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Introduction

The theory of inverse scattering for acoustic and electromagnetic waves, is an active area of research with significant developments in the past few years. Inverse problems consist in getting informations on a physical object from measurement data. More specifically, the inverse scattering problem is the problem of finding characteristics of an unknown object referred to as scatterer (location, shape, material properties,...) from measurement data of acoustic or electromagnetic waves scattered by this object. The question is not only to detect objects like radar and sonar can do, but also to identify them.

Inverse problems are not easy to solve since they belong to the class of ill-posed problems as defined by Hadamard. Indeed, a solution may not exist but even if it is the case, the solution does not depend continuously on the data. Such problems require the use of regularisation schemes to be solved numerically. The first successful algorithms for solving inverse problems of target identification are based on either weak-scattering approximation or on non-linear optimization techniques. The main problem of the weak-scattering approximation is that it ignores polarization effects and consequently, it cannot be used in complex environments. Both the above methods also rely on a priori knowledge on physical properties of the scatterer (for instance if the object is penetrable or not) and such information is not in general available. Moreover, nonlinear optimization techniques are numerically expensive. A short survey of these methods can be found in [22].

These issues have naturally brought to search for new target identification algorithms that are easy to implement and need little a priori information on the scatterer. This has led to develop a new class of methods called qualitative methods in inverse scattering theory [8] to solve time harmonic inverse scattering problems for acoustic and electromagnetic waves. The main representatives of this class are the linear sampling method (LSM) [24, 29], the factorization method [40, 36] and the method of singular sources [47, 48]. These methods allow the reconstruction of the shape of an obstacle from a knowledge of multi-static data at a fixed frequency. They are based on suitably solving a linear ill-posed integral equation, known as the far field equation. Another advantage of these methods compared to iterative methods for instance is that they avoid solving the direct scattering problem and they do not make use of any a priori information on the geometry or physics of scatterers.

The theoretical study of the LSM for impenetrable obstacles with perfectly conducting boundary leads to investigate the eigenvalue problem for $-\Delta$ in the case of acoustic waves and for $\operatorname{curl}\operatorname{curl}$ in the case of electromagnetic waves in the scatterer and with Dirichlet boundary conditions. It is pointed out that the method fails when the square of the wave number is an eigenvalue for this eigenvalue problem i.e. k^2 is a Dirichlet eigenvalue or a

Maxwell eigenvalue. Since it is well-known that Dirichlet and Maxwell eigenvalues exist and more importantly, form a discrete set, then they are easy to avoid in order to use the linear sampling method. In the case of the scattering by penetrable objects, Colton and Kirsch [23, 37] show that the linear sampling method leads to study a new type of problem called the *interior transmission eigenvalue problem*. The eigenvalues of this interior problem are called *transmission eigenvalues* and need to be excluded in the theory of the LSM similarly to Dirichlet and Maxwell eigenvalues in the case of impenetrable obstacles. Naturally, the study of the interior transmission problem has become a subject of great interest, first to clarify the role of transmission eigenvalues in the LSM but later it became also interesting in the identification problem of getting information on physical properties of the scatterer [14, 11]. Here, the main feature of transmission eigenvalues is that not only they can give information on the physical properties of the scattering object [7, 15, 11] but they can also be computed from the far field data [13].

Thus, three main questions can be asked about transmission eigenvalues. The first two questions are related to the use of the LSM. It is essential to know if they exist and if they behave like Dirichlet or Maxwell eigenvalue by forming a discrete set. Finally, the last question to be asked is if it is possible to take benefit of these transmission eigenvalues to derive some estimates on the material characteristics such as its index of refraction or identify the presence of faults like cavities of inclusions and this could be useful in non-destructive testing.

Although simply stated, the interior transmission problem is not covered by the standard theory of elliptic partial differential equations since as it stands it is neither elliptic nor self-adjoint. Two main approaches have arouse for the study of the interior transmission problem: integral equation methods [25, 37] and variational methods [10, 14, 49]. Whereas the discreteness of the set of transmission eigenvalues can "easily" be proven using the analytical Fredholm theory, a method to prove existence has been more laborious to find. The existence of transmission eigenvalues has first been proven for spherically stratified medium in [26], and much later Päivärinta and Sylvester proved it in the general case of scalar isotropic media in [43] provided the index of refraction is bounded away from 1. Several results on the existence of an infinite discrete set of transmission eigenvalues have now been established in more general cases for both acoustic and electromagnetic waves and with less restrictive hypothesis on the index of refraction [17]. It has also been shown in the case where the medium contains a cavity [12] i.e. subregions with index of refraction the same as of the background medium. Moreover, there still persists a restriction on the index of refraction n: $n-1$ cannot change sign. However, recently, Sylvester [50] and Bonnet-Ben Dhia Chesnel and Haddar [5] have been able to prove the discreteness of the set of transmission eigenvalues provided the sign requirement holds only on n in a neighborhood of the boundary of the scatterer.

The aim of this thesis is to contribute to the study of the interior transmission problem and to answer some open problems on this subject.

Outline of the thesis

In the first two chapters, we give the basis of the interior transmission problem for Maxwell's equations. Chapter 1 is devoted to introduce the notions of the interior transmission problem and transmission eigenvalues and see how they arise in the inverse scattering theory. After recalling the direct scattering problem for electromagnetic waves, we give a general survey of the linear sampling method and make the parallel between Maxwell eigenvalues for impenetrable obstacles and transmission eigenvalues for penetrable objects. This first chapter is ended by giving the state of the art on the interior transmission problem, difficulties that can be encountered and the different methods that have been developed to study this problem. The chapter 2 is dedicated to the study of the simple case of an inhomogeneous isotropic medium. The variational method used with a fourth order formulation of the problem and the obtained results are described in details. Existence of an infinite discrete set of transmission eigenvalues is established as well as estimates on the first transmission eigenvalue with respect to the index of refraction. A new result of continuity of the first transmission eigenvalue with respect to n is also given as well as a theorem that characterizes transmission eigenvalues by far field data.

Chapters 3 and 4 study the interior transmission problem for two new types of obstacles. Chapter 3 considers inhomogeneous media containing a cavity and Chapter 4, inhomogeneous media containing a perfect conductor. From practical point of view, the importance of these problems lies in the possibility of using transmission eigenvalues to detect anomalies inside inhomogeneous media in non-destructive testing. This type of problem is considered in [41] where the authors recover the obstacle embedded in an inhomogeneous medium. In Chapter 3, the results on transmission eigenvalues of a domain containing a cavity for acoustic waves in [12] are extended to Maxwell's equations. Besides the technicality inherent to Maxwell's equations, the main difficulty here is in proving the equivalence between weak and variational solutions and also lies in a second part in the fact that the variational space depends of the frequency. Chapter 4 is devoted to the study of the interior transmission problem corresponding to the scattering of an inhomogeneous (possibly anisotropic) medium of $\mathbb{R}^d$ ($d = 2$ or $d = 3$) containing a perfect conductor. Existence and discreteness of transmission eigenvalues are established for both the isotropic and anisotropic case. In the first case, the main difficulty of the problem is to define the appropriate space in which the interior transmission problem is well-posed whereas for the anisotropic case, the difficulty is to find an equivalent Fredholm formulation of the problem.

In Chapter 5, we develop a new approach for the study of the interior transmission problem based on a surface integral equation formulation which for the moment is only done in the scalar case. The main original motivation behind this study was the design of a numerical method to solve ITP in the case of piece-wise constant index of refraction and compute transmission eigenvalues for general geometries. This numerical study is presented in Chapter 6. We adopted the integral equation approach since an efficient forward solver for scattering problems based on this technique is already developed at CERFACS, namely the CESC software. This study presents some theoretical interests in the use of non standard results on potentials established with the theory of pseudo-differential operators. Another important interest is related to the study of the ITP for relaxed assumptions on the sign of the contrasts. However, this approach only enables to show the discreteness of the set of transmission eigenvalues.

The last chapter is devoted to the numerical computation of transmission eigenvalues by two methods. The first one is inspired by the approach of the previous chapter and computes transmission eigenvalues by solving an eigenvalue problem for a surface integral operator. The second method uses the characterization of transmission eigenvalues from far field data established in Chapter 2. Numerical examples are given first for electromagnetic waves in 2 dimensions and finally for 3D electromagnetic waves and for both homogeneous media and for media containing a cavity.

Chapter 1

From the scattering problem to the interior transmission problem

This chapter is devoted to introduce the notions of the interior transmission problem and transmission eigenvalues and see how they arise in the inverse scattering theory. While studying the problem of finding the shape of a penetrable obstacle, it appears that the obstacle can be invisible to some incident waves at particular frequencies. Those frequencies that are called transmission eigenvalues can be compared in some sense to Maxwell eigenvalues that describe a resonance phenomenon in the case of a bounded inclusion.

After recalling the context of the direct and inverse scattering problems for penetrable and impenetrable objects, we give main results on particular entire solutions to Maxwell's equations called Herglotz wave pairs.

Next, we introduce an effective method to retrieve the shape of an obstacle from far field measurements called the Linear Sampling Method first described by Colton and Kirsch in 1996 in [24]. One positive point of this method is that it requires a few a priori knowledge on the obstacle: for instance we do not need to know if the obstacle is penetrable or impenetrable. However, in both cases, the method fails for some particular frequencies that correspond with Maxwell eigenvalues when the obstacle is impenetrable and the so-called transmission eigenvalues when the obstacle is penetrable.

Transmission eigenvalues are defined from a singular transmission problem where two fields with same boundary data satisfy both Maxwell's equations for two different wave numbers. Of particular interest is the existence of such eigenvalues but also the distribution of the spectrum to make sure that transmission eigenvalues can be easily avoided in the use of the Linear Sampling Method. It also appears that they provide some qualitative information on the index of refraction of the medium. In the last section of this chapter, we shall see that this non classical transmission problem is not that easy to solve in a way that the usual variational formulation is not appropriate in this case. Indeed, this problem is not covered by the standard theory of elliptic partial differential equations since as it stands it is neither elliptic nor self-adjoint. Finally, we give a general survey of the state of the art concerning the interior transmission problem and transmission eigenvalues.

1.1 Direct scattering problems

1.1.1 Maxwell's equations

The following has been taken from [26] and [45]. Consider electromagnetic wave propagation in an isotropic medium in $\mathbb{R}^3$ with space independent electric permittivity ε, magnetic permeability μ and electric conductivity σ. The electromagnetic wave is described by the electric field $\mathcal{E}$ and the magnetic field $\mathcal{H}$ satisfying *Maxwell's equations*

$$\operatorname{curl}\mathcal{E} + \mu\frac{\partial \mathcal{H}}{\partial t} = 0,$$

$$\operatorname{curl}\mathcal{H} - \varepsilon\frac{\partial \mathcal{E}}{\partial t} = \sigma\mathcal{E}.$$

For time-harmonic electromagnetic waves of the form

$$\mathcal{E}(x,t) = \Re\left\{\varepsilon_0^{-1/2}\mathbf{E}(x)e^{-\mathrm{i}\omega t}\right\},$$

$$\mathcal{H}(x,t) = \Re\left\{\mu_0^{-1/2}\mathbf{H}(x)e^{-\mathrm{i}\omega t}\right\},$$

with frequency $\omega > 0$, it implies that the complex valued space dependent parts $\mathbf{E}$ and $\mathbf{H}$ satisfy the reduced Maxwell's equations

$$\operatorname{curl}\mathbf{E} - \mathrm{i}\omega\frac{\sqrt{\varepsilon_0}}{\sqrt{\mu_0}}\mu(x)\mathbf{H} = 0, \quad \operatorname{curl}\mathbf{H} + (\mathrm{i}\omega\varepsilon(x) - \sigma(x))\frac{\sqrt{\mu_0}}{\sqrt{\varepsilon_0}}\mathbf{E} = 0.$$

Now consider the scattering of time-harmonic waves by obstacles surrounded by a homogeneous medium with vanishing conductivity $\sigma = 0$ and constant magnetic permeability μ_0 and electric conductivity ε_0.

Let us assume that the obstacle occupies a domain D which is a bounded domain such that $\mathbb{R}^3\backslash\bar{D}$ is connected and $\Gamma := \partial D$ is piece-wise smooth. Let us denote by ν the unit outward normal to the boundary Γ.

We must distinguish between the two cases of *penetrable* or *impenetrable* objects. First consider the scattering by a perfectly conducting obstacle i.e. where the tangential component of the electric field of the total wave vanishes on Γ. In this case, Maxwell's equations become

$$\operatorname{curl}\mathbf{E} - \mathrm{i}k\mathbf{H} = 0, \quad \operatorname{curl}\mathbf{H} + \mathrm{i}k\mathbf{E} = 0 \tag{1.1}$$

where $k^2 := \omega^2\mu_0\varepsilon_0$. More precisely, let us consider the scattering of a given incoming wave $\mathbf{E}^i$, $\mathbf{H}^i$ by a perfect conductor D. Assume that $\mathbf{E}^i$, $\mathbf{H}^i$ is a solution to Maxwell's equations (1.1) in all $\mathbb{R}^3$ and that the total field is defined by

$$\mathbf{E} := \mathbf{E}^i + \mathbf{E}^s,$$

$$\mathbf{H} := \mathbf{H}^i + \mathbf{H}^s,$$

where $\mathbf{E}^s$, $\mathbf{H}^s$ is the scattered field satisfying *Silver-Müller radiation condition*

$$\lim_{r\to+\infty}(\mathbf{H}^s \times x - r\mathbf{E}^s) = 0$$

uniformly in all directions $\hat{x} := x/|x|$ and where $r = |x|$. The total wave must satisfy Maxwell's equations (1.1) in the exterior domain $\mathbb{R}^3\backslash\bar{D}$ and the perfect conductor boundary condition $\nu \times \mathbf{E} = 0$ on Γ. To summarize, we have the following system

$$\begin{cases} \operatorname{curl}\mathbf{E} - \mathrm{i}k\mathbf{H} = 0, \quad \operatorname{curl}\mathbf{H} + \mathrm{i}k\mathbf{E} = 0 & \text{in } \mathbb{R}^3\backslash\bar{D} \\ \nu \times \mathbf{E} = 0 & \text{on } \partial D \\ \mathbf{E} := \mathbf{E}^i + \mathbf{E}^s, \quad \mathbf{H} := \mathbf{H}^i + \mathbf{H}^s & \\ \lim\limits_{r\to+\infty} (\mathbf{H}^s \times x - r\mathbf{E}^s) = 0. & \end{cases} \tag{1.2}$$

Remark 1.1.1. *The Silver-Müller radiation condition plays the same role as the Sommerfeld radiation condition for Helmholtz equation and ensures uniqueness for the solutions to scattering problems. It characterizes outgoing waves. A solution to Maxwell's equations satisfying the Silver-Müller radiation condition is called a radiating solution.*

In the case of a penetrable object, the total wave must also satisfy Maxwell's equations in D but with a different wave number. Consider an obstacle D with variable magnetic permeability $\mu(x)$, electric permittivity $\varepsilon(x)$ and electric conductivity $\sigma(x) > 0$ for $x \in D$ different from the magnetic permeability μ_0, the electric permittivity ε_0 and the electric conductivity $\sigma = 0$ of the surrounded medium $\mathbb{R}^3\backslash\bar{D}$. The magnetic permeability $\mu(x)$ and the electric permittivity $\varepsilon(x)$ are 3×3 real symmetric matrix valued functions. In this case, the equations satisfied by the total fields are

$$\operatorname{curl}\mathbf{E} - \mathrm{i}\omega\sqrt{\frac{\varepsilon_0}{\mu_0}}\mu(x)\mathbf{H} = 0$$

and

$$\operatorname{curl}\mathbf{H} + (\mathrm{i}\omega\varepsilon(x) - \sigma(x))\sqrt{\frac{\mu_0}{\varepsilon_0}}\mathbf{E} = 0.$$

Let us define the relative permittivity and permeability by

$$\varepsilon_r := \frac{1}{\varepsilon_0}\left(\varepsilon + \mathrm{i}\frac{\sigma}{\omega}\right) \quad \text{and} \quad \mu_r = \frac{\mu}{\mu_0}.$$

If we still denote $\sqrt{\varepsilon_0}\mathbf{E}$ and $\sqrt{\mu_0}\mathbf{H}$ by respectively $\mathbf{E}$ and $\mathbf{H}$, we obtain the final version of the direct scattering problem for penetrable objects

$$\begin{cases} \operatorname{curl}\mathbf{E} - \mathrm{i}k\mu_r(x)\mathbf{H} = 0, \quad \operatorname{curl}\mathbf{H} + \mathrm{i}k\varepsilon_r(x)\mathbf{E} = 0 \quad \text{in } \mathbb{R}^3 \\ \mathbf{E} := \mathbf{E}^i + \mathbf{E}^s, \quad \mathbf{H} := \mathbf{H}^i + \mathbf{H}^s \\ \lim\limits_{r\to+\infty} (\mathbf{H}^s \times x - r\mathbf{E}^s) = 0 \end{cases} \tag{1.3}$$

where k is the wave number $k = \sqrt{\varepsilon_0\mu_0}\omega$. Note that $\varepsilon_r = \mu_r = 1$ in $\mathbb{R}^3\backslash\bar{D}$.

