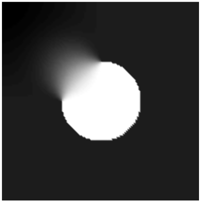
Black-White Inpainting

Comparison to Other Methods

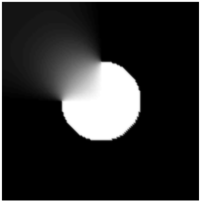
Figure: Initial state.

Black-White Inpainting

Comparison to Other Methods



(a) MATLAB



(b) Heat

(c) TV

(d) TV4

(a) Nonsmooth, FEM

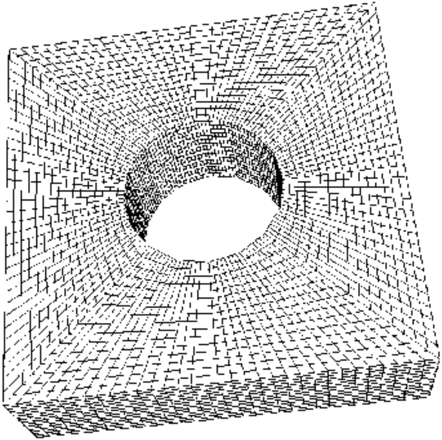
(b) Smooth, FEM

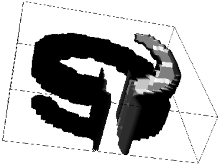
(c) Nonsmooth, FFT

(d) Smooth, FFT

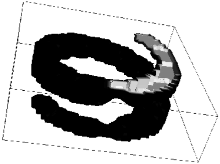
Black-White Inpainting

FEM vs. FFT – Smooth Potential





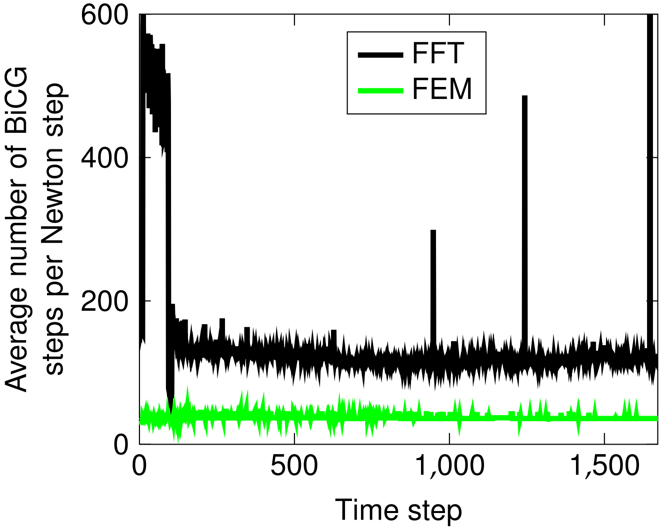
$n = 0$



$n = 880$

Black-White Inpainting

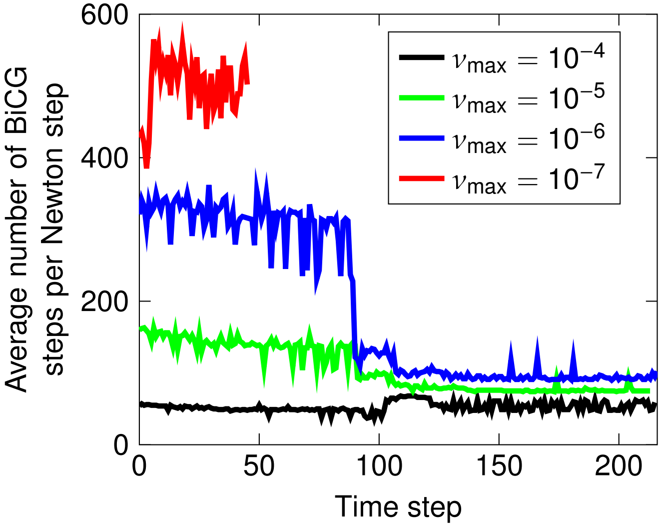
FEM vs. FFT – Nonsmooth Potential



- $\varepsilon = 0.8 \rightarrow 0.01$
- $\tau = 1$
- $\omega_0 = 10^5$
- $\nu_{max} = 10^{-7}$
- $h = 2^{-7}$

Black-White Inpainting

FFT with Regularization



- $\varepsilon = 0.8 \to 0.01$
- $\tau = 1$
- $\omega_0 = 10^5$
- $h = 2^{-7}$

Multi Phases

Final Steps to the Linear Systems

- Time discretization: Implicit Euler scheme

$$\tau < \frac{4\varepsilon^2}{\lambda_{\max}^2(A)\,\|L\|} \quad \text{(nonsmooth)} \quad \Rightarrow \quad \text{here: } \tau < 4\varepsilon^2$$

[Blowey/Copetti/Elliott '96]

- Nonlinear systems: (Semismooth) Newton method

$$(\min(0, v))' \rightsquigarrow \chi_{\mathcal{A}(v)},$$

where $\mathcal{A}(v) := \{\mathbf{x} \in \Omega : v(\mathbf{x}) < 0\}$

- Space discretization: Finite element method

Multi Phases

FFT based Preconditioner [Stoll '13]

We formulate each of the N complex valued systems to 2×2 real valued block systems. As $\lambda_1 = 0$ and $\lambda_2 = \ldots = \lambda_N = 1$ we get two types

$$\begin{bmatrix} \frac{N}{N-1} M (+\tau K) & 0 \\ 0 & \frac{N}{N-1} M (+\tau K) \end{bmatrix} \begin{bmatrix} \tilde{y}_r \\ \tilde{y}_c \end{bmatrix} = \begin{bmatrix} \tilde{g}_r \\ \tilde{g}_c \end{bmatrix}$$

which are solved with a fixed number of steps of

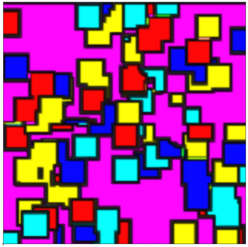
Inexact Uzawa method

$$\tilde{y}^{(l+1)} = \tilde{y}^{(l)} + \omega \mathcal{P}_1^{-1} r^{(l)},$$

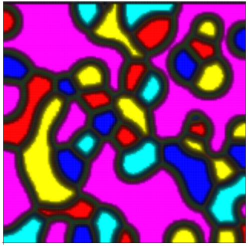
where $\mathcal{P}_1$ is a block-diagonal AMG preconditioner.

