Max-plus algebra

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Let $G = (g_{ij}) \in \mathbb{R}_{\max}^{n\times m}$ and $H = (h_{ij}) \in \mathbb{R}_{\max}^{l\times n}$, then $G \otimes H \in \mathbb{R}_{\max}^{l\times n}$ is the $l \times m$ max-plus matrix with

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Max-plus linear systems are used to model certain queuing systems and scheduling problems.

Max-plus polynomials

For $q_0, \ldots, q_d \in \mathbb{R}_{\max}$, let

$$q(x) = \bigoplus_{k=0}^{d} x^{\otimes k} \otimes q_k = \max\{kx + q_k : 0 \leq k \leq d\},$$

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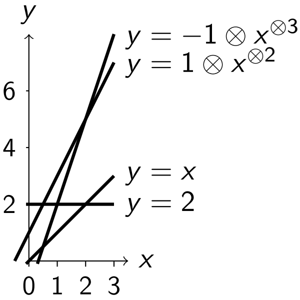
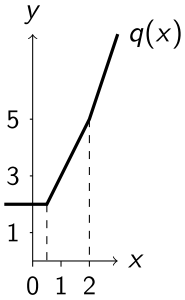
We also include $-\infty$ as a root with multiplicity k, whenever $q_0, \ldots, q_{k-1}$ are all equal to $-\infty$.

Max-plus polynomials II

For example, let $q(x) = -1 \otimes x^{\otimes 3} \oplus 1 \otimes x^{\otimes 2} \oplus x \oplus 2$.





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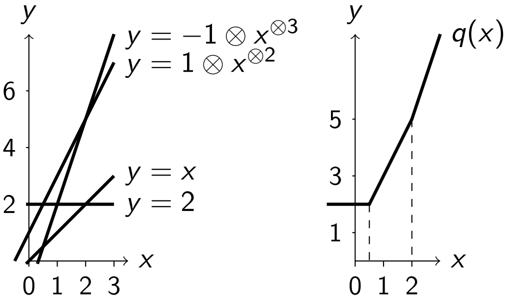

A max-plus polynomial can be factorized using its roots, so that

$$q(x) = 2\max\{x, 1/2\} + \max\{x, 2\} = (x \oplus 1/2)^{\otimes 2} \otimes (x \oplus 2).$$

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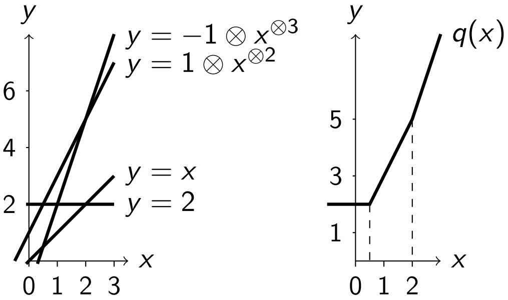

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Multivariate max-plus polynomials are of great interest in Algebraic Geometry.

Max-plus polynomials and classical polynomials I

Theorem (Ostrowski)

Let $p(z) = \sum_{k=0}^{d} p_k z^k \in \mathbb{C}[z]$ *be a classical polynomial with roots* $|z_1| \geq \cdots \geq |z_d|$ *and define* $q(z)$ *to be the max-plus polynomial*

$$q(z) = \bigoplus_{k=0}^{d} z^{\otimes k} \log |p_k|,$$

with max-plus roots $r_1 \geq \cdots \geq r_d$*. Then*

$$\begin{aligned} \tfrac{1}{2}\exp(r_1) &< |z_1| \leq && d\exp(r_1), \\ [1-(\tfrac{1}{2})^{\frac{1}{k}}]\exp(r_k) &\leq |z_k| \leq && \exp(r_k)[1-(\tfrac{1}{2})^{\frac{1}{d-k+1}}]^{-1}, \quad \text{for } 2 \leq k \leq d-1, \\ \tfrac{1}{d}\exp(r_d) &\leq |z_d| < && 2\exp(r_d). \end{aligned}$$

Max-plus polynomials and classical polynomials II

Corollary

Let $p^t(z) = \sum_{k=0}^{d} z^d p_k(t)$ *be a parameterized polynomial with roots* $|z_1(t)| \geq \cdots \geq |z_d(t)|$ *then for each* i *the limit*

$$r_i = \lim_{t\to\infty} \frac{1}{t} \log |z_i(t)|,$$

exists and is equal to the i*th max-plus root of the max-plus polynomial*

$$q(z) = \bigoplus_{k=0}^{d} z^{\otimes k} q_k,$$

where

$$q_k = \lim_{t\to\infty} \frac{1}{t} \log |p_k(t)|.$$

Max-plus eigenvalues I

Let $G \in \mathbb{R}_{\max}^{n\times n}$ be a max-plus matrix. The *max-plus eigenvalues* $\mu_1, \dots, \mu_n$ of G are the max-plus roots of the *max-plus characteristic polynomial*

$$t\chi_G(z) = \mathrm{perm}(G \oplus z \otimes I),$$

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$$\mathrm{perm}(M) = \bigoplus_{\pi \in P_n} \bigotimes_{k=1}^{n} m_{\pi(k),k} = \max_{\pi \in P_n} \sum_{k=1}^{n} m_{\pi(k),k},$$

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... and I is the $n \times n$ max-plus identity matrix with zeros on the diagonal and $-\infty$ off the diagonal.