It can be shown [26] that the scattered field has the following asymptotic expansion, far from the scatterer, given by

$$\mathbf{E}^s(x) = \frac{e^{\mathrm{i}k|x|}}{|x|}\mathbf{E}_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \; |x| \to \infty,$$

$$\mathbf{H}^s(x) = \frac{e^{\mathrm{i}k|x|}}{|x|}\mathbf{H}_\infty(\hat{x}) + O\left(\frac{1}{|x|^2}\right),\ |x| \to \infty,$$

uniformly in $\hat{x} = x/|x| \in \Omega$ where

$$\Omega := \left\{\hat{x} \in \mathbb{R}^3/|\hat{x}| = 1\right\}$$

is the unit sphere in $\mathbb{R}^3$. $\mathbf{E}_\infty$ and $\mathbf{H}_\infty$ are respectively called the *electric far field pattern* and *magnetic far field pattern.* They satisfy

$$\mathbf{H}_\infty = \nu \times \mathbf{E}_\infty \quad \text{and} \quad \nu \cdot \mathbf{E}_\infty = \nu \cdot \mathbf{H}_\infty = 0 \tag{1.4}$$

with the unit outward normal ν on Ω. Moreover, we have

$$\mathbf{E}_\infty(\hat{x}) = \frac{\mathrm{i}k}{4\pi}\hat{x} \times \int_{\partial D} \left(\nu(y) \times \mathbf{E}^s(y) + (\nu(y) \times \mathbf{H}^s(y)) \times \hat{x}\right) e^{-\mathrm{i}k\hat{x}\cdot y} ds(y), \tag{1.5}$$

$$\mathbf{H}_\infty(\hat{x}) = \frac{\mathrm{i}k}{4\pi}\hat{x} \times \int_{\partial D} \left(\nu(y) \times \mathbf{H}^s(y) - (\nu(y) \times \mathbf{E}^s(y)) \times \hat{x}\right) e^{-\mathrm{i}k\hat{x}\cdot y} ds(y). \tag{1.6}$$

The direct scattering problem consists in finding the solution $\mathbf{E}$, $\mathbf{H}$ to the systems (1.2) or (1.3) when the obstacle is known i.e. from the knowledge of D, ε_r and μ_r. As the purpose of this work is not solving the direct problem, we just recall here the main results. It is shown in [26] that this problem is well-posed and that there exists a unique solution that depends continuously on the data. We can state this property in the Hilbert space

$$H(\operatorname{curl}, D) := \left\{\mathbf{u} \in L^2(D)^3/\operatorname{curl}\mathbf{u} \in L^2(D)^3\right\}.$$

The following theorem is extracted from [45].

Theorem 1.1.1. *Assume that ε_r and μ_r are piece-wise smooth functions (in $\mathcal{C}^1$ for instance) such that their discontinuity surfaces are Lipschitz. Then there exist a unique solution $\mathbf{E} \in H(\operatorname{curl}, \mathbb{R}^3)$ to the scattering problem* (1.3) *and a unique solution $\mathbf{E} \in H_{\mathrm{loc}}(\operatorname{curl}, \mathbb{R}^3 \backslash \overline{D})$ to the scattering problem* (1.9)*.*

Let us now recall one useful theorem concerning radiating solutions to Maxwell's equations in the exterior domain ([26] for continuous solutions, [45] for solutions in $H_{loc}(\operatorname{curl}, \mathbb{R}^3\backslash\bar{D})$).

Theorem 1.1.2. *Let $\mathbf{E}, \mathbf{H} \in H_{loc}(\operatorname{curl}, \mathbb{R}^3\backslash\bar{D})$ be a radiating solution to Maxwell's equations for which either the electric or the magnetic far field pattern vanishes identically. Then, $\mathbf{E} = \mathbf{H} = 0$ in $\mathbb{R}^3\backslash\bar{D}$.*

On the contrary, concerning the inverse problem, from the knowledge of the far field pattern, we want to recover the shape of the obstacle in the case of impenetrable objects, but also information on the physical properties of the object, for instance, the index of refraction or the presence of defects when the obstacle is penetrable. The main difficulty is that we want to use a few a priori knowledge on the obstacle.

1.1.2 Herglotz wave pairs

In this section, we give some results on a particular entire solution to Maxwell's equations and some approximation properties. Let $g \in L^2_t(\Omega)$ where $L^2_t(\Omega)$ is the set of tangential functions in $L^2(\Omega)$

$$L^2_t(\Omega) := \left\{u : \Omega \to \mathbb{R}^3 / u \in L^2(\Omega), u(d) \cdot d = 0,\, d \in \Omega\right\},$$

where Ω is the unit sphere in $\mathbb{R}^3$.

Definition 1.1.1. *An electromagnetic Herglotz pair is a pair of vector fields of the form*

$$\begin{aligned}\mathbf{E}_g(x) &:= \int_\Omega e^{\mathrm{i}kx\cdot d} g(d) ds(d),\\ \mathbf{H}_g(x) &:= \frac{1}{\mathrm{i}k} \operatorname{curl} \mathbf{E}_g(x)\end{aligned}$$

for all $x \in \mathbb{R}^3$ where the square integrable tangential field $g \in L^2_t(\Omega)$ on the unit sphere is called the Herglotz kernel of the pair $\mathbf{E}_g$, $\mathbf{H}_g$.

Electromagnetic Herglotz pairs obviously represent entire solutions to Maxwell's equations. Furthermore, considering the vector Herglotz wave function

$$\mathbf{E}_g(x) := \int_\Omega e^{\mathrm{i}kx\cdot d} g(d) ds(d),$$

we have that

$$\operatorname{div} \mathbf{E}_g(x) = \mathrm{i}k \int_\Omega e^{\mathrm{i}kx\cdot d} d \cdot g(d) ds(d)$$

and we deduce that the property of the kernel g to be tangential is equivalent to $\operatorname{div} \mathbf{E}_g = 0$ in $\mathbb{R}^3$. The next result is also shown in [26].

Theorem 1.1.3. *Assume that the Herglotz wave pair is identically equal to zero in all $\mathbb{R}^3$ i.e. $\mathbf{E}_g = \mathbf{H}_g = 0$, then $g = 0$.*

The following lemma gives a result on superposition of solutions to Maxwell's equations. We consider the scattering of electromagnetic plane waves

$$\mathbf{E}^i(x, d, p) := \frac{\mathrm{i}}{k} \operatorname{curl} \operatorname{curl} p e^{\mathrm{i}kx\cdot d} = \mathrm{i}k(d \times p) \times d e^{\mathrm{i}kx\cdot d}, \tag{1.7}$$

$$\mathbf{H}^i(x, d, p) := \operatorname{curl} p e^{\mathrm{i}kx\cdot d} = \mathrm{i}kd \times p e^{\mathrm{i}kx\cdot d} \tag{1.8}$$

where

▷ the constant unit vector d gives the direction of propagation and

▷ the constant vector p gives the polarization.

Let us denote by $\mathbf{E}^s(x, d, p)$, $\mathbf{H}^s(x, d, p)$ the corresponding scattered field and by $\mathbf{E}^\infty(\hat{x}, d, p)$, $\mathbf{H}^\infty(\hat{x}, d, p)$ the corresponding far field pattern.

Theorem 1.1.4. *[26] Given $g \in L^2_t(\Omega)$, the solution to the perfect conductor scattering problem for the incident wave*

$$\tilde{\mathbf{E}}^i(x) = \int_\Omega \mathbf{E}^i(x, d, g(d))ds(d),$$
$$\tilde{\mathbf{H}}^i(x) = \int_\Omega \mathbf{H}^i(x, d, g(d))ds(d)$$

is given by

$$\tilde{\mathbf{E}}^s(x) = \int_\Omega \mathbf{E}^s(x, d, g(d))ds(d),$$
$$\tilde{\mathbf{H}}^s(x) = \int_\Omega \mathbf{H}^s(x, d, g(d))ds(d)$$

for $x \in \mathbb{R}^3 \backslash \bar{D}$ and has the far field pattern

$$\tilde{\mathbf{E}}^\infty(\hat{x}) = \int_\Omega \mathbf{E}^\infty(\hat{x}, d, g(d))ds(d),$$
$$\tilde{\mathbf{H}}^\infty(\hat{x}) = \int_\Omega \mathbf{H}^\infty(\hat{x}, d, g(d))ds(d)$$

for $\hat{x} \in \Omega$.

Remark 1.1.2. *In particular, for $g \in L^2_t(\Omega)$, we can write*

$$\tilde{\mathbf{E}}^i(x) = \mathrm{i}k \int_\Omega g(d)e^{\mathrm{i}kx\cdot d}ds(d),$$
$$\tilde{\mathbf{H}}^i(x) = \mathrm{curl} \int_\Omega g(d)e^{\mathrm{i}kx\cdot d}ds(d)$$

for $x \in \mathbb{R}^3$, i.e. $\tilde{\mathbf{E}}^i$, $\tilde{\mathbf{H}}^i$ represents an electromagnetic Herglotz pair with kernel $\mathrm{i}kg$.

Let us show that solutions to Maxwell's equations in D can be approximated by Herglotz wave functions. We recall the definition of the Hilbert space

$$H(\mathrm{curl}\,, D) := \left\{ \mathbf{u} \in L^2(D)^3 / \mathrm{curl}\, \mathbf{u} \in L^2(D)^3 \right\}$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathrm{curl}} = (\mathbf{u}, \mathbf{v})_{L^2(D)} + (\mathrm{curl}\, \mathbf{u}, \mathrm{curl}\, \mathbf{v})_{L^2(D)}$ and the corresponding norm $|| \cdot ||_{\mathrm{curl}}$.

First remark that $\mathbf{E}$ solution to $\mathrm{curl}\,\mathrm{curl}\, \mathbf{E} - k^2\mathbf{E} = 0$ is equivalent to $\mathbf{E}$, $\mathbf{H} := \frac{1}{\mathrm{i}k}\mathrm{curl}\, \mathbf{E}$ solutions to Maxwell's equations

$$\mathrm{curl}\, \mathbf{E} - \mathrm{i}k\mathbf{H} = 0,$$
$$\mathrm{curl}\, \mathbf{H} + \mathrm{i}k\mathbf{E} = 0.$$

Now define

$$M(D) := \left\{ \mathbf{u} \in H(\mathrm{curl}\,, D) / \mathrm{curl}\,\mathrm{curl}\, \mathbf{u} - k^2\mathbf{u} = 0 \text{ in } D \right\}.$$

We can now state the following theorem, proven in [27].

Theorem 1.1.5. *Assume that $\mathbb{R}^3 \backslash \bar{D}$ is connected. Then the set of electric Herglotz functions $\mathbf{E}_g$ with $g \in L^2_t(\Omega)$ is dense in the space $M(D)$ with respect to the $H(\mathrm{curl}\,, D)$ norm.*

1.2 The Linear Sampling Method

The Linear Sampling Method (LSM) has been first introduced by D. Colton and A. Kirsch in [24] and is a method used to find the shape of an obstacle. The advantages of that method is that it requires a few a priori information on the scatterer and it also avoids solving the direct scattering problem. We will consider here both cases of a penetrable and an impenetrable obstacle. The study of the LSM will highlight resonance phenomena described by Maxwell eigenvalues in the case of an impenetrable object and by transmission eigenvalues in the case of a penetrable object. We refer for instance to [29, 26, 4, 21, 3] for more details on the linear sampling method.

1.2.1 Presentation of the method for impenetrable obstacles

Let us first consider the direct scattering problem by a perfect conductor D. We recall that the total field satisfies

$$\begin{cases} \operatorname{curl}\mathbf{E} - \mathrm{i}k\mathbf{H} = 0,\ \operatorname{curl}\mathbf{H} + \mathrm{i}k\mathbf{E} = 0 & \text{in } \mathbb{R}^3\backslash\bar{D} \\ \nu\times\mathbf{E} = 0 & \text{on } \partial D \\ \mathbf{E} := \mathbf{E}^i + \mathbf{E}^s,\ \mathbf{H} := \mathbf{H}^i + \mathbf{H}^s & \\ \lim\limits_{r\to+\infty}(\mathbf{H}^s\times x - r\mathbf{E}^s) = 0 & \end{cases} \tag{1.9}$$

where $\mathbf{E}^s$, $\mathbf{H}^s$ is the scattered field and $\mathbf{E}^i$, $\mathbf{H}^i$ is an incident plane wave defined by (1.7) and (1.8).

We assume that we know the far field pattern $\mathbf{E}_\infty(\hat{x}, d, p)$ for all $d, x \in \Omega$ generated by $\mathbf{E}^i$, $\mathbf{H}^i$. Since far field patterns are tangential fields (see (1.4)), we can define the *far field operator* $\mathcal{F} : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(\mathcal{F}g)(\hat{x}) := \int_\Omega \mathbf{E}_\infty(\hat{x}, d, g(d))ds(d),\ \hat{x}\in\Omega,$$

for all $g \in L^2_t(\Omega)$.

Since the scattered field depends linearly on the polarization of the incident field, $\mathcal{F}$ is a linear operator.

Remark 1.2.1. *By superposition and using Theorem 1.1.2, $\mathcal{F}g$ is the electric far field pattern of the scattered field $\mathbf{E}^s_g$ generated by the incident electric field of an electromagnetic Herglotz pair with kernel* $\mathrm{i}kg$ *i.e.* $\mathbf{E}^s$ *is solution to* (1.9) *with* $\nu\times\mathbf{E}^s = -\mathrm{i}k\nu\times\mathbf{E}_g$ *on* Γ *where* $\mathbf{E}_g$ *is the Herglotz wave function with kernel* g*.*

An electric dipole with polarization q is defined by

$$\mathbf{E}_e(x, z, q) := \frac{\mathrm{i}}{k}\operatorname{curl}_x\operatorname{curl}_x q\Phi_k(x, z), \qquad \mathbf{H}_e(x, z, q) := \operatorname{curl}_x q\Phi_k(x, z), \tag{1.10}$$

where

$$\Phi_k(x, z) := \frac{e^{\mathrm{i}k|x-z|}}{4\pi|x-z|}$$

is the fundamental solution to the Helmholtz equation. In particular, $\mathbf{E}_e(\cdot, z, q)$ is a radiating solution to Maxwell's equations outside a neighborhood of z and the corresponding far field pattern is given by

$$\mathbf{E}_{e,\infty}(\hat{x}, z, q) = \frac{\mathrm{i}k}{4\pi}(\hat{x} \times q) \times \hat{x} e^{-\mathrm{i}k\hat{x}\cdot z}. \tag{1.11}$$

The LSM relies on the *far field equation* defined by

$$(\mathcal{F}g)(\hat{x}) = \mathbf{E}_{e,\infty}(\hat{x}, z, q). \tag{1.12}$$

If $z \in D$ and g_z is a solution to the far field equation (1.12), since the two far fields are equal, we deduce that the scattered field $\mathbf{E}_g^s$ corresponding to the incident wave $\mathrm{i}k\mathbf{E}_g$ (Herglotz wave function with kernel $\mathrm{i}kg$) coincides with the electric dipole $\mathbf{E}_e(\cdot, z, q)$ in $\mathbb{R}^3 \backslash \bar{D}$ i.e.

$$\mathbf{E}_g^s(x) = \mathbf{E}_e(x, z, q) \quad \text{for } x \in \mathbb{R}^3 \backslash \bar{D}.$$

From the trace theorem, they also coincide on the boundary Γ and as a consequence, we have

$$-\mathrm{i}k\nu \times \mathbf{E}_g = \nu \times \mathbf{E}_e(\cdot, z, q) \text{ on } \Gamma.$$

However, $||\nu \times \mathbf{E}_e(\cdot, z, q)||_{H^{-1/2}(\mathrm{div}\,,\Gamma)}$ is not bounded as $z \in D$ tends to Γ. As a consequence,

$$\lim_{z \to \Gamma} ||\nu \times \mathbf{E}_g||_{H^{-1/2}(\mathrm{div}\,,\Gamma)} = \infty.$$

Hence,

$$||g_z||_{L_t^2(\Omega)} \to \infty$$

and we see that the boundary of D is indicated by the growth of $||g_z||_{L_t^2(\Omega)}$. Later, after we have discussed in more details how to solve the far field pattern, we shall show that if $z \notin \overline{D}$, the procedure for computing g will also result in a function with large norm. Thus, we can say that the behavior of $||g_z||_{L_t^2(\Omega)}$ determines Γ and consequently the shape of the obstacle D.

