Improving Communication Lower Bounds for Matrix-Matrix Multiplication

Sparse Days 2014 - Toulouse, France

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- Algorithms have two costs (cost in time, energy, power):
  (1) *Computation:* Cost to perform computation
    - # of operations to be performed
  (2) *Communication:* Cost to move data
    - volume of data to be moved (bandwidth)
    - # of messages (latency)
- Motivations
  (1) *Time.* On current architecture, communication is much slower than computation. Trend is not in favor of communication.
  (2) *Energy/Power.* Communication (moving data) consumes a lots of energy, power
  (3) *Co-design.*

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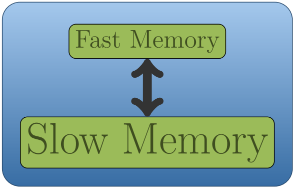

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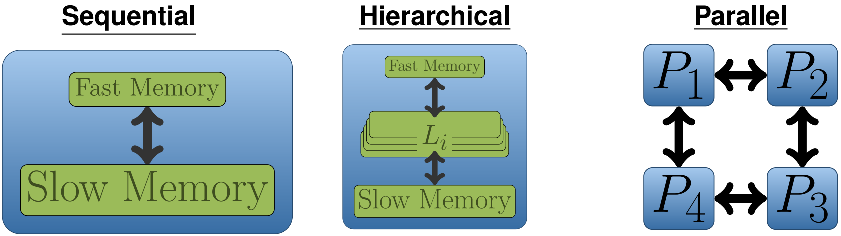
- **ordinary:** we compute all (n^3)

$$c_{ijk} = a_{ik} \cdot b_{kj}$$

(consequence: Strassen-like matrix-matrix multilplications are not allowed.)

Important to realize that this generalizes to

- **# of messages** (latency related) (as opposed to “total volume of messages”, bandwidth related)
- **parallel distributed**
- **hierarchical memories**



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Communication Cost for (OD) Matrix-Matrix Multiplication

Dense matrix-matrix multiplication moves n^2 data for n^3 computation.

$$n\left\{\underbrace{\boxed{C}}_{n}\right. += \boxed{A} \times \boxed{B}$$

- Computation cost is $2n^3$

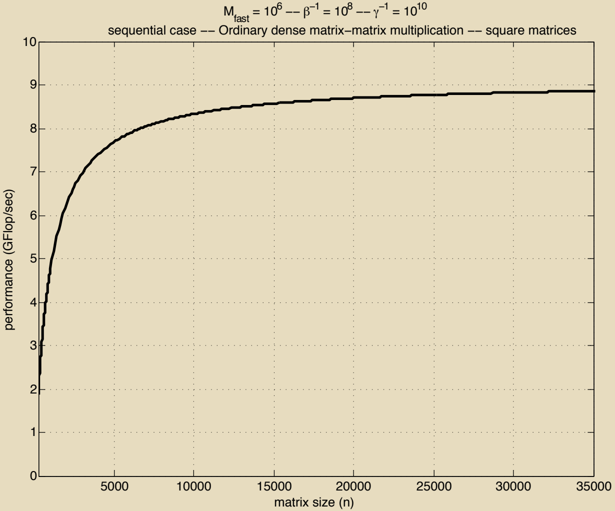
  for i=1:n, for j=1:n, for k=1:n, $c_{ij} = c_{ij} + a_{ik} b_{kj}$; end; end; end;

- Communication cost is $3n^2$

Conclusion of the study

When n increases, communication cost (n^2) becomes negligible with respect to computation cost (n^3).

$$\beta^{-1} = 10^8 \text{words/sec} \quad \gamma^{-1} = 10^{10} \text{flops/sec} \quad M = 10^6 \text{words}$$



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- **Easy fix:** A common easy fix is to block the matrix-matrix multiplication with square blocks so that the square blocks fit in cache.
Let M be the size of our cache. Let $b = \sqrt{\frac{M}{3}}$ (so that $3b^2 = M$). Then,

for i=1:n/b, for j=1:n/b, for k=1:n/b,

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Then, at each loop, we are moving $2b^2$ data and computing $2b^3$ so ...
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We see that the previous algorithm

- performs $2n^3$ floating point operations
- performs a volume of data movement of

$$\left(\frac{2\sqrt{3}}{\sqrt{M}}\right) n^3.$$

Therefore the time of a OD matrix-matrix multiplication is

$$\left(\frac{2\sqrt{3}}{\sqrt{M}}\right) \beta n^3 + 2\gamma n^3$$

(1) assuming no overlap between communication and computations; (2) with β being the time to move one unit of data (inverse of bandwidth) and γ being the time to perform one floating-point operation.

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If $\beta/\sqrt{M} << \gamma$ then, communication is negligible against computation.

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the number of words transferred between slow and fast memory is at least

$$\frac{1}{2\sqrt{2}} \frac{mnp}{\sqrt{M}} - M.$$

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Upper bound :: square tile matrix-matrix multiplication

The number of words transferred between slow and fast memory is at most

$$3.46\left(\frac{n^3}{\sqrt{M}}\right).$$

Lower Bound :: Irony, Toledo, and Tiskin, 2004

The number of words transferred between slow and fast memory is at least

$$0.35\left(\frac{n^3}{\sqrt{M}}\right) - M.$$

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We know that $(?)$ is between 0.35 and 3.46.