Max-plus eigenvalues II

Theorem (Akian, Gaubert, Bupat)

Let $G = (g_{ij}) \in \mathbb{R}_{\max}^{n \times n}$ *be a max-plus matrix and let* $B = (b_{ij}) \in \mathbb{C}^{n \times n}$ *be a complex matrix. Now let* $A(t) = (a_{ij}(t))$ *be the parameterized matrix with*

$$a_{ij}(t) = b_{ij} \exp(g_{ij} t),$$

where by convention $\exp(-\infty) = 0$. *Let* $\lambda_1(t), \ldots, \lambda_n(t)$ *be the analytic eigenvalues of* A, *with* $\lambda_{n-k+1}(t), \ldots, \lambda_n(t) \equiv 0$. *For all* G *and generic* B, *including generic symmetric* B, *and for* $i = 1, \ldots n-k$

$$lim_{t \to \infty} \frac{\log |\lambda_i(t)|}{t} = \mu_i$$

exists and is independent B.
Moreover these limits are equal to G*'s finite max-plus eigenvlaues, while* G*'s full spectrum of max-plus eigenvalues is given by* $\mu_1 \ldots \mu_n$, *with* $\mu_1, \ldots \mu_{n-k}$ *defined as above and* $\mu_{n-k+1}, \ldots \mu_n = -\infty$.

Max-plus eigenvalues: Example 1 - I

For generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, consider

$$A(t) = \begin{bmatrix} \alpha \exp(3t) & \beta \exp(5t) \\ \gamma \exp(0t) & \delta \exp(3t) \end{bmatrix}.$$

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$$\chi_A(z) = z^2 - (\alpha + \delta) \exp(3t) z + \alpha\delta \exp(6t) - \beta\gamma \exp(5t).$$

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Assuming $\alpha + \delta \neq 0$, $\alpha\delta \neq 0$, by Corollary 1 we have

$$\lim_{t \to \infty} \frac{\log |\lambda_i(t)|}{t} = r_i,$$

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$$q(x) = x^{\otimes 2} \oplus 3 \otimes z \oplus 6$$

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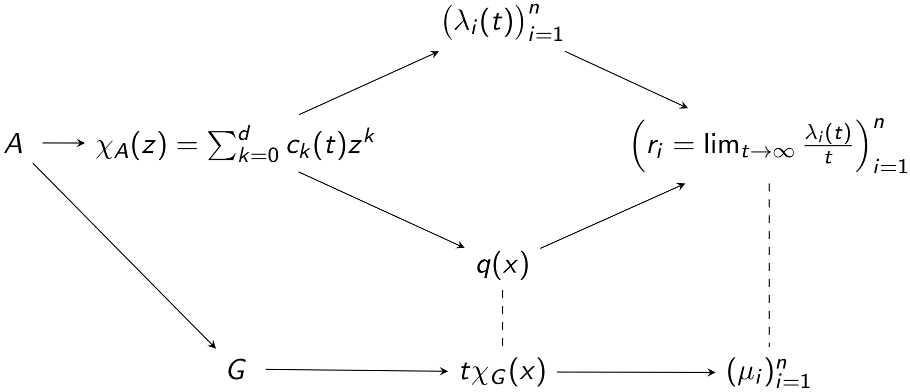
So that $r_1 = 3$ and $r_2 = 3$.

What's going on

For $A(t) = \big(a_{ij}(t)\big) \in \mathbb{C}[[t]]^{n\times n}$, with $a_{ij}(t) = b_{ij}\exp(g_{ij}t)$, where $B = (b_{ij}) \in \mathbb{C}^{n\times n}$ and $G = (g_{ij}) \in \mathbb{R}_{\max}^{n\times n}$



where

$$q(x) = \bigoplus_{k=0}^{d} \left(\lim_{t\to\infty} \frac{c_k(t)}{t} \right) \otimes x^{\otimes k}.$$

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$$A(t) = \begin{bmatrix} \alpha \exp(3t) & \beta \exp(5t) \\ \gamma \exp(0t) & \delta \exp(3t) \end{bmatrix}.$$

Theorem 2 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the eigenvalues of A are equal to the max-plus eigenvalues of the max-plus matrix

$$G = \begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix}.$$

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which gives $\mu_1 = 3$ and $\mu_2 = 3$. Agreeing with the previous calculation.