The general scheme to find Γ is now clear. We take a sample of points z in a region of $\mathbb{R}^3$ where we expect D to lie. An outline of D is then established by regions where the norm of g is small.

The problem is that in general there does not exist a solution g_z to (1.12). This follows from the fact that if g_z is a solution to the far field equation, then the Herglotz wave function $\mathrm{i}k\mathbf{E}_{g_z}$ is the solution to the interior boundary value problem

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_z - k^2\mathbf{u}_z = 0 & \text{in } D \\ \nu \times \mathbf{u}_z = -\nu \times \mathbf{E}_e(\cdot, z, q) & \text{on } \Gamma \end{cases} \tag{1.13}$$

which is in general not possible. However, we can prove (see Theorem 1.2.2) that the operator $\mathcal{F}$ is injective provided k^2 is not a Maxwell eigenvalue whose definition is given below.

Definition 1.2.1. *A real λ is a called a Maxwell eigenvalue for D if there exists $\mathbf{v} \in H(\operatorname{curl}, D)$ a non trivial solution to*

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{v} - \lambda\mathbf{v} = 0 & \text{in } D \\ \mathbf{v} \times \nu = 0 & \text{on } \Gamma. \end{cases} \tag{1.14}$$

Remark 1.2.2. *We remark that if $\lambda \neq 0$, then existence of a non trivial solution $\mathbf{v} \in H(\operatorname{curl}, D)$ to (1.14) is equivalent to existence of non trivial solution $\mathbf{w} = \operatorname{curl}\mathbf{v} \in H(\operatorname{curl}, D)$ to*

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{w} - \lambda\mathbf{w} = 0 & \text{in } D \\ \operatorname{curl}\mathbf{w} \times \nu = 0 & \text{on } \Gamma. \end{cases}$$

Properties of Maxwell eigenvalues are well-known and can be found for example in [45]. We recall here the main properties of these eigenvalues.

Theorem 1.2.1. *There is an infinite discrete set of eigenvalues $\lambda_j > 0$, $j = 1, 2, \ldots$ and corresponding eigenfunctions $\mathbf{v}_j \in H(\operatorname{curl}, D)$, $\mathbf{v}_j \neq 0$ with tangential trace in $L^2(\Gamma)^3$, such that*

(a) (1.14) *is satisfied,*

(b) $0 < \lambda_1 \leq \lambda_2 \leq \ldots$,

(c) $\lim_{j\to\infty} \lambda_j = \infty$,

(d) $\mathbf{v}_j$ *is orthogonal to* $\mathbf{v}_\ell$ *in the* $(\cdot,\cdot)_{L^2(D)}$ *inner product if* $j \neq \ell$.

Then, we can state the following theorem.

Theorem 1.2.2. *$\mathcal{F}$ is injective provided k^2 is not a Maxwell eigenvalue for D.*

Proof. Assume that k^2 is not a Maxwell eigenvalue for D and that $\mathcal{F}g = 0$. From Remark 1.2.1 and Theorem 1.1.2, we deduce that on the boundary Γ, $-ik\nu \times \mathbf{E}_g = 0$ and since k^2 is not a Maxwell eigenvalue for D, we deduce that $\mathbf{E}_g = 0$ is equal to zero. From Theorem 1.1.3, $g = 0$ and consequently $\mathcal{F}$ is injective. □

We are confronted to an ill-posed problem and since the operator is injective, the natural approach is to treat this problem using a regularization method.

1.2.2 The main theorem

We first recall that the trace $\nu \times \mathbf{u}|_{\partial D}$ of a function $\mathbf{u} \in H(\operatorname{curl}, D)$ is in the Hilbert space defined by

$$H^{-1/2}_{\operatorname{div}}(\Gamma) := \left\{\mathbf{u} \in H^{-1/2}(\Gamma) / \operatorname{div}\mathbf{u} \in H^{-1/2}(\Gamma)\right\}.$$

Its dual is $H^{-1/2}_{\operatorname{curl}}(\Gamma)$ defined by

$$H^{-1/2}_{\operatorname{curl}}(\Gamma) := \left\{\mathbf{u} \in H^{-1/2}(\Gamma) / \operatorname{curl}\mathbf{u} \in H^{-1/2}(\Gamma)\right\}.$$

Let us now define the bounded linear operator $\mathcal{B} : H^{-1/2}_{\mathrm{div}}(\Gamma) \to L^2_t(\Omega)$ which maps the boundary data $f \in H^{-1/2}_{\mathrm{div}}(\Gamma)$ to the far field pattern $\mathbf{E}_\infty$ of the radiating solution $\mathbf{E}^s$ to

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E}^s - k^2\mathbf{E}^s = 0 & \text{in } \mathbb{R}^3\backslash\bar{D} \\ \nu \times \mathbf{E}^s = f & \text{on } \Gamma. \end{cases}$$

Then the far field pattern can be written in terms of this operator:

$$\mathcal{F}g = -\mathrm{i}k\mathcal{B}(\nu \times \mathbf{E}_g)$$

and the far field equation becomes

$$\mathcal{B}(\nu \times \mathbf{E}_g) = -\frac{1}{\mathrm{i}k}\mathbf{E}_{e,\infty}(\cdot, z, q), \quad z \in \mathbb{R}^3.$$

Lemma 1.2.3. $\mathcal{B} : H^{-1/2}_{\mathrm{div}}(\Gamma) \to L^2_t(\Omega)$ *is compact.*

Proof. Assume that $\bar{D} \subset B_R := \{x \in \mathbb{R}^3/|x| < R\}$ and define $\Omega_R := \{x \in \mathbb{R}^3/|x| = R\}$. Then, $\mathcal{B}$ is the composition of the bounded linear operator which maps the boundary data f onto $(\nu \times \mathbf{E}^s, \nu \times \mathbf{H}^s) \in (H^{-1/2}_{\mathrm{div}}(\Omega_R))^2$ with the operator which takes this data onto the electric far field pattern given by

$$\mathbf{E}_\infty(\hat{x}) = \frac{\mathrm{i}k}{4\pi}\hat{x} \times \int_{\partial B_R} ((\nu_y \times \mathbf{E}^s(y)) + (\nu_y \times \mathbf{H}^s(y)) \times \hat{x})\, e^{\mathrm{i}k\hat{x}\cdot y} ds(y).$$

The latter is compact due to the regularity of the kernel. □

Lemma 1.2.4. $\mathcal{B} : H^{-1/2}_{\mathrm{div}}(\Gamma) \to L^2_t(\Omega)$ *is injective with dense range.*

Proof. Injectivity is a direct consequence of Theorem 1.1.2 and the fact that if the far field pattern of a radiating solution to Maxwell's equations vanishes then the solution is equals to zero in $\mathbb{R}^3\backslash\bar{D}$.

To show that $\mathcal{B}$ has dense range, we consider the dual operator $\mathcal{B}^\top : L^2_t(\Omega) \to H^{-1/2}_{\mathrm{curl}}(\Gamma)$ given by

$$\langle \mathcal{B}f, g\rangle_{L^2_t(\Omega), L^2_t(\Omega)} = \langle f, \mathcal{B}^\top g\rangle_{H^{-1/2}_{\mathrm{div}}(\Gamma), H^{-1/2}_{\mathrm{curl}}(\Gamma)},$$

where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between the denoted spaces. By changing order of integration and integrating by parts it can be shown that

$$\langle \mathcal{B}f, g\rangle_{L^2_t(\Omega), L^2_t(\Omega)} = \frac{1}{4\pi}\int_{\partial D} f \cdot (\operatorname{curl}\mathbf{E}_g - \operatorname{curl}\tilde{\mathbf{E}})ds, \tag{1.15}$$

where $\tilde{\mathbf{E}} \in H_{loc}(\operatorname{curl}, \mathbb{R}^3\backslash\overline{D})$ is the solution of

$$\begin{cases} \operatorname{curl}\operatorname{curl}\tilde{\mathbf{E}} - k^2\tilde{\mathbf{E}} = 0 & \text{in } \mathbb{R}^3\backslash\overline{D} \\ \nu \times \left(\tilde{\mathbf{E}} - \mathbf{E}_g\right) = 0 & \text{on } \partial D. \end{cases} \tag{1.16}$$

Hence, noting that the integral (1.15) is interpreted in the sense of duality between $H^{-1/2}_{\mathrm{div}}(\Gamma)$ and $H^{-1/2}_{\mathrm{curl}}(\Gamma)$, we have that

$$(\mathcal{B}^\top g)(x) = \nu \times (\operatorname{curl}\mathbf{E}_g(x) - \operatorname{curl}\tilde{\mathbf{E}}(x)) \times \nu, \quad x \in \partial D.$$

To show that $\mathcal{B}$ has dense range, it suffices to show that $\mathcal{B}^\top$ is injective. To this end, $\mathcal{B}^\top g = 0$ implies that $\nu \times \operatorname{curl} \mathbf{E}_g = \nu \times \operatorname{curl} \tilde{\mathbf{E}}$ on ∂D and by definition we have that $\nu \times \mathbf{E}_g = \nu \times \tilde{\mathbf{E}}$ on ∂D. Now let $B_R := \{x/|x| < R\}$ be a ball containing $\overline{D}$ in its interior and consider the solution $\hat{\mathbf{E}}$, $\hat{\mathbf{H}}$ of Maxwell's equations in B_R defined by

$$\hat{\mathbf{E}}(x) := \begin{cases} 0, & x \in D \\ \mathbf{E}_g(x) - \tilde{\mathbf{E}}(x), & x \in B_R \backslash \overline{D}, \end{cases}$$
$$\hat{\mathbf{H}}(x) := \frac{1}{\mathrm{i}k} \operatorname{curl} \hat{\mathbf{E}}(x).$$

Then using the Stratton-Chu formula (see Appendix B), we see that $\hat{\mathbf{E}}(x) = 0$ for $x \in B_R$, and, since R is arbitrary, $\hat{\mathbf{E}}(x) = 0$ for $x \in \mathbb{R}^3$, i.e., $\mathbf{E}_g(x) = \tilde{\mathbf{E}}(x)$ for $x \in \mathbb{R}^3 \backslash \overline{D}$. By Theorem 1.1.2, this is a contradiction unless $\mathbf{E}_g(x) = \tilde{\mathbf{E}}(x) = 0$ for $x \in \mathbb{R}^3 \backslash \overline{D}$. Then $\mathbf{E}_g(x) = 0$ for $x \in \mathbb{R}^3$ and hence $g = 0$, i.e. $\mathcal{B}^\top$ is injective. □

Lemma 1.2.5. *$\mathbf{E}_{e,\infty}(\hat{x}, z, q)$ is in the range of $\mathcal{B}$ if and only if $z \in D$.*

Proof. If $z \in D$ then $\mathcal{B}(-\nu \times \mathbf{E}_e(\cdot, z, q)) = \mathbf{E}_{e,\infty}(\cdot, z, q)$.

Now let $z \in \mathbb{R}^3 \backslash \bar{D}$ and assume that there exists a tangential vector field $f \in H^{-1/2}_{\mathrm{div}}(\Gamma)$ such that $\mathcal{B}f = \mathbf{E}_{e,\infty}(\cdot, z, q)$. Then from Theorem 1.1.2, the scattered field $\mathbf{E}^s$ corresponding to the boundary data f and the electric dipole $\mathbf{E}_e(\cdot, z, q)$ coincide in $\{x \in \mathbb{R}^3 \backslash \bar{D}/x \neq z\}$. This contradicts the fact that $\mathbf{E}^s \in H_{loc}(\operatorname{curl}, \mathbb{R}^3 \backslash \bar{D})$ but $\mathbf{E}_e(\cdot, z, q)$ is not. □

Remark 1.2.3. *If k^2 is not a Maxwell eigenvalue, for an arbitrary tangential vector $f \in H^{-1/2}_{\mathrm{div}}(\Gamma)$, there exists $\mathbf{u}$ solution to*

$$\begin{cases} \operatorname{curl} \operatorname{curl} \mathbf{u} - k^2 \mathbf{u} = 0 & \text{in } D \\ \nu \times \mathbf{u} = f & \text{on } \Gamma. \end{cases} \tag{1.17}$$

From the denseness of Herglotz wave functions in $M(D)$, we get that for all $\varepsilon > 0$, there exists $g_\varepsilon \in L^2_t(\Omega)$ such that

$$||\nu \times \mathbf{E}_{g_\varepsilon} - f||_{H^{-1/2}_{\mathrm{div}}(\Gamma)} < \varepsilon. \tag{1.18}$$

Thus, every tangential vector $f \in H^{-1/2}_{\mathrm{div}}(\Gamma)$ can be approximated by the tangential trace of Herglotz wave functions with kernel $g_\varepsilon \in L^2_t(\Omega)$.

Now, let us consider the ill-posed equation

$$\mathcal{B}f_z = -\frac{1}{\mathrm{i}k} \mathbf{E}_{e,\infty}(\cdot, z, q), \quad z \in \mathbb{R}^3. \tag{1.19}$$

From the previous lemma, if $z \in D$ then the tangential vector field $f_z = \nu \times \mathbf{E}_e(\cdot, z, q))$ is the unique solution to (1.19). In particular, as $z \to \Gamma$, we have that

$$||f_z||_{H^{-1/2}_{\mathrm{div}}(\Gamma)} \to \infty.$$

From Remark 1.2.3, f_z can be approximated by the trace of Herglotz wave functions with kernels $\mathrm{i}kg_z^\varepsilon$ and as a consequence we have

$$\begin{aligned}||\mathcal{F}g_z^\varepsilon - \mathbf{E}_{e,\infty}||_{L_t^2(\Omega)} &= ||-\mathrm{i}k\mathcal{B}(\nu \times \mathbf{E}_{g_z^\varepsilon}) + \mathrm{i}k\mathcal{B}f_z||_{L_t^2(\Omega)} \\ &< \varepsilon\end{aligned}$$

since $\mathcal{B}$ is continuous. g_z^ε is consequently an approximated solution to the far field equation and the behavior of f_z on the boundary Γ implies that the norm of $\mathbf{E}_{g_z^\varepsilon}$ in $H(\mathrm{curl}\,, D)$ and the norm of g_z^ε in $L_t^2(\Omega)$ explodes when $z \to \Gamma$.

Now, for $z \in \mathbb{R}^3\backslash\bar{D}$, $\mathbf{E}_{e,\infty}(\cdot, z, q)$ is not in the range of $\mathcal{B}$ but from Lemma 1.2.3 and 1.2.4, we can use Tikhonov regularization to construct a regularized solution f_z^α to (1.19) corresponding to the regularization parameter α. We may choose α small enough so that

$$||\mathcal{B}f_z^\alpha - \frac{1}{\mathrm{i}k}\mathbf{E}_{e,\infty}(\cdot, z, q)||_{L_t^2(\Omega)} < \delta,$$

for an arbitrary small $\delta > 0$. Again, f_z^α can be approximated by the trace of a Herglotz wave functions with kernel $\mathrm{i}kg_{z,\delta}^\varepsilon$. From the inequality

$$||\mathcal{F}g_{z,\alpha}^\varepsilon - \mathbf{E}_{e,\infty}||_{L_t^2(\Omega)} \leq ||-\mathrm{i}k\mathcal{B}(\nu \times \mathbf{E}_{g_{z,\alpha}^\varepsilon}) + \mathrm{i}k\mathcal{B}f_z^\alpha||_{L_t^2(\Omega)} + ||-\mathrm{i}k\mathcal{B}f_z^\alpha - \mathbf{E}_{e,\infty}(\cdot, z, q)||_{L_t^2(\Omega)},$$

we deduce that $g_{z,\alpha}^\varepsilon$ is an approximated solution to the far field equation. Because $\mathbf{E}_{e,\infty}(\cdot, z, q)$ is not in the range of $\mathcal{B}$,

$$||f_z^\alpha|| \to \infty \quad \text{when } \alpha \to 0.$$

Consequently we also have that the norm of $\mathbf{E}_{g_{z,\alpha}^\varepsilon}$ in $H(\mathrm{curl}\,, D)$ and the norm of $g_{z,\alpha}^\varepsilon$ in $L_t^2(\Omega)$ explodes when $\alpha \to 0$.

To summarize, we can now state the main theorem of the LSM.