Multi Phases

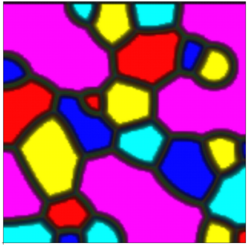
Evolution of Multi Phases



$n = 0$



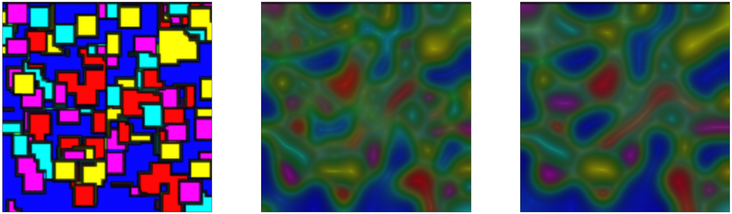
$n = 50$



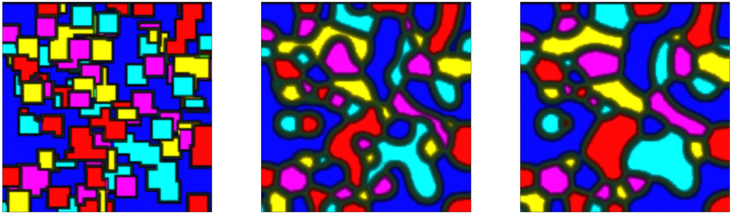
$n = 500$

Multi Phases

Smooth vs. Nonsmooth Evolution



$n = 0$ $n = 20$ $n = 100$



Multi Phases

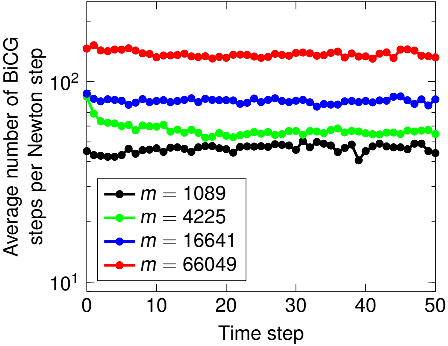
Smooth vs. Nonsmooth Evolution

		Time step 20	60	100
Min	Smooth	-0.02771	-0.02439	-0.02627
0	Nonsmooth	$-1.186 \cdot 10^{-7}$	$-1.172 \cdot 10^{-7}$	$-1.178 \cdot 10^{-7}$
Max	Smooth	0.9764	1.001	0.9972
1	Nonsmooth	1	1	1

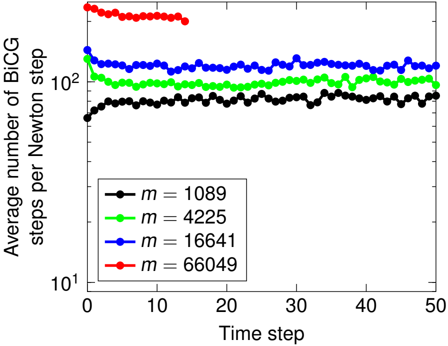
Table: Minimum and maximum values of the phase variable u_1 in the smooth and nonsmooth model.

Multi Phases

Nonsmooth Potential – BiCG Iteration Numbers



(a) $N = 5$



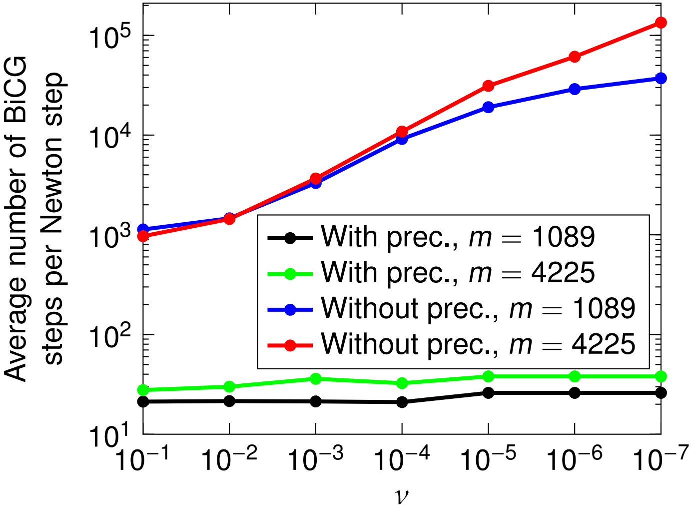
(b) $N = 20$

Figure: $L = I - \frac{1}{N}\mathbf{1}\mathbf{1}^T$, $\varepsilon \approx \frac{9h}{\pi}$, $\tau \approx 4\varepsilon^2$

$$\boxed{\text{Whole system size:} \quad 2 \cdot m \cdot N}$$

Multi Phases

Nonsmooth Potential – Preconditioning vs. No Preconditioning



- $N = 3$
- $\varepsilon \approx \frac{9h}{\pi}$
- $\tau \approx 4\varepsilon^2$
- $L = I$