Max-plus singular values: Example 1 - III

We can compute the singular values, $\sigma_1(t), \sigma_2(t)$ of $A(t)$ as the square roots of the eigenvalues $\lambda_1(t), \lambda_2(t)$ of

$$A(t)A^*(t) = \left[\begin{array}{cc} \alpha\overline{\alpha}\exp(6t) + \beta\overline{\beta}\exp(10t) & \alpha\overline{\gamma}\exp(3t) + \beta\overline{\delta}\exp(8t) \\ \gamma\overline{\alpha}\exp(3t) + \delta\overline{\beta}\exp(8t) & \gamma\overline{\gamma}\exp(0t) + \delta\overline{\delta}\exp(6t) \end{array} \right],$$

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which has characteristic polynomial

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which has characteristic polynomial

$$\begin{aligned} \chi_{AA^*}(z) &= z^2 + [\beta\overline{\beta}\exp(10t) + (\alpha\overline{\alpha}\delta\overline{\beta}) + \exp(6t) + \gamma\overline{\gamma}\exp(0t)]z \\ &+ (\beta\overline{\beta}\delta\overline{\delta} - \beta\overline{\delta}\delta\overline{\beta})\exp(16t) + (\alpha\overline{\alpha}\gamma\overline{\gamma} - \alpha\overline{\gamma}\gamma\overline{\alpha})\exp(6t) \\ &+ \alpha\overline{\alpha}\delta\overline{\delta}\exp(12t) + \beta\overline{\beta}\gamma\overline{\gamma}\exp(10t) - (\alpha\overline{\gamma}\delta\overline{\beta} + \beta\overline{\delta}\gamma\overline{\alpha})\exp(11t). \end{aligned}$$

Assuming $\beta\overline{\beta} \neq 0$, $\alpha\overline{\alpha}\delta\overline{\delta} \neq 0$, by Corollary 1 we have

$$\lim_{t\to\infty} \frac{2\log\sigma_i(t)}{t} = \lim_{t\to\infty} \frac{\log|\lambda_i(t)|}{t} = r_i,$$

where r_1, r_2 are the roots of the max-plus polynomial

$$q(x) = x^{\otimes 2} \oplus 10 \otimes z \oplus 12$$

Max-plus singular values: Example 1 - III

We can compute the singular values, $\sigma_1(t), \sigma_2(t)$ of $A(t)$ as the square roots of the eigenvalues $\lambda_1(t), \lambda_2(t)$ of

$$A(t)A^*(t) = \begin{bmatrix} \alpha\overline{\alpha}\exp(6t) + \beta\overline{\beta}\exp(10t) & \alpha\overline{\gamma}\exp(3t) + \beta\overline{\delta}\exp(8t) \\ \gamma\overline{\alpha}\exp(3t) + \delta\overline{\beta}\exp(8t) & \gamma\overline{\gamma}\exp(0t) + \delta\overline{\delta}\exp(6t) \end{bmatrix},$$

which has characteristic polynomial

$$\begin{aligned} \chi_{AA^*}(z) &= z^2 + [\beta\overline{\beta}\exp(10t) + (\alpha\overline{\alpha}\delta\overline{\beta}) + \exp(6t) + \gamma\overline{\gamma}\exp(0t)]z \\ &+ (\beta\overline{\beta}\delta\overline{\delta} - \beta\overline{\delta}\delta\overline{\beta})\exp(16t) + (\alpha\overline{\alpha}\gamma\overline{\gamma} - \alpha\overline{\gamma}\gamma\overline{\alpha})\exp(6t) \\ &+ \alpha\overline{\alpha}\delta\overline{\delta}\exp(12t) + \beta\overline{\beta}\gamma\overline{\gamma}\exp(10t) - (\alpha\overline{\gamma}\delta\overline{\beta} + \beta\overline{\delta}\gamma\overline{\alpha})\exp(11t). \end{aligned}$$

Assuming $\beta\overline{\beta} \neq 0$, $\alpha\overline{\alpha}\delta\overline{\delta} \neq 0$, by Corollary 1 we have

$$\lim_{t\to\infty} \frac{2\log\sigma_i(t)}{t} = \lim_{t\to\infty} \frac{\log|\lambda_i(t)|}{t} = r_i,$$

where r_1, r_2 are the roots of the max-plus polynomial

$$q(x) = x^{\otimes 2} \oplus 10 \otimes z \oplus 12 = (x \oplus 10) \otimes (x \oplus 2).$$

Max-plus singular values: Example 1 - III

We can compute the singular values, $\sigma_1(t), \sigma_2(t)$ of $A(t)$ as the square roots of the eigenvalues $\lambda_1(t), \lambda_2(t)$ of

$$A(t)A^*(t) = \left[\begin{array}{cc} \alpha\overline{\alpha}\exp(6t) + \beta\overline{\beta}\exp(10t) & \alpha\overline{\gamma}\exp(3t) + \beta\overline{\delta}\exp(8t) \\ \gamma\overline{\alpha}\exp(3t) + \delta\overline{\beta}\exp(8t) & \gamma\overline{\gamma}\exp(0t) + \delta\overline{\delta}\exp(6t) \end{array} \right],$$

which has characteristic polynomial

$$\begin{aligned} \chi_{AA^*}(z) = {} & z^2 + [\beta\overline{\beta}\exp(10t) + (\alpha\overline{\alpha}\delta\overline{\beta}) + \exp(6t) + \gamma\overline{\gamma}\exp(0t)]z \\ & + (\beta\overline{\beta}\delta\overline{\delta} - \beta\overline{\delta}\delta\overline{\beta})\exp(16t) + (\alpha\overline{\alpha}\gamma\overline{\gamma} - \alpha\overline{\gamma}\gamma\overline{\alpha})\exp(6t) \\ & + \alpha\overline{\alpha}\delta\overline{\delta}\exp(12t) + \beta\overline{\beta}\gamma\overline{\gamma}\exp(10t) - (\alpha\overline{\gamma}\delta\overline{\beta} + \beta\overline{\delta}\gamma\overline{\alpha})\exp(11t). \end{aligned}$$