Theorem 1.2.6. *Assume that k^2 is not a Maxwell eigenvalue for D and that $\mathcal{F}$ is the far field operator corresponding to the scattering problem for a perfect conductor. Then the following hold*

1. *For $z \in D$ and a given $\varepsilon > 0$, there exists a $g_z^\varepsilon \in L_t^2(\Omega)$ such that*

   $$||\mathcal{F}g_z^\varepsilon - \mathbf{E}_{e,\infty}(\cdot, z, q)||_{L_t^2(\Omega)} < \varepsilon$$

   and the corresponding Herglotz wave function $\mathrm{i}k\mathbf{E}_{g_z^\varepsilon}$ converges to a solution to (1.13) *in $H(\mathrm{curl}\,, D)$ as $\varepsilon \to 0$.*

   Moreover, for a fixed $\varepsilon > 0$, we have that

   $$\lim_{z\to\Gamma} ||\mathbf{E}_{g_z^\varepsilon}||_{H(\mathrm{curl}\,,D)} = \infty \text{ and } \lim_{z\to\Gamma} ||g_z^\varepsilon||_{L_t^2(\Omega)} = \infty.$$

2. *For $z \in \mathbb{R}^3\backslash\bar{D}$ and a given $\varepsilon > 0$, every $g_z^\varepsilon \in L_t^2(\Omega)$ that satisfies*

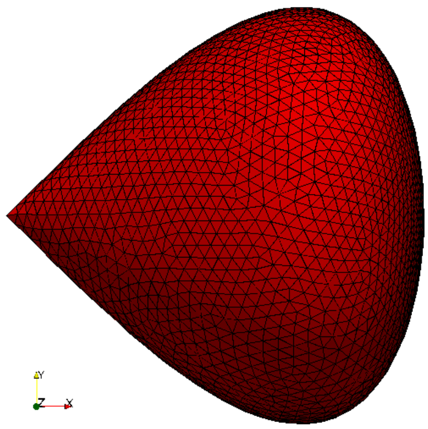
   $$||\mathcal{F}g_z^\varepsilon - \mathbf{E}_{e,\infty}(\cdot, z, q)||_{L_t^2(\Omega)} < \varepsilon$$

   is such that

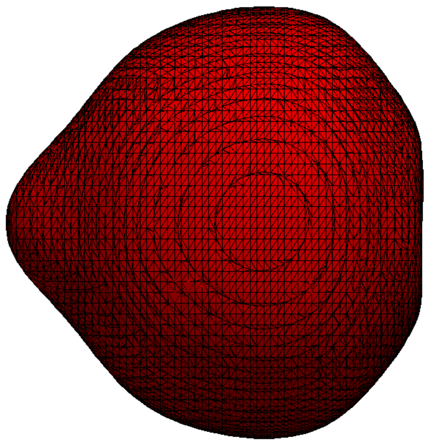
   $$\lim_{\varepsilon\to 0} ||\mathbf{E}_{g_z^\varepsilon}||_{H(\mathrm{curl}\,,D)} = \infty \text{ and } \lim_{\varepsilon\to 0} ||g_z^\varepsilon||_{L_t^2(\Omega)} = \infty.$$

This theorem characterizes the boundary of the obstacle by the behavior of the approximated solution to the far field pattern. In practice, the linear sampling method consists in solving the far field equation using Tikhonov regularization together with the Morozov discrepancy principle for a sample of points z. However, even if this solution behaves the same way as the function in the previous theorem, there is no mathematical justification except for the case of Helmholtz equation [4].

Example of reconstruction using the LSM



(a) TRUE OBJECT



(b) RECONSTRUCTION

1.2.3 Determination of Maxwell eigenvalues from far field data

We just showed how the LSM works when k^2 is not a Maxwell eigenvalue. It is then natural to wonder what happens when k^2 is a Maxwell eigenvalue. In this section, we extend in the electromagnetic case the result proven in [13] by Cakoni, Colton and Haddar for the acoustic case. We show that the norm of the regularized solution to the far field equation explodes when k^2 is a Maxwell eigenvalue and this proves that the LSM fails in this case.

Let us remark that the direct scattering problem can be expressed as a boundary value problem for the scattered electric field $\mathbf{E}^s$:

$$\operatorname{curl}\operatorname{curl}\mathbf{E}^s - k^2\mathbf{E}^s = 0 \text{ in } \mathbb{R}^3\backslash\bar{D} \tag{1.20}$$

$$\nu \times \mathbf{E}^s = f \text{ on } \Gamma \tag{1.21}$$

where k in the wave number, $f = -\nu \times \mathbf{E}^i$ where $\mathbf{E}^i$ is the incident field given by

$$\mathbf{E}^i(x, d, p) = \frac{\mathrm{i}}{k}\operatorname{curl}\operatorname{curl} p e^{\mathrm{i}kx\cdot d}. \tag{1.22}$$

$p \in \mathbb{R}^3$ is a polarization vector and $d \in \Omega := \{x \in \mathbb{R}^3 / |x| = 1\}$ is the direction of propagation. Finally, $\mathbf{E}^s$ is required to satisfy the Silver-Müller radiation condition

$$\lim_{|x|\to\infty}(\operatorname{curl}\mathbf{E}^s \times x - \mathrm{i}k|x|\mathbf{E}^s) = 0 \tag{1.23}$$

uniformly for all direction $\hat{x} = x/|x|$.

Let $\mathcal{F}^\delta$ denote the noisy operator corresponding to noisy measurements $\mathbf{E}^\delta_\infty(\hat{x}, d, q)$. We define the noisy bounded operator $\mathcal{B}^\delta$ associated with $\mathcal{B}$ for all $g \in L^2_t(\Omega)$ by

$$\mathcal{F}^\delta g = -\mathcal{B}^\delta(\nu \times \mathbf{E}_g)$$

and assume that

$$||\mathcal{B}^\delta - \mathcal{B}|| \leq \delta$$

where $\delta > 0$ is a measure of the noise level. In particular, $\mathcal{F}^\delta$ is a bounded and compact linear operator.

For each fixed z and q we now determine $g_{z,q,\delta}$ by minimizing the Tikhonov functional

$$||\mathcal{F}^\delta g_{z,q,\delta} - \mathbf{E}_{e,\infty}(., z, q)||^2_{L^2_t(\Omega)} + \varepsilon ||g_{z,q,\delta}||^2_{L^2_t(\Omega)} \tag{1.24}$$

where $\varepsilon := \varepsilon(\delta) \to 0$ as $\delta \to 0$ is the regularization parameter. We assume that $\varepsilon(\delta)$ is such that

$$\lim_{\delta \to 0} ||\mathcal{F}^\delta g_{z,q,\delta} - \mathbf{E}_{e,\infty}(\cdot, z, q)||_{L^2_t(\Omega)} = 0 \tag{1.25}$$

Theorem 1.2.7. *We assume that k^2 is a Maxwell eigenvalue in D and that* (1.25) *is verified. Then for almost every $z \in D$, there exists q such that $||\mathbf{E}_{g_{z,q,\delta}}||_{H(\mathrm{curl}, D)}$ cannot be bounded when $\delta \to 0$.*

Proof. Assume that for a set $\mathcal{A}$ of points $z \in D$ which has a positive measure, there exists a constant $M > 0$ such that for all $q \in \mathbb{R}^3$,

$$||\mathbf{E}_{g_{z,q,\delta}}||_{H(\mathrm{curl}, D)} \leq M. \tag{1.26}$$

Then we have

$$||\mathcal{F}^\delta g_{z,q,\delta} - \mathcal{F} g_{z,q,\delta}||_{L^2_t(\Omega)} \leq ||\mathcal{B}^\delta - \mathcal{B}|| \, ||\nu \times \mathbf{E}_{g_{z,q,\delta}}||_{H^{-1/2}_{\mathrm{div}}} \leq M\delta.$$

Using (1.25) and the previous inequality, we deduce that

$$\lim_{\delta \to 0} ||\mathcal{F} g_{z,q,\delta} - \mathbf{E}_{e,\infty}(\cdot, z, q)||_{L^2_t(\Omega)} = 0.$$

Moreover, from (1.26), there exists a subsequence $\mathbf{E}_n := \mathbf{E}_{g_{z,q,\delta_n}}$ which weakly converges to $\mathbf{E} \in H(\mathrm{curl}, D)$ such that $\mathrm{curl}\,\mathrm{curl}\,\mathbf{E} - k^2\mathbf{E} = 0$ in D. We deduce that $\nu \times \mathbf{E}_n$ weakly converges to $\nu \times \mathbf{E}$ in $H^{-1/2}_{\mathrm{div}}(\Gamma)$, and by the compactness of $\mathcal{B}$ we conclude that $||\mathcal{B}(\nu \times \mathbf{E}_n) - \mathcal{B}(\nu \times \mathbf{E})||_{L^2_t(\Omega)} \to 0$ as $n \to \infty$ i.e.

$$\lim_{n \to \infty} ||\mathcal{F} g_{z,q,\delta_n} + \mathcal{B}(\nu \times \mathbf{E})||_{L^2_t(\Omega)} = 0.$$

Then $\mathbf{E}_{e,\infty}(\cdot, z, q) = -\mathcal{B}(\nu \times \mathbf{E})$. From the injectivity of $\mathcal{B}$, we deduce that $\nu \times \mathbf{E} = -\nu \times \mathbf{E}_e$ on Γ. We have found $\mathbf{E} \in H(\mathrm{curl}, D)$ solution to (1.20)-(1.21) with $f = -\nu \times \mathbf{E}_e$.

Now let $\mathbf{u}$ be a Maxwell eigenvalue associated with k^2. From the second Green's formula, we have the following equality

$$\int_\Gamma \nu \times \mathbf{E}_e(x, z, q) \cdot \mathrm{curl}\,\mathbf{u}(x) ds(x) = 0, \quad \forall z \in \mathcal{A}, \quad q \in \mathbb{R}^3. \tag{1.27}$$

Let $\mathbf{G}$ a function defined in $\mathbb{R}^3 \backslash D$ by

$$\mathbf{G}(z) := \int_\Gamma \Phi_k(x,z) \operatorname{curl} \mathbf{u}(x) \times \nu(x) ds(x) + \frac{1}{k^2} \nabla \operatorname{div} \int_\Gamma \operatorname{curl} \mathbf{u}(x) \times \nu(x) \Phi_k(x,z) ds(x)$$

where Φ_k is the fundamental solution to Helmholtz equation. Remark that $\mathbf{G}$ is a radiating solution to Maxwell's equations.

From Lemma B.2.1 in Appendix B, (1.27) is equivalent to $\mathrm{i}kq \cdot \mathbf{G}(z) = 0$, for all $z \in \mathcal{A}$ and all $q \in \mathbb{R}^3$. As a consequence, using the unique continuation principle, $\mathbf{G} = 0$ in D. Furthermore, $\mathbf{G}$ is a radiating solution to Maxwell's equations such that $\nu \times \mathbf{G} = 0$ on Γ. Hence, by Rellich's lemma $\mathbf{G} = 0$ in $\mathbb{R}^3 \backslash D$ and from the jump properties of $\nu \times \operatorname{curl} \mathbf{G}$ on Γ (see [26]) we deduce that $\nu \times \operatorname{curl} \mathbf{u} = 0$ on Γ. Finally, the representation formula for Maxwell's equations of $\mathbf{u}$ shows that $\mathbf{u} = 0$ which contradicts the fact that $\mathbf{u}$ is an eigenvector. □

1.2.4 Case of penetrable objects

We now consider the inverse problem of finding the shape of a penetrable object by using again the linear sampling method. We will see that in this case the role of Maxwell eigenvalues will be replaced by the so-called transmission eigenvalues whose definition is given in the following.

Let us consider in this section the case where the magnetic permeability μ_r is equal to 1 inside the object.

Similarly to the case of an impenetrable object, the LSM consists in solving the far field equation

$$(\mathcal{F}g)(\hat{x}) = \mathbf{E}_{e,\infty}(\hat{x}, z, q) \tag{1.28}$$

where $\mathbf{E}_{e,\infty}(\cdot, z, q)$ is the far field pattern of an electric dipole located at a point z given by (1.11). We still assume that the far field pattern corresponding to the scattering problem

$$\begin{cases} \operatorname{curl} \mathbf{E} - \mathrm{i}k\mathbf{H} = 0, \ \operatorname{curl} \mathbf{H} + \mathrm{i}k\varepsilon_r(x)\mathbf{E} = 0 & \text{in } \mathbb{R}^3 \backslash \bar{D} \\ \mathbf{E} := \dfrac{\mathrm{i}}{k} \operatorname{curl} \operatorname{curl} p e^{\mathrm{i}kx\cdot d} + \mathbf{E}^s, \ \mathbf{H} := \operatorname{curl} p e^{\mathrm{i}kx\cdot d} + \mathbf{H}^s & \\ \lim\limits_{r \to +\infty} (\mathbf{H}^s \times x - r\mathbf{E}^s) = 0 & \end{cases} \tag{1.29}$$

is known for all d and $\hat{x} \in \Omega$.

In the case of an impenetrable object, we have seen that the far field equation was solvable if and only if there exists a Herglotz wave function with kernel $\mathrm{i}kg$ solution to (1.13). This led to exclude Maxwell eigenvalues to study the LSM. In the case of penetrable object, this role is replaced by the so-called transmission eigenvalues. Before giving the definition, let us introduce the interior transmission problem with the following theorem that links the resolvability of the far field equation to the interior transmission problem. In the following sections, we shall give the right setting for the study of the interior transmission problem.

Theorem 1.2.8. *There exists a solution $g \in L^2_t(\Omega)$ to the far field equation for an inhomogeneous anisotropic medium if and only if there exists a solution $\mathbf{E}_0^z$ and $\mathbf{E}^z$ to the interior transmission problem*

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E}^z - k^2\varepsilon_r\mathbf{E}^z = 0 & \text{in } D \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0^z - k^2\mathbf{E}_0^z = 0 & \text{in } D \\ \nu \times \mathbf{E}^z - \nu \times \mathbf{E}_0^z = \nu \times \mathbf{E}_e(\cdot, z, q) & \text{on } \Gamma \\ \nu \times \operatorname{curl}\mathbf{E}^z - \nu \times \operatorname{curl}\mathbf{E}_0^z = \nu \times \operatorname{curl}\mathbf{E}_e(\cdot, z, q) & \text{on } \Gamma \end{cases} \tag{1.30}$$

and $\mathbf{E}_0^z$ is the electric field of an electromagnetic Herglotz pair with kernel $\mathrm{i}kg$.

We now can define the transmission eigenvalues.

Definition 1.2.2. *Transmission eigenvalues are values of k for which the homogeneous interior transmission problem*

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\varepsilon_r\mathbf{E} = 0 & \text{in } D \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 & \text{in } D \\ \nu \times \mathbf{E} - \nu \times \mathbf{E}_0 = 0 & \text{on } \Gamma \\ \nu \times \operatorname{curl}\mathbf{E} - \nu \times \operatorname{curl}\mathbf{E}_0 = 0 & \text{on } \Gamma \end{cases}$$

has a non trivial solution.

Using same arguments as in the case of impenetrable obstacles, it can be shown that the far field operator $\mathcal{F}$ is injective provided k is not a transmission eigenvalue. We refer to [15] for more details and to Chapter 2 for the study of the interior transmission problem.

Theorem 1.2.9. *Assume that k is not a transmission eigenvalue for D and that $\mathcal{F}$ is the far field operator corresponding to the scattering problem* (1.29). *Then*

1. *For $z \in D$ and a given $\varepsilon > 0$, there exists a $g_z^\varepsilon \in L^2_t(\Omega)$ such that*

   $$||\mathcal{F}g_z^\varepsilon - \mathbf{E}_{e,\infty}(\cdot, z, q)||_{L^2_t(\Omega)} < \varepsilon$$

   and the corresponding Herglotz wave function $\mathbf{E}_{g_z^\varepsilon}$ converges to $\mathbf{E}_0^z$ in the $L^2(D)$ norm as $\varepsilon \to 0$ where $\mathbf{E}_0^z, \mathbf{E}^z$ is the solution to (1.30).

   Moreover, for a fixed $\varepsilon > 0$, we have that

   $$\lim_{z\to\Gamma} ||\mathbf{E}_{g_z^\varepsilon}||_{L^2(D)} = \infty \text{ and } \lim_{z\to\Gamma} ||g_z^\varepsilon||_{L^2_t(\Omega)} = \infty.$$

2. *For $z \in \mathbb{R}^3\backslash\bar{D}$ and a given $\varepsilon > 0$, there exists $g_z^\varepsilon \in L^2_t(\Omega)$ that satisfies*

   $$||\mathcal{F}g_z^\varepsilon - \mathbf{E}_{e,\infty}(\cdot, z, q)||_{L^2_t(\Omega)} < \varepsilon$$

   such that

   $$\lim_{\varepsilon\to 0} ||\mathbf{E}_{g_z^\varepsilon}||_{L^2(D)} = \infty \text{ and } \lim_{\varepsilon\to 0} ||g_z^\varepsilon||_{L^2_t(\Omega)} = \infty.$$

1.3 The interior transmission problem

1.3.1 Motivation and questions

In the previous theorem, we have excluded particular frequencies called transmission eigenvalues for which there exists an incident wave that does not scatter. We can wonder what happens when the wave number is a transmission eigenvalue and we will see later in Chapters 2 and 3 that the LSM fails in this case.

One can hope that, similarly to Maxwell eigenvalues for impenetrable objects, transmission eigenvalues form at most a discrete set. More precisely, there exists an infinite discrete sequence of real transmission eigenvalues. We will see in the following that this is the case in all the studied configurations. Since the existence of such transmission eigenvalues cannot be avoided, another point to view is to try to take benefit of them. Indeed, it has been recently noticed that they also give information on the physical properties of the scatterer. As a consequence, the study of transmission eigenvalues has been a subject of great interest in the past few years.

The goal of this thesis is to contribute to the study of the interior transmission problem and to answer to some open problems on this subject. To summarize, three questions can be asked:

- Do transmission eigenvalues exist ?
- Do they form a discrete set ?
- Can we find estimates on the parameters of the scatterer ε_r and μ_r with respect to transmission eigenvalues ?

We will consider in the following both scalar and vector equations. Our results are of two types:

- in Chapter 3, we have extended results concerning dielectrics with cavities obtained in the scalar case in [12] to the electromagnetic case,
- in Chapters 4 and 5, we consider new configurations and/or methods that we present for the scalar case. The extension of these results to the electromagnetic case is one of the main perspectives of this thesis.

Let us now present the three main approaches that we will use in the following to study the ITP. In order to highlight the difficulties of the study of the interior transmission problem, we consider here the simpler case of the scalar equations (see Appendix A). The first remark that can be made is that there does not exist a general method to study the interior transmission problem and every configuration needs to be treated separately.

1.3.2 Second order formulation

First, consider a scatterer characterized by two contrasts μ and n. The problem consists in finding k for which the ITP

$$\begin{cases} \nabla \cdot \dfrac{1}{\mu} \nabla w + k^2 n(x) w = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \partial D \\ \dfrac{1}{\mu} \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} & \text{on } \partial D. \end{cases} \tag{1.31}$$

has non trivial solution. A natural approach consists in looking for a variational formulation in $H^1(D)$. By multiplying the first equation satisfied by w by a test function w' and the second equation satisfied by v by a test function v' such that $w' = v'$ on the boundary ∂D, and integrating by parts, we get

$$\int_D \frac{1}{\mu} \nabla w \cdot \nabla \bar{w}' dx - \int_D k^2 n w \bar{w}' dx - \int_{\partial D} \frac{1}{\mu} \frac{\partial w}{\partial \nu} \bar{w}' ds(x) = 0$$

and

$$\int_D \nabla v \cdot \nabla \bar{v}' dx - \int_D k^2 v \bar{v}' dx - \int_{\partial D} \frac{\partial v}{\partial \nu} \bar{v}' ds(x) = 0.$$

Now, using the boundary conditions $\frac{1}{\mu} \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu}$ and $w' = v'$ on Γ, the variational formulation becomes: find $(w, v) \in \mathbb{H}$ such that

$$\int_D \left(\frac{1}{\mu} \nabla w \cdot \nabla \bar{w}' - \nabla v \cdot \nabla \bar{v}' \right) dx - k^2 \int_D (n w \bar{w}' - v \bar{v}') dx = 0 \tag{1.32}$$

for all $(w', v') \in \mathbb{H}$ where

$$\mathbb{H} := \left\{ (w, v) \in H^1(D) \times H^1(D) / w = v \text{ on } \partial D \right\}.$$

At first sight, the problem looks like a linear problem with respect to k^2 with bilinear symmetric forms. However, one can remark that the problem is not that simple since the bilinear forms do not have a constant sign. Moreover, the corresponding operators are not self-adjoint. This is confirmed by the fact that complex transmission eigenvalues can be found numerically.

Nevertheless, this problem can be solved using the T-coercivity approach that will also be used in Chapter 4 for the study of transmission eigenvalues when the scatterer contains a perfect conductor. This method has been introduced in [5].

If we define

$$a_k((v, w), (v', w')) := \int_D \left(\frac{1}{\mu} \nabla w \cdot \nabla \bar{w}' - \nabla v \cdot \nabla \bar{v}' \right) dx - k^2 \int_D (n w \bar{w}' - v \bar{v}') dx,$$

the idea is to get an equivalent formulation of (1.32) by considering $\tilde{a}_k$ instead of a_k defined by

$$\tilde{a}_k((v, w), (v', w')) := a_k((v, w), T(v', w'))$$

where T is an isomorphism. It is easily verified that $(w, v) \in \mathbb{H}$ satisfies $a_k((v, w), (v', w')) = 0$ for all $(w', v') \in \mathbb{H}$ if and only if it satisfies $\tilde{a}_k((v, w), (v', w')) = 0$ for all $(w', v') \in \mathbb{H}$. By taking the isomorphism $T : \mathbb{H} \to \mathbb{H}$ defined by

$$T(w, v) := (w - 2v, -w),$$

the variational formulation becomes: find $(w, v) \in \mathbb{H}$ such that

$$\int_D \frac{1}{\mu} \nabla w \cdot \nabla \bar{w}' dx + \int_D \nabla v \cdot \nabla \bar{v}' dx - 2 \int_D \frac{1}{\mu} \nabla w \cdot \nabla \bar{v}' dx \\ - k^2 \int_D n w \bar{w}' dx - k^2 \int_D v \bar{v}' dx + 2k^2 \int_D n w v' dx = 0$$

for all $(w', v') \in \mathbb{H}$. One can remark that by making this change of variable, we loose the symmetric property of the formulation, but we obtain the Fredholm property of the formulation. Indeed, if $\mu_* := inf_{x \in D} \inf_{|\xi|=1} \left(\xi \cdot \frac{1}{\mu} \xi \right) > 1$ and $n^* = \sup_{x \in D} n(x) < 1$, for $k = \mathrm{i}\kappa$, $\kappa \in \mathbb{R}^*$ and using Young's inequality, we obtain

$$\begin{aligned} |\tilde{a}_{\mathrm{i}\kappa}((v, w), (v, w))| &= | \left(\mu^{-1} \nabla w, \nabla w\right)_D + (\nabla v, \nabla v)_D - 2 \left(\mu^{-1} \nabla w, \nabla v\right) \\ &\quad + \kappa^2 (nw, w)_D + \kappa^2 (v, v)_D - 2\kappa^2 (nw, v)_D \\ &\geq \left(\mu^{-1} \nabla w, \nabla w\right)_D + (\nabla v, \nabla v)_D + \kappa^2 (nw, w)_D + \kappa^2 (v, v)_D \\ &\quad - 2 \left(\mu^{-1} \nabla w, \nabla v\right) - 2\kappa^2 (nw, v)_D \\ &\geq \left((1 - \alpha) \mu^{-1} \nabla w, \nabla w\right)_D + ((1 - (\alpha \mu_*)^{-1} \nabla v, \nabla v)_D \\ &\quad + \kappa^2 ((1 - \beta) nw, w)_D + \kappa^2 ((1 - \beta^{-1} n^*) v, v)_D. \end{aligned}$$

Taking α and β such that $\frac{1}{\mu_*} < \alpha < 1$ and $n^* < \beta < 1$, the previous estimates proves that $\tilde{a}_{\mathrm{i}\kappa}$ is coercive over $\mathbb{H}$. Finally, the embedding of $\mathbb{H}$ in $L^2(D) \times L^2(D)$ shows we can decompose the formulation into a compact and a coercive part. Nevertheless, this implies that we will only be able to prove the discreteness but not the existence.

It is shown in [5] that this T-coercivity approach allows to prove the discreteness of the set of transmission eigenvalues and also gives estimates on the first transmission eigenvalue with respect to the contrasts μ and n in the case where $\mu - 1$ and $n - 1$ have a constant sign at least in a neighborhood of the boundary ∂D. However, this method does not give the existence of transmission eigenvalues. We refer to [19] for the proof of existence of transmission eigenvalues using another approach for both scalar and vector cases. This method will be described and used in Chapter 4 in the scalar case to study the transmission eigenvalues of a scatterer containing a perfect conductor.

1.3.3 Fourth order formulation

In the case where $\mu = 1$, the T-coercivity method does not work anymore since the Fredholm property does not hold anymore in $H^1(D)$. Despite the fact that this case looks more complicated to treat, it has been studied first and more widely in the literature by using a fourth order formulation.

In the case where $\mu = 1$, the interior transmission problem is as follow

$$\begin{cases} \Delta w + k^2 n w = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \partial D \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial w}{\partial \nu} & \text{on } \partial D. \end{cases} \tag{1.33}$$

Let us consider the new variable $u := w - v$. First remark that in D, u satisfies

$$\Delta u + k^2 u = -k^2(n-1)w \tag{1.34}$$

and

$$\Delta u + k^2 n u = -k^2(n-1)v. \tag{1.35}$$

The fourth order formulation satisfied by u now appears when applying, for instance, the operator $\Delta + k^2$ to (1.34) after dividing both sides by $n-1$. Then, we can rewrite (1.33) as a fourth order equation for u

$$\begin{cases} (\Delta + k^2)\dfrac{1}{n-1}(\Delta + k^2 n)u = 0 & \text{in } D \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial D. \end{cases}$$

The natural space in which we study this fourth order problem is $u \in H_0^2(D)$. Consequently, the solution (w, v) of (1.33) is defined in $L^2(D) \times L^2(D)$ such that $w - v \in H^2(D)$. However, due to the term $\dfrac{1}{n-1}$, we need to be careful with the sign of $n-1$ and assume that $n > 1$ or $n < 1$. In this case, existence and discreteness of the set of transmission eigenvalues can be shown [18] using the following variational formulation

$$(A_k u - k^2 B u, v)_{H_0^2(D)} = 0$$

for all $v \in H_0^2(D)$ where

$$(A_k u, v)_{H_0^2(D)} = \int_D \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{v} + k^2 \overline{v})dx + k^4 \int_D u\overline{v}dx$$

with A_k positive definite and self-adjoint on $H_0^2(D) \times H_0^2(D)$ and

$$(Bu, v)_{H_0^2(D)} = \int_D \nabla u \cdot \nabla \overline{v} dx$$

with $B : H_0^2(D) \to H_0^2(D)$ compact. One can remark that the operators are self-adjoint which enables to use the spectral theory of self-adjoint operators and the min-max principle. It can be shown that transmission eigenvalues exist and form a discrete set [18] but only if $n-1$ is either strictly positive or strictly negative. We can also get estimates on the first transmission eigenvalue with respect to n. These results have been extended recently by Cakoni-Colton-Haddar [12] in the case where D contains cavities that is to say that $n = 1$ in D_0 where $D_0 \subset D$.

We refer to the following chapter for details on the study of the vector case and to Chapter 3 for the extension to the case where D contains cavities in the vector case.

1.3.4 Surface integral formulation

This new approach using surface integral equations is detailed in Chapter 5 in the scalar case. It is recalled in Appendix B that solutions to Helmholtz equation can be represented using surface integral operators. We assume here that the contrasts μ and n are constant and we denote $k_0 := k$ and $k_1 := k\sqrt{n\mu}$. If SL_k and DL_k are the classical single and double layer potentials (see Appendix B), the solutions to the interior transmission problem

$$\begin{cases} \nabla \cdot \frac{1}{\mu}\nabla w + k^2 n w = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \partial D \\ \frac{1}{\mu}\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} & \text{on } \partial D. \end{cases} \tag{1.36}$$

can be expressed using the integral representation

$$\begin{array}{ll} v = \mathrm{SL}_{k_0}\alpha - \mathrm{DL}_{k_0}\beta & \text{in } D, \\ w = \mu\mathrm{SL}_{k_1}\alpha - \mathrm{DL}_{k_1}\beta & \text{in } D \end{array} \tag{1.37}$$

where

$$\alpha := \frac{\partial v}{\partial \nu}|_\Gamma = \frac{1}{\mu}\frac{\partial w}{\partial \nu}|_\Gamma \in H^{-1/2}(\Gamma)$$

and

$$\beta := v|_\Gamma = w|_\Gamma \in H^{1/2}(\Gamma).$$

The boundary conditions of (1.36) $w = v$ and $\frac{1}{\mu}\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu}$ on ∂D and the jump properties of the single and double layer potentials yield the system satisfied by α and β

$$Z(k)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \tag{1.38}$$

where

$$Z(k) := \begin{pmatrix} \mu S_{k_1} - S_{k_0} & -K_{k_1} + K_{k_0} \\ K'_{k_1} - K'_{k_0} & -1/\mu T_{k_1} + T_{k_0} \end{pmatrix}.$$

Despite the fact that this method only answers the question of the discreteness of the set of transmission eigenvalues, the main interest is that it also enables to extend this result to the more general case of $n-1$ having a constant sign on a neighborhood of the boundary ∂D. In Chapter 5, we expose the difficulties of this method, in particular in the case where $\mu = 1$ since the space of solutions is not classical for the integral operators.

Chapter 2

Electromagnetic interior transmission problem - The model case of dielectric inclusion

In this chapter, we study the interior problem for an inhomogeneous medium. This is the first case that has been studied and the simplest one. The first results have been proven in [26] and show the existence of an infinite set of transmission eigenvalues but only for spherical stratified media. Since 2008 and the paper of Päivärinta and Sylvester [43], new results have been proven for general geometries. In this chapter, we focus on the main properties of existence and discreteness of the set of transmission eigenvalues.

After giving the notations used in this chapter and recalling the spaces in which we study the problem, we give the basic theorems used all along this thesis to get the theoretical results. We recall the main theorems of the Fredholm theory : the Fredholm alternative used in the study of the well-posedness of the interior transmission problem and the analytic Fredholm theorem that gives the discreteness of the set of transmission eigenvalues. The last important tool is based on generalized eigenvalue problems and its use with an intermediate value theorem provides a powerful method to get existence but also estimates for transmission eigenvalues.

Next, we resume some results from [34] and [18]. We first consider the well-posedness of the interior transmission problem. In particular, by using a fourth order formulation and a variational approach, we show the existence of solutions in $L^2(D)$ to the interior transmission problem provided the wave number k is not a transmission eigenvalue.

The last sections of this chapter focus on the properties of transmission eigenvalues. We first show existence of an infinite discrete set of transmission eigenvalues using the analytic Fredholm theorem and an auxiliary eigenvalue problem. Next, we show that the first transmission eigenvalue is a continuous function of the index of refraction in the case of an isotropic medium. To conclude, we make the parallel between Maxwell eigenvalues for an impenetrable obstacles and transmission eigenvalues for penetrable objects by considering the far field equation and giving an equivalent theorem to Theorem 1.2.7. This property extend the same result proven in [13] for acoustic waves.

2.1 Notations and definitions

In all this chapter, the obstacle denoted by $D \subset \mathbb{R}^3$ is a bounded simply connected region of $\mathbb{R}^3$ with piece-wise smooth boundary $\Gamma := \partial D$ and index of refraction $N(x)$. N is supposed to be a 3×3 symmetric matrix whose entries are bounded complex valued functions in $\mathbb{R}^3$ and such that $N = I$ in $\mathbb{R}^3 \backslash D$. We denote by ν the outward normal vector to Γ.

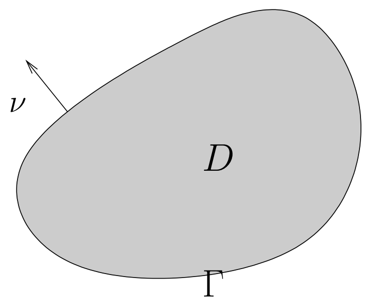


Figure 2.1: Geometry and notations

Let $(.,.)_D$ denote the $L^2(D)^3$ scalar product with the corresponding norm $||\cdot||_D$ and consider the Hilbert spaces

$$H(\mathrm{curl}\,, D) := \left\{ \mathbf{u} \in L^2(D)^3 / \mathrm{curl}\,\mathbf{u} \in L^2(D)^3 \right\},$$
$$H_0(\mathrm{curl}\,, D) := \{ \mathbf{u} \in H(\mathrm{curl}\,, D) / \nu \times \mathbf{u} = 0 \text{ on } \Gamma \}$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathrm{curl}} = (\mathbf{u}, \mathbf{v})_D + (\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v})_D$ and the corresponding norm $||.||_{\mathrm{curl}}$. Next, we define

$$\mathcal{U}(D) := \{ \mathbf{u} \in H(\mathrm{curl}\,, D) / \mathrm{curl}\,\mathbf{u} \in H(\mathrm{curl}\,, D) \},$$
$$\mathcal{U}_0(D) := \{ \mathbf{u} \in H_0(\mathrm{curl}\,, D) / \mathrm{curl}\,\mathbf{u} \in H_0(\mathrm{curl}\,, D) \}$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathcal{U}} = (\mathbf{u}, \mathbf{v})_{\mathrm{curl}} + (\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v})_{\mathrm{curl}}$ and the corresponding norm $||.||_{\mathcal{U}}$.