Assuming $\beta\overline{\beta} \neq 0$, $\alpha\overline{\alpha}\delta\overline{\delta} \neq 0$, by Corollary 1 we have

$$\lim_{t\to\infty} \frac{2\log\sigma_i(t)}{t} = \lim_{t\to\infty} \frac{\log|\lambda_i(t)|}{t} = r_i,$$

where r_1, r_2 are the roots of the max-plus polynomial

$$q(x) = x^{\otimes 2} \oplus 10 \otimes z \oplus 12 = (x \oplus 10) \otimes (x \oplus 2).$$

So that $r_1 = 10$ and $r_2 = 2$.

Max-plus singular values: Example 1 - IV

$$A(t)A^*(t) = \left[\begin{array}{cc} \alpha\overline{\alpha}\exp(6t) + \beta\overline{\beta}\exp(10t) & \alpha\overline{\gamma}\exp(3t) + \beta\overline{\delta}\exp(8t) \\ \gamma\overline{\alpha}\exp(3t) + \delta\overline{\beta}\exp(8t) & \gamma\overline{\gamma}\exp(0t) + \delta\overline{\delta}\exp(6t) \end{array} \right],$$

(careless application of) Theorem 2 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the eigenvalues of AA^* are equal to the max-plus eigenvalues of the max-plus matrix

$$G = \left[\begin{array}{cc} 10 & 8 \\ 8 & 6 \end{array} \right].$$

Max-plus singular values: Example 1 - IV

$$A(t)A^*(t) = \left[\begin{array}{cc} \alpha\overline{\alpha}\,\exp(6t) + \beta\overline{\beta}\,\exp(10t) & \alpha\overline{\gamma}\,\exp(3t) + \beta\overline{\delta}\,\exp(8t) \\ \gamma\overline{\alpha}\,\exp(3t) + \delta\overline{\beta}\,\exp(8t) & \gamma\overline{\gamma}\,\exp(0t) + \delta\overline{\delta}\,\exp(6t) \end{array} \right],$$

(careless application of) Theorem 2 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the eigenvalues of AA^* are equal to the max-plus eigenvalues of the max-plus matrix

$$G = \left[\begin{array}{cc} 10 & 8 \\ 8 & 6 \end{array} \right].$$

The max-plus eigenvalues μ_1, μ_2 of G are the roots of the max-plus characteristic polynomial

$$t\chi_G(x) = \text{perm}\left(\left[\begin{array}{cc} 10 & 8 \\ 8 & 6 \end{array} \right] \oplus x \otimes \left[\begin{array}{cc} 0 & -\infty \\ -\infty & 0 \end{array} \right] \right)$$

Max-plus singular values: Example 1 - IV

$$A(t)A^*(t) = \begin{bmatrix} \alpha\overline{\alpha}\exp(6t) + \beta\overline{\beta}\exp(10t) & \alpha\overline{\gamma}\exp(3t) + \beta\overline{\delta}\exp(8t) \\ \gamma\overline{\alpha}\exp(3t) + \delta\overline{\beta}\exp(8t) & \gamma\overline{\gamma}\exp(0t) + \delta\overline{\delta}\exp(6t) \end{bmatrix},$$

(careless application of) Theorem 2 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the eigenvalues of AA^* are equal to the max-plus eigenvalues of the max-plus matrix

$$G = \begin{bmatrix} 10 & 8 \\ 8 & 6 \end{bmatrix}.$$

The max-plus eigenvalues μ_1, μ_2 of G are the roots of the max-plus characteristic polynomial

$$t\chi_G(x) = \text{perm}\left(\begin{bmatrix} 10 & 8 \\ 8 & 6 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & -\infty \\ -\infty & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} 10 \oplus x & 8 \\ 8 & 6 \oplus x \end{bmatrix}\right)$$

Max-plus singular values: Example 1 - IV

$$A(t)A^*(t) = \begin{bmatrix} \alpha\overline{\alpha}\exp(6t) + \beta\overline{\beta}\exp(10t) & \alpha\overline{\gamma}\exp(3t) + \beta\overline{\delta}\exp(8t) \\ \gamma\overline{\alpha}\exp(3t) + \delta\overline{\beta}\exp(8t) & \gamma\overline{\gamma}\exp(0t) + \delta\overline{\delta}\exp(6t) \end{bmatrix},$$

(careless application of) Theorem 2 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the eigenvalues of AA^* are equal to the max-plus eigenvalues of the max-plus matrix

$$G = \begin{bmatrix} 10 & 8 \\ 8 & 6 \end{bmatrix}.$$

The max-plus eigenvalues μ_1, μ_2 of G are the roots of the max-plus characteristic polynomial

$$t\chi_G(x) = \text{perm}\left(\begin{bmatrix} 10 & 8 \\ 8 & 6 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & -\infty \\ -\infty & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} 10 \oplus x & 8 \\ 8 & 6 \oplus x \end{bmatrix}\right)$$
$$= (10 \oplus x) \otimes (6 \oplus x) \oplus 8 \otimes 8$$

Max-plus singular values: Example 1 - IV

$$A(t)A^*(t) = \left[\begin{array}{cc} \alpha\overline{\alpha}\exp(6t) + \beta\overline{\beta}\exp(10t) & \alpha\overline{\gamma}\exp(3t) + \beta\overline{\delta}\exp(8t) \\ \gamma\overline{\alpha}\exp(3t) + \delta\overline{\beta}\exp(8t) & \gamma\overline{\gamma}\exp(0t) + \delta\overline{\delta}\exp(6t) \end{array} \right],$$

(careless application of) Theorem 2 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the eigenvalues of AA^* are equal to the max-plus eigenvalues of the max-plus matrix

$$G = \left[\begin{array}{cc} 10 & 8 \\ 8 & 6 \end{array} \right].$$

The max-plus eigenvalues μ_1, μ_2 of G are the roots of the max-plus characteristic polynomial

$$t\chi_G(x) = \text{perm}\left(\left[\begin{array}{cc} 10 & 8 \\ 8 & 6 \end{array} \right] \oplus x \otimes \left[\begin{array}{cc} 0 & -\infty \\ -\infty & 0 \end{array} \right] \right) = \left(\left[\begin{array}{cc} 10 \oplus x & 8 \\ 8 & 6 \oplus x \end{array} \right] \right)$$
$$= (10 \oplus x) \otimes (6 \oplus x) \oplus 8 \otimes 8 = x^{\otimes 2} \oplus 10 \otimes x \oplus 16$$

Max-plus singular values: Example 1 - IV

$$A(t)A^*(t) = \begin{bmatrix} \alpha\overline{\alpha}\exp(6t) + \beta\overline{\beta}\exp(10t) & \alpha\overline{\gamma}\exp(3t) + \beta\overline{\delta}\exp(8t) \\ \gamma\overline{\alpha}\exp(3t) + \delta\overline{\beta}\exp(8t) & \gamma\overline{\gamma}\exp(0t) + \delta\overline{\delta}\exp(6t) \end{bmatrix},$$

(careless application of) Theorem 2 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the eigenvalues of AA^* are equal to the max-plus eigenvalues of the max-plus matrix

$$G = \begin{bmatrix} 10 & 8 \\ 8 & 6 \end{bmatrix}.$$

The max-plus eigenvalues μ_1, μ_2 of G are the roots of the max-plus characteristic polynomial

$$\begin{aligned} t\chi_G(x) &= \text{perm}\left(\begin{bmatrix} 10 & 8 \\ 8 & 6 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & -\infty \\ -\infty & 0 \end{bmatrix} \right) = \left(\begin{bmatrix} 10 \oplus x & 8 \\ 8 & 6 \oplus x \end{bmatrix} \right) \\ &= (10 \oplus x) \otimes (6 \oplus x) \oplus 8 \otimes 8 = x^{\otimes 2} \oplus 10 \otimes x \oplus 16 \\ &= (x \oplus 10) \otimes (x \oplus 6). \end{aligned}$$

Max-plus singular values: Example 1 - IV

$$A(t)A^*(t) = \begin{bmatrix} \alpha\overline{\alpha}\exp(6t) + \beta\overline{\beta}\exp(10t) & \alpha\overline{\gamma}\exp(3t) + \beta\overline{\delta}\exp(8t) \\ \gamma\overline{\alpha}\exp(3t) + \delta\overline{\beta}\exp(8t) & \gamma\overline{\gamma}\exp(0t) + \delta\overline{\delta}\exp(6t) \end{bmatrix},$$

(careless application of) Theorem 2 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the eigenvalues of AA^* are equal to the max-plus eigenvalues of the max-plus matrix

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The max-plus eigenvalues μ_1, μ_2 of G are the roots of the max-plus characteristic polynomial

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So that $\mu_1 = 10$ and $\mu_2 = 6$. Disagreeing with the previous calculation!

Max-plus singular values

Let $G, H \in \mathbb{R}_{\max}^{n\times n}$ be max-plus matrices. The *max-plus eigenvalues* of the *max plus pencil*

$$Q(z) = G \oplus z \otimes H,$$

are the max-plus roots of the max-plus characteristic polynomial

$$t\chi_Q(z) = \text{perm}(G \oplus z \otimes H).$$

Theorem (Max-Plus Singular Values)

Let $G \in \mathbb{R}_{\max}^{n\times n}$ be a max-plus matrix. The max-plus singular values of G are given by the max-plus eigenvalues of the max-plus pencil,

$$Q(z) = G \oplus z \otimes \underline{O},$$

where $\underline{O}$ is an $n \times n$ matrix of zeros.