Let $s \geq 0$ be a given real number and $H^s(\Gamma)$ be the usual Sobolev space. We define

$$TH^s(\Gamma) = \left\{ \boldsymbol{\varphi} \in H^s(\Gamma)^3 / \boldsymbol{\varphi} \cdot \nu = 0 \right\}.$$

Definition 2.1.1. *A matrix field K is said to be bounded positive definite on D if $K \in L^\infty(D, \mathbb{C}^3)^{3\times 3}$ and if there exists a constant $\gamma > 0$ such that*

$$\Re(K\xi, \xi) \geq \gamma |\xi|^2, \ \forall \xi \in \mathbb{C}^3 \text{ and a.e. in } D.$$

In the following, we assume that N, N^{-1} and either $(N - I)^{-1}$ or $(I - N)^{-1}$ are bounded positive definite matrix fields on D.

2.2 Important theorems

All the theorem we recall here are the theoretical basis of the study of transmission eigenvalues that is used in almost every type of obstacles.

2.2.1 Fredholm theory

We recall here two important theorems in the Fredholm theory: the Fredholm alternative and the analytic Fredholm theorem. The first theorem will be useful in the following in order to show the existence and uniqueness of solutions to the interior transmission problem and the second one will be used to show the discreteness of the set of transmission eigenvalues.

Let us first recall the Fredholm alternative. It is a well-known theorem in functional analysis for compact operators.

Theorem 2.2.1. *Let $A : X \to X$ be a compact operator on a Banach space X. Then either*

1. *the homogeneous equation*
$$\varphi - A\varphi = 0$$
has a nontrivial solution $\varphi \in X$

   or

2. *for each $f \in X$, the equation*
$$\varphi - A\varphi = f$$
has a unique solution $\varphi \in X$. If $I - A$ is injective (and hence bijective), then $(I - A)^{-1} : X \to X$ is bounded.

The next theorem considers compact operators on a Banach space depending on a complex parameter. Let us denote by $\mathcal{L}(X)$ the Banach space of bounded linear operators mapping the Banach space X into itself.

Theorem 2.2.2. *Let R be a domain in $\mathbb{C}$ and let $A : R \to \mathcal{L}(X)$ be an operator valued analytic function such that $A(z)$ is compact for each $z \in R$. Then either*

a) $(I - A(z))^{-1}$ *does not exist for any $z \in R$*

b) $(I - A(z))^{-1}$ *exists for all $z \in R \backslash S$ where S is a discrete subset of R.*

This theorem says that if we can find at least one z for which an analytic Fredholm operator is injective then it is always injective except for a discrete set of values of z.

2.2.2 Generalized eigenvalue problem

The theorem we shall use in all the study of existence of transmission is the last one of this section. To establish this theorem, we need some results on the spectral decomposition of a bounded, positive definite and self-adjoint operator on a Hilbert space with respect to a self-adjoint, non negative, compact operator. The following has been taken from [18].

Let U be a separable Hilbert space with scalar product $(\cdot, \cdot)$ and associated norm $||\cdot||$. First, we recall the min-max formulae of Courant-Fréchet.

Theorem 2.2.3. *Let A be a bounded, positive definite and self-adjoint operator on U and B be a non negative, self-adjoint and compact linear operator on U. Then there exists an increasing sequence of positive real numbers $(\lambda_k)_{k\geq 1}$ and a sequence $(u_k)_{k\geq 1}$ elements of U such that $Au_k = \lambda_k B u_k$.*

Furthermore if we define the Rayleigh quotient as

$$R(u) := \frac{(Au, u)}{(Bu, u)}$$

for $u \notin \ker(B)$, where $(\cdot, \cdot)$ is the inner product in U, the min-max principles hold

$$\lambda_k = \min_{W \in \mathcal{U}_k} \left(\max_{u \in W \setminus \{0\}} R(u) \right) \tag{2.1}$$

where $\mathcal{U}_k$ denotes the set of all k-dimensional subspaces W of U such that $W \cap \ker(B) = \{0\}$.

Now, we can formulate the main result that will be useful to show the existence of transmission eigenvalues.

Theorem 2.2.4. *Let $\tau \mapsto A_\tau$ be a continuous mapping from $]0, \infty[$ to the set of self-adjoint and positive definite bounded linear operators on U and let B be a self-adjoint and non-negative compact linear operator on U. We assume that there exist two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ satisfying*

1. *$A_{\tau_0} - \tau_0 B$ is positive on U,*

2. *$A_{\tau_1} - \tau_1 B$ is non positive on a p-dimensional subspace W_p of U.*

Then, each equation $\lambda_j(\tau) = \tau$ for $j = 1, ..., p$, has at least one solution in $[\tau_0, \tau_1]$ where $\lambda_j(\tau)$ is the j^{th} eigenvalue (counting multiplicity) of A_τ with respect to B, i.e. $\ker(A_\tau - \lambda_j(\tau)B) \neq \{0\}$.

Proof. The min-max formulae (2.1) ensures the continuity of each $\lambda_j(\tau)$ with respect to τ. The proof now relies on the intermediate value theorem. From assumption 1., we get that $\lambda_j(\tau_0) > \tau_0$ for all $j \geq 1$. Now, assumption 2. implies in particular that $W_p \cap \ker(B) = \{0\}$ and as a consequence, we have $\lambda_j(\tau_1) \leq \tau_1$ for all $j \geq 1$ which concludes the proof. $\square$

Remark 2.2.1. *The multiplicity of a transmission eigenvalue k is defined as the dimension of the kernel of $A_k - kB$.*

2.3 The interior transmission problem

In this section, we study the well-posedness of the non homogeneous interior transmission problem.

Let $\mathbf{G} \in TH^{3/2}(\Gamma)$ and $\mathbf{H} \in TH^{1/2}(\Gamma)$ be two given boundary data. We consider the following interior transmission problem

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 & \text{in } D \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 & \text{in } D \\ \nu \times \mathbf{E} - \nu \times \mathbf{E}_0 = \mathbf{G} & \text{on } \Gamma \\ \nu \times \operatorname{curl}\mathbf{E} - \nu \times \operatorname{curl}\mathbf{E}_0 = \mathbf{H} & \text{on } \Gamma. \end{cases} \tag{ITP2.1}$$

First let us give the appropriate definition of solutions to the previous problem.

Definition 2.3.1. *A strong solution to* (ITP2.1) *is a pair* $(\mathbf{E}, \mathbf{E}_0) \in L^2(D)^3 \times L^2(D)^3$ *that satisfies*

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 & \textit{in } D \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 & \textit{in } D \end{cases}$$

in the sense of distributions and such that $\mathbf{E} - \mathbf{E}_0 \in \mathcal{U}(D)$ *and* $\mathbf{E} - \mathbf{E}_0$ *satisfies the boundary conditions of* (ITP2.1)*.*

The aim of this section is to show that there exists a unique solution $\mathbf{E}$, $\mathbf{E}_0$ in $L^2(D)^3$ to (ITP2.1) depending continuously on the data $\mathbf{G}$ and $\mathbf{H}$. To this end, we first need to find the appropriate equivalent formulation. We consider a new unknown $\mathbf{F} := \mathbf{E} - \mathbf{E}_0$ which will satisfy a fourth order equation.

2.3.1 Fourth order formulation

To study the existence and uniqueness of solutions to (ITP2.1), we rewrite it as a fourth order boundary value problem on $\mathbf{F} := \mathbf{E} - \mathbf{E}_0$.

Let us set $\mathbf{F} := \mathbf{E} - \mathbf{E}_0$. First, remark that $\mathbf{F}$ satisfies

$$\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2 N\mathbf{F} = k^2(N - I)\mathbf{E}_0 \quad \text{in } D \tag{2.2}$$

or

$$\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F} = k^2(N - I)\mathbf{E} \quad \text{in } D. \tag{2.3}$$

Multiplying both sides of (2.3) by $(N - I)^{-1}$ and applying the operator $(\operatorname{curl}\operatorname{curl} - k^2 N)$, we get that $\mathbf{F}$ satisfies the fourth order equation

$$(\operatorname{curl}\operatorname{curl} - k^2 N)(N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}) = 0 \quad \text{in } D \tag{2.4}$$

together with

$$\nu \times \mathbf{F} = \mathbf{G}, \quad \nu \times \operatorname{curl}\mathbf{F} = \mathbf{H} \quad \text{on } \Gamma. \tag{2.5}$$

It is obvious that finding a solution $\mathbf{F} \in \mathcal{U}(D)$ to (2.4)-(2.5) is equivalent to finding a strong solution $(\mathbf{E}, \mathbf{E}_0)$ to (ITP2.1) by setting

$$\mathbf{E} := \frac{1}{k^2}(N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}) \text{ and } \mathbf{E}_0 := \frac{1}{k^2}(N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2 N\mathbf{F}).$$

The study of (2.4)-(2.5) will be done using a variational framework. Let $\boldsymbol{\varphi} \in \mathcal{U}_0(D)$ be a test function. Using the following equality, valid for all $\mathbf{u} \in \mathcal{U}(D)$ and $\mathbf{v} \in \mathcal{U}_0(D)$,

$$\int_D \operatorname{curl}\operatorname{curl}\mathbf{u} \cdot \overline{\mathbf{v}} dx = \int_D \operatorname{curl}\mathbf{u} \cdot \operatorname{curl}\overline{\mathbf{v}} dx,$$

and integrating by parts, we obtain

$$\begin{aligned}
0 &= \int_D (\operatorname{curl}\operatorname{curl} \, - k^2 N)(N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}) \cdot \overline{\boldsymbol{\varphi}} dx \\
&= \int_D (N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}) \cdot (\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\varphi}} - k^2 N\overline{\boldsymbol{\varphi}}) dx \\
&= \int_D (N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}) \cdot (\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\varphi}} - k^2\overline{\boldsymbol{\varphi}}) dx \\
&\quad - k^2 \int_D (\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F})\overline{\boldsymbol{\varphi}} dx \\
&= \int_D (N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}) \cdot (\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\varphi}} - k^2\overline{\boldsymbol{\varphi}}) dx + k^4 \int_D \mathbf{F} \cdot \overline{\boldsymbol{\varphi}} dx \\
&\quad - k^2 \int_D \operatorname{curl}\mathbf{F} \cdot \operatorname{curl}\overline{\boldsymbol{\varphi}} dx.
\end{aligned}$$

Then $\mathbf{F} \in \mathcal{U}(D)$ satisfies (2.4) if and only if

$$\mathcal{A}_k(\mathbf{F}, \boldsymbol{\varphi}) - k^2 \mathcal{B}(\mathbf{F}, \boldsymbol{\varphi}) = 0 \text{ for all } \boldsymbol{\varphi} \in \mathcal{U}_0(D) \tag{2.6}$$

where $\mathcal{A}_k$ and $\mathcal{B}$ are sesquilinear forms on $\mathcal{U}(D) \times \mathcal{U}(D)$ defined by

$$\mathcal{A}_k(\mathbf{u}, \mathbf{v}) := \big((N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}), (\operatorname{curl}\operatorname{curl}\mathbf{v} - k^2\mathbf{v})\big)_D + k^4(\mathbf{u}, \mathbf{v})_D$$

and

$$\mathcal{B}(\mathbf{u}, \mathbf{v}) := (\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v})_D \, .$$

The variational formulation (2.6) is also equivalent to

$$\tilde{\mathcal{A}}_k(\mathbf{F}, \boldsymbol{\varphi}) - k^2 \mathcal{B}(\mathbf{F}, \boldsymbol{\varphi}) = 0 \text{ for all } \boldsymbol{\varphi} \in \mathcal{U}_0(D) \tag{2.7}$$

where $\tilde{\mathcal{A}}_k$ is a sesquilinear form on $\mathcal{U}(D) \times \mathcal{U}(D)$ defined by

$$\begin{aligned}
\tilde{\mathcal{A}}_k(\mathbf{u}, \mathbf{v}) &:= \big((I-N)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2 N\mathbf{u}), (\operatorname{curl}\operatorname{curl}\mathbf{v} - k^2 N\mathbf{v})\big)_D + k^4(N\mathbf{u}, \mathbf{v})_D \\
&= \big(N(I-N)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}), (\operatorname{curl}\operatorname{curl}\mathbf{v} - k^2\mathbf{v})\big)_D \\
&\quad + (\operatorname{curl}\operatorname{curl}\mathbf{u}, \operatorname{curl}\operatorname{curl}\mathbf{v})_D.
\end{aligned}$$

To establish the appropriate variational formulation, we need the following lifting result (see [34]).

Lemma 2.3.1. *Let $\boldsymbol{\varphi} \in TH^{3/2}(\Gamma)$ and $\boldsymbol{\psi} \in TH^{1/2}(\Gamma)$ be two given boundary data. There exists $\mathbf{w} \in H^2(D)^3$ such that*

$$\nu \times \mathbf{w} = \boldsymbol{\varphi} \quad \textit{and} \quad \nu \times \operatorname{curl}\mathbf{w} = \boldsymbol{\psi} \quad \textit{on } \Gamma$$

and

$$||\mathbf{w}||_{H^2(D)^3} \le c\,(||\boldsymbol{\varphi}||_{H^{3/2}} + ||\boldsymbol{\psi}||_{H^{1/2}})$$

where c is a constant independent of $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$.

Let $\mathbf{w} \in H^2(D)^3 \subset \mathcal{U}(D)$ be a lifting function associated with $\mathbf{G}$ and $\mathbf{H}$ as in Lemma 2.3.1. Therefore, finding a solution $\mathbf{F} \in \mathcal{U}(D)$ to (2.6)-(2.5) is equivalent to finding a function $\mathbf{F}_0 = \mathbf{F} - \mathbf{w} \in \mathcal{U}_0(D)$ satisfying

$$\mathcal{A}_k(\mathbf{F}_0, \boldsymbol{\varphi}) - k^2\mathcal{B}(\mathbf{F}_0, \boldsymbol{\varphi}) = \ell_k(\boldsymbol{\varphi}) \text{ for all } \boldsymbol{\varphi} \in \mathcal{U}_0(D) \tag{2.8}$$

where

$$\ell_k(\boldsymbol{\varphi}) := -\mathcal{A}_k(\mathbf{w}, \boldsymbol{\varphi}) + k^2\mathcal{B}(\mathbf{w}, \boldsymbol{\varphi})$$

or

$$\tilde{\mathcal{A}}_k(\mathbf{F}_0, \boldsymbol{\varphi}) - k^2\mathcal{B}(\mathbf{F}_0, \boldsymbol{\varphi}) = \tilde{\ell}_k(\boldsymbol{\varphi}) \text{ for all } \boldsymbol{\varphi} \in \mathcal{U}_0(D) \tag{2.9}$$

where

$$\tilde{\ell}_k(\boldsymbol{\varphi}) := -\tilde{\mathcal{A}}_k(\mathbf{w}, \boldsymbol{\varphi}) + k^2\mathcal{B}(\mathbf{w}, \boldsymbol{\varphi}).$$

2.3.2 Existence of solutions

The theorem that ensures the existence of transmission eigenvalues is the Fredholm alternative. In the following, the variational formulation (2.8) will be used when $(N-I)^{-1}$ is bounded positive definite while the variational formulation (2.9) will be used when $(I-N)^{-1}$ is bounded positive definite to show that the left-hand side is of Fredholm type.

Lemma 2.3.2. *Assume that there exists a constant $\gamma > 0$ such that*

$$\Re\left((N-I)^{-1}\xi, \xi\right) \geq \gamma|\xi|^2, \forall \in \mathbb{C}^3 \textit{ and a.e. in } D, \tag{2.10}$$

$$\left(\textit{respectively, } \Re\left(N(I-N)^{-1}\xi, \xi\right) \geq \gamma|\xi|^2, \forall \in \mathbb{C}^3 \textit{ and a.e. in } D\right). \tag{2.11}$$

Then $\mathcal{A}_k$ (respectively $\tilde{\mathcal{A}}_k$) is a coercive sesquilinear form on $\mathcal{U}_0(D) \times \mathcal{U}_0(D)$.

Proof. Assume first that (2.10) holds. In this case, we can show that $\mathcal{A}_k$ is coercive on $\mathcal{U}_0(D) \times \mathcal{U}_0(D)$. Let $\mathbf{u} \in \mathcal{U}_0(D)$. From (2.10), we deduce that

$$\begin{aligned}\mathcal{A}_k(\mathbf{u}, \mathbf{u}) &\geq \gamma||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u}||_D^2 + k^4||\mathbf{u}||_D^2 \\ &\geq \gamma||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2 + k^4(\gamma+1)||\mathbf{u}||_D^2 - 2\gamma k^2||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D||\mathbf{u}||_D.\end{aligned}$$

Using the identity

$$\gamma X^2 - 2\gamma XY + (\gamma+1)Y^2 = \left(\gamma + \frac{1}{2}\right)\left(Y - \frac{\gamma}{\gamma+\frac{1}{2}}X\right)^2 + \frac{1}{2}Y^2 + \frac{\gamma}{1+2\gamma}X^2 \tag{2.12}$$

with $\varepsilon = \gamma + 1/2$, $X = ||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D$ and $Y = k^2||\mathbf{u}||_D$, we get

$$\mathcal{A}_k(\mathbf{u}, \mathbf{u}) \geq \frac{\gamma}{1+2\gamma}\left(||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2 + k^2||\mathbf{u}||_D^2\right). \tag{2.13}$$

Furthermore, since for all $\mathbf{u} \in \mathcal{U}_0(D)$,

$$||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u}||_D^2 = ||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2 + k^4||\mathbf{u}||_D^2 - 2k^2||\mathrm{curl}\,\mathbf{u}||_D^2,$$

we obtain

$$2k^2||\mathrm{curl}\,\mathbf{u}||_D^2 \leq ||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2 + k^4||\mathbf{u}||_D^2,$$

which combined with (2.13) yields the existence of a constant $c_k > 0$ (independent of $\mathbf{u}$ and γ) such that

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}) \geq c_k \frac{\gamma}{1+2\gamma}||\mathbf{u}||_{\mathcal{U}(D)}^2.$$

Now, assume that (2.11) holds and show that $\tilde{\mathcal{A}}_k$ is coercive on $\mathcal{U}_0(D) \times \mathcal{U}_0(D)$. From (2.11), we deduce that

$$\begin{aligned}\tilde{\mathcal{A}}_k(\mathbf{u},\mathbf{u}) &\geq \gamma||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u}||_D^2 + ||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2 \\ &\geq k^4\gamma||\mathbf{u}||_D^2 + (\gamma+1)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2 - 2\gamma k^2||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D||\mathbf{u}||_D.\end{aligned}$$

Using the same method as in the previous case with this time, $X = k^2||\mathbf{u}||_D$ and $Y = ||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D$, we also get that $\tilde{\mathcal{A}}_k$ is coercive on $\mathcal{U}_0(D) \times \mathcal{U}_0(D)$. □

From the Riesz representation theorem, we can define the operators $A_k : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$, $\tilde{A}_k : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ and $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ by

$$(A_k\mathbf{u},\mathbf{v})_{\mathcal{U}(D)} = \mathcal{A}_k(\mathbf{u},\mathbf{v}), \quad \left(\tilde{A}_k\mathbf{u},\mathbf{v}\right)_{\mathcal{U}(D)} = \tilde{\mathcal{A}}_k(\mathbf{u},\mathbf{v})$$

and

$$(B\mathbf{u},\mathbf{v})_{\mathcal{U}(D)} = \mathcal{B}(\mathbf{u},\mathbf{v})$$

for all $\mathbf{v} \in \mathcal{U}_0(D)$.

Lemma 2.3.3. *The operator $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is compact.*

Proof. Let $(\mathbf{u}_n)$ be a bounded sequence in $\mathcal{U}_0(D)$. We can extract a subsequence, still denoted by $(\mathbf{u}_n)$, that weakly converges to some $\mathbf{u}_0$ in $\mathcal{U}_0(D)$. We recall that if the boundary Γ of D is sufficiently smooth, the space of functions

$$\{\mathbf{u} \in H_0(\mathrm{curl}\,, D)/\mathrm{div}\,\mathbf{u} = 0 \text{ in } D\}$$

is continuously embedded into $H^1(D)$. We deduce that the sequence $(\mathrm{curl}\,\mathbf{u}_n)$ is bounded in $H^1(D)$. By the Rellich compact embedding theorem, we deduce that $(\mathrm{curl}\,\mathbf{u}_n)$ converges strongly to $\mathrm{curl}\,\mathbf{u}_0$ in $L^2(D)$. From the definition of B and using Schwarz inequality, we get

$$||B(\mathbf{u}_n - \mathbf{u}_0)||_{\mathcal{U}(D)} \leq ||\mathrm{curl}\,(\mathbf{u}_n - \mathbf{u}_0)||_{L^2(D)}.$$

Hence, $B\mathbf{u}_n$ converges strongly to $B\mathbf{u}_0$ in $\mathcal{U}_0(D)$ and we can conclude on the compactness of the operator B. □

Now we can state the main theorem of this section on the the well-posedness of (ITP2.1).

Theorem 2.3.4. *Assume that N satisfies* (2.10) *or* (2.11) *and that k is not a transmission eigenvalue. Let $\mathbf{G} \in TH^{3/2}(\Gamma)$ and $\mathbf{H} \in TH^{1/2}(\Gamma)$. Then there exists a unique solution $\mathbf{F} \in \mathcal{U}(D)$ to* (2.6) *such that*

$$||\mathbf{F}||_{\mathcal{U}(D)} \leq C\,(||\mathbf{G}||_{TH^{3/2}} + ||\mathbf{H}||_{TH^{1/2}})$$

where $C > 0$ is a constant independent of $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$.

Proof. Since A_k or $\tilde{A}_k$ is invertible (as a consequence of Lemma 2.3.2) and B is compact, the Fredholm alternative ensures the existence of a unique solution $\mathbf{F}_0$ to (2.8) in both cases (2.10) and (2.11). This solution depends continuously on the data $\mathbf{G}$ and $\mathbf{H}$ which also yields the a priori estimate. □

2.4 Transmission eigenvalues

In this section, we consider the homogeneous interior transmission problem i.e. $\mathbf{G} = \mathbf{H} = 0$,

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 & \text{in } D \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 & \text{in } D \\ \nu\times\mathbf{E} - \nu\times\mathbf{E}_0 = 0 & \text{on } \Gamma \\ \nu\times\operatorname{curl}\mathbf{E} - \nu\times\operatorname{curl}\mathbf{E}_0 = 0 & \text{on } \Gamma \end{cases}$$

and we are interested in the values of k for which the previous problem has a non trivial solution $\mathbf{E}$, $\mathbf{E}_0 \in L^2(D)^3$ such that $\mathbf{E} - \mathbf{E}_0 \in \mathcal{U}_0(D)$.

2.4.1 Discreteness of the set of transmission eigenvalues

To show the discreteness, we use the analytic Fredholm theorem that is recalled in Theorem 2.2.2. The Fredholm property of the operator $A_k - k^2 B$ when $(N-I)^{-1}$ is definite positive and $\tilde{A}_k - k^2 B$ when $N(I-N)^{-1}$ is definite positive has been proven in the previous section. To conclude, it only remains to show that this operator is invertible for at least one wave number k.

Let $0 < \eta_1(x) < \eta_2(x) < \eta_3(x)$ be the eigenvalues of the positive definite matrix N. We recall that the smallest eigenvalue is given by $\eta_1(x) = \inf_{||\xi||=1}(N(x)\xi,\xi)$ and the largest eigenvalue is given by $\eta_3(x) = \sup_{||\xi||=1}(N(x)\xi,\xi)$. We denote by $N^* = \sup_D \eta_3(x)$ and $N_* = \inf_D \eta_1(x)$. Let $\lambda_0(D)$ be the first Dirichlet eigenvalue for $-\Delta$ in D.

This first lemma shows that the operators $A_k - k^2 B$ or $\tilde{A}_k - k^2 B$ are injective provided k is small enough.

Lemma 2.4.1. *If $(N-I)^{-1}$ is a bounded positive definite matrix field on D, then*

$$\left(A_k\mathbf{u} - k^2 B\mathbf{u}, \mathbf{u}\right)_{\mathcal{U}(D)} \geq \alpha||\mathbf{u}||_{\mathcal{U}(D)} \textit{ for all } 0 < k^2 < \frac{\lambda_0(D)}{N^*} \textit{ and } \mathbf{u} \in \mathcal{U}_0.$$

If $N(I-N)^{-1}$ is a bounded positive definite matrix field on D, then

$$\left(\tilde{A}_k\mathbf{u} - k^2 B\mathbf{u}, \mathbf{u}\right)_{\mathcal{U}(D)} \geq \alpha||\mathbf{u}||_{\mathcal{U}(D)} \textit{ for all } 0 < k^2 < \lambda_0(D) \textit{ and } \mathbf{u} \in \mathcal{U}_0.$$

Proof. Assume that $(N-I)^{-1}$ is bounded positive definite. For $\gamma = \dfrac{1}{N^*-1}$,

$$\begin{aligned} \left(A_k\mathbf{u} - k^2 B\mathbf{u}, \mathbf{u}\right)_{\mathcal{U}(D)} &\geq \gamma||\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}||_D^2 + k^4||\mathbf{u}||_D^2 - k^2||\operatorname{curl}\mathbf{u}||_D^2 \\ &\geq \gamma||\operatorname{curl}\operatorname{curl}\mathbf{u}||_D^2 - 2k^2\gamma||\operatorname{curl}\operatorname{curl}\mathbf{u}||_D||\mathbf{u}||_D + k^4(\gamma+1)||\mathbf{u}||_D^2 \\ &\quad - k^2||\operatorname{curl}\mathbf{u}||_D^2. \end{aligned}$$

From the following identity, valid for all $\gamma < \varepsilon < \gamma + 1$,

$$\gamma X^2 - 2\gamma XY + (\gamma+1)Y^2 = \varepsilon\left(Y - \frac{\gamma}{\varepsilon}X\right)^2 + (1+\gamma-\varepsilon)Y^2 + \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)X^2 \tag{2.14}$$

with $X = ||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D$ and $Y = k^2||\mathbf{u}||_D$, we obtain

$$\left(A_k\mathbf{u} - k^2B\mathbf{u}, \mathbf{u}\right)_{\mathcal{U}(D)} \geq \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2 + k^4(1+\gamma-\varepsilon)||\mathbf{u}||_D^2 - k^2||\mathrm{curl}\,\mathbf{u}||_D^2. \tag{2.15}$$

First, we observe that since $\mathbf{u}\times\nu = 0$ on Γ, then $\mathrm{curl}\,\mathbf{u}\cdot\nu = 0$ on Γ. This holds true by interpreting the relationship $\mathrm{curl}\,\mathbf{u}\cdot\nu = 0$ in the weak sense [46]. On the other hand, the continuous embedding of

$$\{\mathbf{u} \in H_0(\mathrm{curl}\,, D)/\mathrm{div}\,\mathbf{u} = 0 \text{ in } D\}$$

into $H^1(D)^3$ implies that $\mathrm{curl}\,\mathbf{u} \in H_0^1(D)$. Then the Poincaré inequality now implies

$$||\mathrm{curl}\,\mathbf{u}||_D^2 \leq \frac{1}{\lambda_0(D)}||\nabla\mathrm{curl}\,\mathbf{u}||_D^2.$$

Let $\tilde{\mathbf{v}}$ be the extension of $\mathrm{curl}\,\mathbf{u}$ by 0 outside D. Then

$$||\nabla\mathrm{curl}\,\mathbf{u}||_D^2 = ||\nabla\tilde{\mathbf{v}}||_{\mathbb{R}^3}^2 = ||\mathrm{curl}\,\tilde{\mathbf{v}}||_{\mathbb{R}^3}^2 + ||\mathrm{div}\,\tilde{\mathbf{v}}||_{\mathbb{R}^3}^2 = ||\mathrm{curl}\,\tilde{\mathbf{v}}||_D^2 + ||\mathrm{div}\,\tilde{\mathbf{v}}||_D^2.$$

We therefore obtain that

$$||\mathrm{curl}\,\mathbf{u}||_D^2 \leq \frac{1}{\lambda_0(D)}||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2. \tag{2.16}$$

Now, from (2.15) and (2.16), we obtain

$$\left(A_k\mathbf{u} - k^2B\mathbf{u}, \mathbf{u}\right)_{\mathcal{U}(D)} \geq \left(\gamma - \frac{\gamma^2}{\varepsilon} - \frac{k^2}{\lambda_0(D)}\right)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2 + k^4(1+\gamma-\varepsilon)||\mathbf{u}||_D^2.$$

Hence, $A_k - k^2B$ is positive as long as $k^2 < \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)\lambda_0(D)$. In particular, letting ε arbitrarily close to $\gamma+1$, the latter becomes $k^2 < \frac{\gamma}{\gamma+1}\lambda_0(D) = \frac{\lambda_0(D)}{N^*}$.

Now assume that $N(I-N)^{-1}$ is bounded positive definite. Then for $\gamma = \frac{N_*}{1-N_*}$,

$$\begin{aligned}\left(\tilde{A}_k\mathbf{u} - k^2B\mathbf{u}, \mathbf{u}\right)_{\mathcal{U}(D)} &\geq \gamma||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u}||_D^2 + ||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2 - k^2||\mathrm{curl}\,\mathbf{u}||_D^2 + k^4||\mathbf{u}||_D^2 \\ &\geq (\gamma+1)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2 - 2k^2\gamma||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D||\mathbf{u}||_D \\ &\quad - k^2||\mathrm{curl}\,\mathbf{u}||_D^2 \\ &\geq \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)k^4||\mathbf{u}||_D^2 + (1+\gamma-\varepsilon)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2 - k^2||\mathrm{curl}\,\mathbf{u}||_D^2 \\ &\geq \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)k^4||\mathbf{u}||_D^2 + \left(1+\gamma-\varepsilon - \frac{k^2}{\lambda_0(D)}\right)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_D^2.\end{aligned}$$

Then, $\tilde{A}_k - k^2B$ is positive as long as $k^2 < (1+\gamma-\varepsilon)\lambda_0(D)$. In particular, letting ε arbitrarily close to γ, the latter becomes $k^2 < \lambda_0(D)$. □

Remark 2.4.1. *This lemma gives a lower bound to the first transmission eigenvalue. Indeed, it implies that the first transmission eigenvalue k_0 is such that*

- $k_0 \geq \dfrac{\lambda_0(D)}{N^*}$ *if $(N-I)^{-1}$ is bounded positive definite on D,*
- $k_0 \geq \lambda_0(D)$ *if $(I-N)^{-1}$ is bounded positive definite on D.*

Note that the kernel of $B : \mathcal{U}_0(D) \to \mathcal{U}_0(D)$ is given by

$$\mathrm{Kernel}(B) = \left\{ \mathbf{u} \in \mathcal{U}_0(D) / \mathbf{u} := \nabla\varphi, \varphi \in H^1(D) \right\}.$$

We can now state the discreteness of the set of transmission eigenvalues. We also give an existence result for particular index of refraction. More specifically, we can show that there are no transmission eigenvalues provided the imaginary part of the index of refraction N is positive definite.

Theorem 2.4.2. *Assume that N satisfies* (2.10) *or* (2.11)*. Then*

(i) *The set of transmission eigenvalues is discrete with $+\infty$ the only accumulation point.*

(ii) *If $\Im(N)$ is positive almost everywhere in D, then the set of transmission eigenvalues is empty.*

Proof. (i) This is a direct consequence to the analytic Fredholm theory. Indeed, the previous lemma shows that $A_k - k^2 B$ is invertible when k^2 is small enough. Then, $(A_k - k^2 B)^{-1}$ exists except for a discrete set of k. We can conclude on the discreteness of the set of transmission eigenvalues.

(ii) Let $k > 0$ be a transmission eigenvalue and $\mathbf{u} \in \mathcal{U}_0(D)$ the corresponding eigenvector which satisfies in particular

$$\int_D (N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u}) \cdot (\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{u}} - k^2\overline{\mathbf{u}}) dx + k^4 \int_D |\mathbf{u}|^2 dx - k^2 \int_D |\mathrm{curl}\,\mathbf{u}|^2 dx = 0.$$

Since $\Im(N-I)^{-1}$ is negative definite in D and all the terms are real except for the first one, by taking the imaginary part, we deduce that $\mathbf{u}$ satisfies Maxwell equations in D. Since $\mathbf{u}$ has zero Cauchy data on Γ, we obtain $\mathbf{u} = 0$ in D which contradicts the fact that k is a transmission eigenvalue. Then there are no transmission eigenvalues. □

Remark 2.4.2. *The property* (ii) *can be extended to the case where $Im(N)$ is positive only in a subset D_0 of D of positive measure and where N is sufficiently regular to allow the use of the unique continuation principle for* curl curl $- k^2 N$*. Indeed, the same argument as the previous proof shows that*

$$\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u} = 0 \quad \textit{in } D_0.$$

Now, if we define $v := \operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}$ in D, then v is equal to zero inside D_0 and satisfies

$$\operatorname{curl}\operatorname{curl}\mathbf{v} - k^2 N\mathbf{v} = 0 \quad \textit{in } D.$$

From the unique continuation principle, $F = 0$ in D. The end of the proof is similar to the previous one. Since $\mathbf{u}$ is now a solution to Maxwell equations in D that has zero Cauchy data on Γ, then $\mathbf{u} = 0$ in D which contradicts the fact that k is a transmission eigenvalue. Then there are no transmission eigenvalues.