Max-plus singular values: Example 1 - V

$$A(t) = \begin{bmatrix} \alpha \exp(3t) & \beta \exp(5t) \\ \gamma \exp(0t) & \delta \exp(3t) \end{bmatrix}.$$

Theorem 4 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the singular values of A are equal to the max-plus eigenvalues of the max-plus matrix pencil $Q(x) = G \oplus x \otimes \underline{O}$ with

$$Q(x) = \begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Max-plus singular values: Example 1 - V

$$A(t) = \begin{bmatrix} \alpha \exp(3t) & \beta \exp(5t) \\ \gamma \exp(0t) & \delta \exp(3t) \end{bmatrix}.$$

Theorem 4 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the singular values of A are equal to the max-plus eigenvalues of the max-plus matrix pencil $Q(x) = G \oplus x \otimes \underline{O}$ with

$$Q(x) = \begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The max-plus eigenvalues s_1, s_2 of Q (the max-plus singular values of G) are the roots of the max-plus characteristic polynomial

$$t\chi_Q(x) = \text{perm}\left(\begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

Max-plus singular values: Example 1 - V

$$A(t) = \begin{bmatrix} \alpha \exp(3t) & \beta \exp(5t) \\ \gamma \exp(0t) & \delta \exp(3t) \end{bmatrix}.$$

Theorem 4 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the singular values of A are equal to the max-plus eigenvalues of the max-plus matrix pencil $Q(x) = G \oplus x \otimes \underline{O}$ with

$$Q(x) = \begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The max-plus eigenvalues s_1, s_2 of Q (the max-plus singular values of G) are the roots of the max-plus characteristic polynomial

$$t\chi_Q(x) = \text{perm}\left(\begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} 3 \oplus x & 5 \oplus x \\ 0 \oplus x & 3 \oplus x \end{bmatrix}\right)$$

Max-plus singular values: Example 1 - V

$$A(t) = \begin{bmatrix} \alpha \exp(3t) & \beta \exp(5t) \\ \gamma \exp(0t) & \delta \exp(3t) \end{bmatrix}.$$

Theorem 4 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the singular values of A are equal to the max-plus eigenvalues of the max-plus matrix pencil $Q(x) = G \oplus x \otimes \underline{O}$ with

$$Q(x) = \begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The max-plus eigenvalues s_1, s_2 of Q (the max-plus singular values of G) are the roots of the max-plus characteristic polynomial

$$t\chi_Q(x) = \text{perm}\left(\begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = \left(\begin{bmatrix} 3 \oplus x & 5 \oplus x \\ 0 \oplus x & 3 \oplus x \end{bmatrix} \right)$$
$$= (3 \oplus x) \otimes (3 \oplus x) \oplus (5 \oplus x) \otimes (0 \oplus x)$$

Max-plus singular values: Example 1 - V

$$A(t) = \begin{bmatrix} \alpha \exp(3t) & \beta \exp(5t) \\ \gamma \exp(0t) & \delta \exp(3t) \end{bmatrix}.$$

Theorem 4 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the singular values of A are equal to the max-plus eigenvalues of the max-plus matrix pencil $Q(x) = G \oplus x \otimes \underline{O}$ with

$$Q(x) = \begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The max-plus eigenvalues s_1, s_2 of Q (the max-plus singular values of G) are the roots of the max-plus characteristic polynomial

$$t\chi_Q(x) = \text{perm}\left(\begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} 3 \oplus x & 5 \oplus x \\ 0 \oplus x & 3 \oplus x \end{bmatrix}\right)$$
$$= (3 \oplus x) \otimes (3 \oplus x) \oplus (5 \oplus x) \otimes (0 \oplus x) = x^{\otimes 2} \oplus 5 \otimes x \oplus 6$$

Max-plus singular values: Example 1 - V

$$A(t) = \begin{bmatrix} \alpha \exp(3t) & \beta \exp(5t) \\ \gamma \exp(0t) & \delta \exp(3t) \end{bmatrix}.$$

Theorem 4 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the singular values of A are equal to the max-plus eigenvalues of the max-plus matrix pencil $Q(x) = G \oplus x \otimes \underline{O}$ with

$$Q(x) = \begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The max-plus eigenvalues s_1, s_2 of Q (the max-plus singular values of G) are the roots of the max-plus characteristic polynomial

$$t\chi_Q(x) = \text{perm}\left(\begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = \left(\begin{bmatrix} 3 \oplus x & 5 \oplus x \\ 0 \oplus x & 3 \oplus x \end{bmatrix} \right)$$
$$= (3 \oplus x) \otimes (3 \oplus x) \oplus (5 \oplus x) \otimes (0 \oplus x) = x^{\otimes 2} \oplus 5 \otimes x \oplus 6$$
$$= (5 \oplus x) \otimes (1 \oplus x).$$

Max-plus singular values: Example 1 - V

$$A(t) = \begin{bmatrix} \alpha \exp(3t) & \beta \exp(5t) \\ \gamma \exp(0t) & \delta \exp(3t) \end{bmatrix}.$$

Theorem 4 tells us that, for generic $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, the asymptotic growth rates of the singular values of A are equal to the max-plus eigenvalues of the max-plus matrix pencil $Q(x) = G \oplus x \otimes \underline{O}$ with

$$Q(x) = \begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The max-plus eigenvalues s_1, s_2 of Q (the max-plus singular values of G) are the roots of the max-plus characteristic polynomial

$$t\chi_Q(x) = \text{perm}\left(\begin{bmatrix} 3 & 5 \\ 0 & 3 \end{bmatrix} \oplus x \otimes \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left(\begin{bmatrix} 3 \oplus x & 5 \oplus x \\ 0 \oplus x & 3 \oplus x \end{bmatrix}\right)$$
$$= (3 \oplus x) \otimes (3 \oplus x) \oplus (5 \oplus x) \otimes (0 \oplus x) = x^{\otimes 2} \oplus 5 \otimes x \oplus 6$$
$$= (5 \oplus x) \otimes (1 \oplus x).$$

So that $s_1 = 5$ and $s_2 = 1$. Agreeing with the original calculation.

Heuristic approximation for parameter independent matrices

Let $A(t) = \big(a_{ij}(t)\big) \in \mathbb{C}[[t]]^{n\times n}$, with $a_{ij}(t) = b_{ij}\exp(g_{ij}t)$, where $B = (b_{ij}) \in \mathbb{C}^{n\times n}$ and $G = (g_{ij}) \in \mathbb{R}_{\max}^{n\times n}$.

Heuristic approximation for parameter independent matrices

Let $A(t) = \big(a_{ij}(t)\big) \in \mathbb{C}[[t]]^{n\times n}$, with $a_{ij}(t) = b_{ij}\exp(g_{ij}t)$, where $B = (b_{ij}) \in \mathbb{C}^{n\times n}$ and $G = (g_{ij}) \in \mathbb{R}_{\max}^{n\times n}$. For generic B we have

$$\lim_{t\to\infty} \frac{\log|\lambda_i(t)|}{t} = \mu_i,$$

where $\lambda_1(t), \dots, \lambda_n(t)$ are the eigenvalues of $A(t)$ and $\mu_1, \dots, \mu_n$ are the max-plus eigenvalues of G.

Heuristic approximation for parameter independent matrices

Let $A(t) = \big(a_{ij}(t)\big) \in \mathbb{C}[[t]]^{n\times n}$, with $a_{ij}(t) = b_{ij}\exp(g_{ij}t)$, where $B = (b_{ij}) \in \mathbb{C}^{n\times n}$ and $G = (g_{ij}) \in \mathbb{R}_{\max}^{n\times n}$. For generic B we have

$$\lim_{t\to\infty} \frac{\log|\lambda_i(t)|}{t} = \mu_i,$$

where $\lambda_1(t), \dots, \lambda_n(t)$ are the eigenvalues of $A(t)$ and $\mu_1, \dots, \mu_n$ are the max-plus eigenvalues of G.

Suppose that $M \in \mathbb{C}^{n\times n}$ has entries that vary a lot in magnitude, then

$$\log|\lambda_i| \approx \mu_i,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of M and $\mu_1, \dots, \mu_n$ are the max-plus eigenvalues of $V = (v_{ij}) = \mathcal{V}(M) \in \mathbb{R}_{\max}^{n\times n}$ the *valuation* of M with

$$v_{ij} = \log|m_{ij}|.$$

Example 2

We use the matrix M of the HB/steam3 problem from the University of Florida Sparse Matrix Collection.

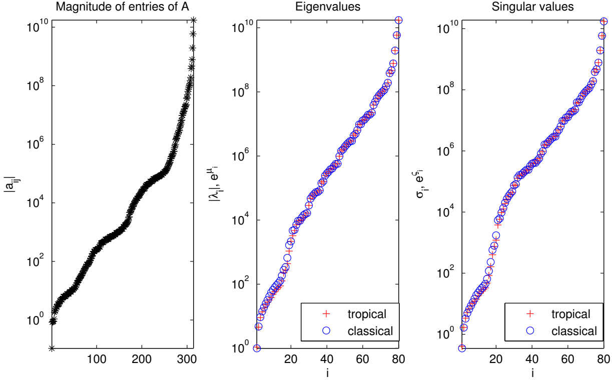

Figure : Left: Magnitude of non-zero entries of M. Right: Log of absolute value of classical eigenvalues and singular values of M (blue circles) and max-plus eigenvalues and singular values of valuation (red crosses).

Example 3

We use the matrix M of the `Rommes/ww_36_pmec_36` problem from the University of Florida Sparse Matrix Collection.

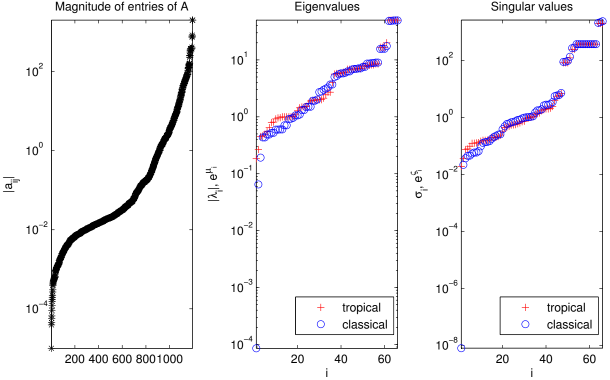

Figure : Left: Magnitude of non-zero entries of M. Right: Log of absolute value of classical eigenvalues and singular values of M (blue circles) and max-plus eigenvalues and singular values of valuation (red crosses).

Example 4 - I

Consider the matrix

$$A = \begin{bmatrix} 100 & 100 \\ 100 & 100 - \epsilon \end{bmatrix},$$

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$$A = \begin{bmatrix} 100 & 100 \\ 100 & 100 - \epsilon \end{bmatrix},$$

The valuation of A is given by

$$G = V\left(\begin{bmatrix} 100 & 100 \\ 100 & 100 - \epsilon \end{bmatrix}\right) = \begin{bmatrix} \log(100) & \log(100) \\ \log(100) & \log(100 - \epsilon) \end{bmatrix},$$

with max-plus characteristic polynomial

$$\chi_G(x) = x^{\otimes 2} \oplus \log(100) \otimes x \oplus \log(10000).$$

Example 4 - I

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with max-plus characteristic polynomial

$$\chi_G(x) = x^{\otimes 2} \oplus \log(100) \otimes x \oplus \log(10000).$$

The max-plus eigenvalues of G are $\mu_1 = \mu_2 = \log(100)$.

Example 4 - I

Consider the matrix

$$A = \begin{bmatrix} 100 & 100 \\ 100 & 100-\epsilon \end{bmatrix},$$

The valuation of A is given by

$$G = V\left(\begin{bmatrix} 100 & 100 \\ 100 & 100-\epsilon \end{bmatrix}\right) = \begin{bmatrix} \log(100) & \log(100) \\ \log(100) & \log(100-\epsilon) \end{bmatrix},$$

with max-plus characteristic polynomial

$$\chi_G(x) = x^{\otimes 2} \oplus \log(100) \otimes x \oplus \log(10000).$$

The max-plus eigenvalues of G are $\mu_1 = \mu_2 = \log(100)$.
From the max-plus analysis we would therefore expect the eigenvalues of A to both be of modulus ≈ 100.

Example 4 - II

However

$$A = \begin{bmatrix} 100 & 100 \\ 100 & 100 - \epsilon \end{bmatrix},$$

has classical characteristic polynomial

$$\begin{aligned} \chi_A(z) &= z^2 - (200 - \epsilon)z + 100 \times (100 - \epsilon) - 100 \times 100 \\ &= z^2 - (200 - \epsilon)z - 100\epsilon. \end{aligned}$$

Example 4 - II

However

$$A = \begin{bmatrix} 100 & 100 \\ 100 & 100 - \epsilon \end{bmatrix},$$

has classical characteristic polynomial

$$\begin{aligned}\chi_A(z) &= z^2 - (200 - \epsilon)z + 100 \times (100 - \epsilon) - 100 \times 100 \\ &= z^2 - (200 - \epsilon)z - 100\epsilon.\end{aligned}$$

The eigenvalues of A are therefore $\lambda_1 \approx 200$, and $\lambda_2 \approx \epsilon$.

Example 5

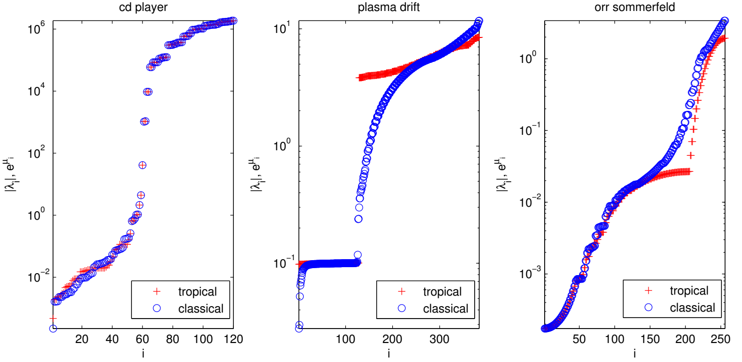

Figure : Comparisons between the classical eigenvalues and the exponential of their max-plus analog for three matrix polynomials from the NLEVP collection.

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- $O(n\tau)$ - computation of elimination tree for full pivoting max-plus Gaussian elimination of an $n \times n$ max-plus matrix.

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- The same ideas apply to matrix polynomial eigenvalues, singular values, LU decomposition...

Thankyou for listening!