2.4.2 Existence of transmission eigenvalues

To show existence of transmission eigenvalues, we use Theorem 2.2.4 and the existence of transmission eigenvalues for spherical geometries that has already been proven in [26].

Let us consider the interior transmission problem corresponding to a ball B_R of radius R centered at zero with constant index of refraction $N = n_0 I$ where I denotes the identity matrix. It is shown in [26] using separation of variables that there exists a discrete set of transmission eigenvalues corresponding to

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 n_0 \mathbf{E} = 0 & \text{in } B_R \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2 \mathbf{E}_0 = 0 & \text{in } B_R \\ \nu \times \mathbf{E} - \nu \times \mathbf{E}_0 = 0 & \text{on } \partial B_R \\ \nu \times \operatorname{curl}\mathbf{E} - \nu \times \operatorname{curl}\mathbf{E}_0 = 0 & \text{on } \partial B_R. \end{cases}$$

Let us denote by $k_0(R, n_0)$ the first transmission eigenvalue corresponding to the previous problem and by $\mathbf{u}^{B_R,n_0} := \mathbf{E}^{B_R,n_0} - \mathbf{E}_0^{B_R,n_0}$ the corresponding eigenfunction. In particular, $\mathbf{u}^{B_R,n_0}$ is in $\mathcal{U}_0(B_R)$ and satisfies

$$\int_{B_R} \frac{1}{n_0 - 1} \left(\operatorname{curl}\operatorname{curl}\mathbf{u}^{B_R,n_0} - k_0(R,n_0)^2 \mathbf{u}^{B_R,n_0}\right) \cdot \left(\operatorname{curl}\operatorname{curl}\overline{\mathbf{u}}^{B_R,n_0} - k_0(R,n_0)^2 n_0 \overline{\mathbf{u}}^{B_R,n_0}\right) dx = 0.$$

Theorem 2.4.3. *Assume that $N \in L^\infty(D, \mathbb{R}^{3\times 3})$ is such that either $(N-I)^{-1}$ or $N(I-N)^{-1}$ is bounded positive definite. Then, there exists an infinite set of transmission eigenvalues.*

Proof. First assume that $(N-I)^{-1}$ is positive definite. From Lemma 2.4.1, we have that the first assumption of Theorem 2.2.4 is satisfied for all $0 < k^2 < \dfrac{\lambda_0(D)}{N^*}$.

Now, let $\varepsilon > 0$ and B_ε^j, $j = 1, \cdots, m(\varepsilon)$, be $m(\varepsilon)$ disjoint balls included in D.

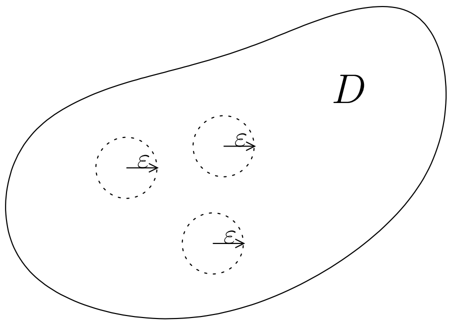


Figure 2.2: Balls of radius ε included in D.

Let $k_0(\varepsilon, N_*)$ be the first transmission eigenvalue for each of these balls with index of refraction N_*. Let us denote by $\mathbf{u}^{B^j_\varepsilon, N_*} \in \mathcal{U}_0(B^j_\varepsilon)$, $j = 1, \cdots, m(\varepsilon)$, the corresponding eigenfunctions. Let $\tilde{\mathbf{u}}^j$, $j = 1, \cdots, m(\varepsilon)$, be the extension of $\mathbf{u}^{B^j_\varepsilon, N_*}$ by zero to the whole D. It is obvious that $\tilde{\mathbf{u}}^j$, $j = 1, \cdots, m(\varepsilon)$, are in $\mathcal{U}_0(D)$ due to the boundary conditions on ∂B^j_ε. Furthermore, we have that

$$
\begin{aligned}
0 &= \int_{B^j_\varepsilon} \frac{1}{N_* - 1} \left(\operatorname{curl}\operatorname{curl}\mathbf{u}^{B^j_\varepsilon, N_*} - k_0(\varepsilon, N_*)^2 \mathbf{u}^{B^j_\varepsilon, N_*}\right) \cdot \left(\operatorname{curl}\operatorname{curl}\overline{\mathbf{u}}^{B^j_\varepsilon, N_*} - k_0(\varepsilon, N_*)^2 N_* \overline{\mathbf{u}}^{B^j_\varepsilon, N_*}\right) dx \\
&= \int_{B^j_\varepsilon} \frac{1}{N_* - 1} \left(\operatorname{curl}\operatorname{curl}\mathbf{u}^{B^j_\varepsilon, N_*} - k_0(\varepsilon, N_*)^2 \mathbf{u}^{B^j_\varepsilon, N_*}\right) \cdot \left(\operatorname{curl}\operatorname{curl}\overline{\mathbf{u}}^{B^j_\varepsilon, N_*} - k_0(\varepsilon, N_*)^2 N_* \overline{\mathbf{u}}^{B^j_\varepsilon, N_*}\right) dx.
\end{aligned}
$$

The functions $\left\{\tilde{\mathbf{u}}^1, ..., \tilde{\mathbf{u}}^{m(\varepsilon)}\right\}$ are linearly independent and orthogonal in $\mathcal{U}_0(D)$ since they have disjoint supports. Let us denote by $\mathcal{V}$ the $m(\varepsilon)$-dimensional subspace of $\mathcal{U}_0(D)$ spanned by $\left\{\tilde{\mathbf{u}}^1, ..., \tilde{\mathbf{u}}^{m(\varepsilon)}\right\}$. Then, if $k_1 := k_0(\varepsilon, N_*)$, for all $\tilde{\mathbf{u}}$ in $\mathcal{V}$

$$
\begin{aligned}
\left(A_{k_1}\tilde{\mathbf{u}} - k_1^2 B\tilde{\mathbf{u}}, \tilde{\mathbf{u}}\right) &= \int_D (N - I)^{-1} (\operatorname{curl}\operatorname{curl}\tilde{\mathbf{u}} - k_1^2 \tilde{\mathbf{u}}) \cdot (\operatorname{curl}\operatorname{curl}\overline{\tilde{\mathbf{u}}} - k_1^2 \overline{\tilde{\mathbf{u}}}) dx \\
&\quad + k_1^4 \int_D |\tilde{\mathbf{u}}|^2 dx - k_1^2 \int_D |\operatorname{curl}\tilde{\mathbf{u}}|^2 dx \\
&\leq \int_D \frac{1}{N_* - 1} (\operatorname{curl}\operatorname{curl}\tilde{\mathbf{u}} - k_1^2 \tilde{\mathbf{u}}) \cdot (\operatorname{curl}\operatorname{curl}\overline{\tilde{\mathbf{u}}} - k_1^2 \overline{\tilde{\mathbf{u}}}) dx \\
&\quad + k_1^4 \int_D |\tilde{\mathbf{u}}|^2 dx - k_1^2 \int_D |\operatorname{curl}\tilde{\mathbf{u}}|^2 dx \\
&= 0.
\end{aligned}
$$

We deduce that there exist $m(\varepsilon)$ transmission eigenvalues in $\left[\dfrac{\lambda_0(D)}{N_*}, k_0(\varepsilon, N_*)\right]$. By letting $\varepsilon \to 0$, we can conclude on the existence of a infinite discrete set of transmission eigenvalues.

Now, if $N(I - N)^{-1}$ is positive definite, the proof goes in the same way as previously by using the second formulation with $\tilde{A}_k$. In this case, from Lemma 2.4.1, the first assumption of Theorem 2.2.4 is satisfied for all $0 < k^2 < \lambda_0(D)$. Using the fact that

$$
\left(N(I - N)^{-1}\xi, \xi\right) \leq \frac{N^*}{1 - N^*} |\xi|^2,
$$

the rest of the proof is similar by replacing N_* by N^*. $\square$

2.4.3 Estimates on the first transmission eigenvalue

In this section, we shall improve the lower bound of the first transmission eigenvalue given by Lemma 2.4.1 and also give an upper bound. This shall be done using again Theorem 2.2.4 by comparing with the transmission eigenvalues of the smallest ball containing D and the largest ball contained in D. We also give a monotonicity result of the first transmission eigenvalue with respect to N.

Let B_{r_1} be the largest ball of radius r_1 included in D and let B_{r_2} be the smallest ball containing D.

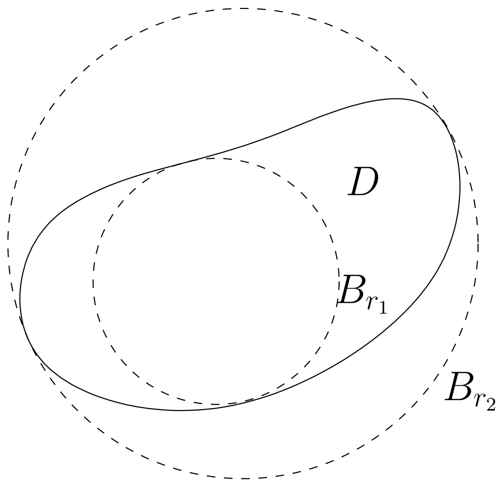


Figure 2.3: Inclusion of the domain D between two spheres of radius r_1 and r_2.

Theorem 2.4.4. *Assume that $N \in L^\infty(D, \mathbb{C}^3)^{3\times 3}$.*

(i) *If $(N-I)^{-1}$ is bounded positive definite and in particular $N_* \geq 1+\alpha > 1$, then*

$$0 < k_0(B_{r_2}, N^*) \leq k_0(D, N) \leq k_0(B_{r_1}, N_*).$$

(ii) *If $N(I-N)^{-1}$ is bounded positive definite and in particular $N^* \leq 1-\beta < 1$, then*

$$0 < k_0(B_{r_2}, N_*) \leq k_0(D, N) \leq k_0(B_{r_1}, N^*).$$

Proof. First, we shall show that for all N in $L^\infty(D, \mathbb{C}^3)^{3\times 3}$, the first transmission eigenvalue decreases with the size of the domain D. In particular, if $D' \subset D$, then

$$k_0(D, N) \leq k_0(D', N).$$

Let $\mathbf{u}^{D',N} \in \mathcal{U}_0(D')$ be the eigenfunction corresponding to $k_0(D', N)$ which satisfies

$$\int_{D'} (N-I)^{-1} \left(\operatorname{curl}\operatorname{curl}\mathbf{u}^{D',N} - k_0(D',N)^2\mathbf{u}^{D',N}\right)\cdot\left(\operatorname{curl}\operatorname{curl}\overline{\mathbf{u}}^{D',N} - k_0(D',N)^2 N\overline{\mathbf{u}}^{D',N}\right) dx = 0$$

Let $\tilde{\mathbf{u}}$ be the extension of $\mathbf{u}^{D',N}$ to all D. Due to the boundary conditions on $\partial D'$, $\tilde{\mathbf{u}}$ is in $\mathcal{U}_0(D)$ and if $k_1 = k_0(D', N)$, we have

$$\begin{aligned}
(A_{k_1}\tilde{\mathbf{u}} - k_1 B\tilde{\mathbf{u}}, \tilde{\mathbf{u}}) &= \int_D (N-I)^{-1}\left(\operatorname{curl}\operatorname{curl}\tilde{\mathbf{u}} - k_1^2\tilde{\mathbf{u}}\right)\cdot\left(\operatorname{curl}\operatorname{curl}\overline{\tilde{\mathbf{u}}} - k_1^2 N\overline{\tilde{\mathbf{u}}}\right) dx \\
&= \int_{D'} (N-I)^{-1}\left(\operatorname{curl}\operatorname{curl}\mathbf{u}^{D',N} - k_1^2\mathbf{u}^{D',N}\right)\cdot\left(\operatorname{curl}\operatorname{curl}\overline{\mathbf{u}}^{D',N} - k_1^2 N\overline{\mathbf{u}}^{D',N}\right) dx \\
&= 0.
\end{aligned}$$

Using the fact that $A_k - k^2B$ is positive for k small enough, applying Theorem 2.2.4, we deduce that $k_0(D,N) \leq k_0(D',N)$.

Now assume that $(N-I)^{-1}$ is positive definite. For all $\mathbf{u} \in \mathcal{U}_0(D)$, we have

$$\frac{\frac{1}{N^*-1}||\operatorname{curl}\operatorname{curl}\mathbf{u} - \tau\mathbf{u}||_D^2 + \tau^2||\mathbf{u}||_D^2}{||\operatorname{curl}\mathbf{u}||_D^2} \leq \frac{(A_k\mathbf{u},\mathbf{u})_{\mathcal{U}}}{(B\mathbf{u},\mathbf{u})_{\mathcal{U}}} \leq \frac{\frac{1}{N_*-1}||\operatorname{curl}\operatorname{curl}\mathbf{u} - \tau\mathbf{u}||_D^2 + \tau^2||\mathbf{u}||_D^2}{||\operatorname{curl}\mathbf{u}||_D^2}$$

and consequently

$$\lambda_1(\tau, N^*) \leq \lambda_1(\tau, N(x)) \leq \lambda_1(\tau, N_*). \tag{2.17}$$

For $\tau = k_0(D,N^*)^2$, we obtain

$$\lambda_1(k_0(D,N^*)^2, N(x)) - k_0(D,N^*)^2 \geq 0,$$

and for $\tau = k_0(D,N_*)^2$

$$\lambda_1(k_0(D,N_*)^2, N(x)) - k_0(D,N_*)^2 \leq 0.$$

From the continuity of $\tau \mapsto \lambda_1(\tau, N(x)) - \tau$, there exists τ_0 between $k_0(D,N^*)^2$ and $k_0(D,N_*)^2$ such that $\lambda_1(\tau_0, N(x)) - \tau_0 = 0$. To conclude, it remains to show that there are no transmission eigenvalues corresponding to $N(x)$ smaller than $k_0(D,N^*)$. To this end, assume that $k_0(D,N(x)) < k_0(D,N^*)^2$. From (2.17), we first have that

$$\lambda_1(k_0(D,N(x))^2, N^*) - k_0(D,N(x))^2 \leq 0.$$

Furthermore, from Lemma 2.4.1, we know that for τ sufficiently small, $\lambda_1(\tau,N^*) - \tau \geq 0$. Then there exists $\tau_1 < k_0(D,N(x)$ such that $\lambda_1(\tau_1,N^*) - \tau_1 \geq 0$. From the continuity of $\tau \mapsto \lambda_1(\tau,N^*) - \tau$, there exists a transmission eigenvalue for N^* in $\left[\sqrt{\tau_1}, k_0(D,N(x))\right]$ which contradicts the fact that $k_0(D,N^*)$ is the smallest transmission eigenvalue for N^*. Then, we conclude that

$$k_0(D,N^*) \leq k_0(D,N(x)) \leq k_0(D,N_*).$$

In the case where $N(I-N)^{-1}$ is positive definite, we use the variational formulation with $\tilde{A}_k$ and the inequality

$$\frac{\frac{N_*}{1-N_*}||\operatorname{curl}\operatorname{curl}\mathbf{u} - \tau\mathbf{u}||_D^2 + ||\operatorname{curl}\operatorname{curl}\mathbf{u}||_D^2}{||\operatorname{curl}\mathbf{u}||_D^2} \leq \frac{(\tilde{A}_k\mathbf{u},\mathbf{u})_{\mathcal{U}}}{(B\mathbf{u},\mathbf{u})_{\mathcal{U}}} \leq \frac{\frac{N^*}{1-N^*}||\operatorname{curl}\operatorname{curl}\mathbf{u} - \tau\mathbf{u}||_D^2 + ||\operatorname{curl}\operatorname{curl}\mathbf{u}||_D^2}{||\operatorname{curl}\mathbf{u}||_D^2}.$$

We deduce that

$$\lambda_1(\tau, N_*) \leq \lambda_1(\tau, N(x)) \leq \lambda_1(\tau, N^*).$$

Now, the rest of the proof is similar and yields the inequality

$$k_0(D,N_*) \leq k_0(D,N(x)) \leq k_0(D,N^*).$$

□

Remark 2.4.3. *Due to the monotonicity of the Dirichlet eigenvalue with respect to the size of the domain, the lower bound is improved. First, since $D \subset B_{r_2}$, we have $\lambda_1(D) \leq \lambda_1(B_{r_2})$. Then, with Lemma 2.4.1 applied to the domain B_{r_2}, we have*

$$\frac{\lambda_1(D)}{N^*} \leq \frac{\lambda_1(B_{r_2})}{N^*} \leq k_0(B_{r_2}, N^*) \leq k_0(D, N(x)).$$

2.4.4 Continuity of the first transmission eigenvalue with respect to N

Numerical computations of transmission eigenvalues have suggested that the first transmission eigenvalue behaved continuously with the index of refraction. Using the implicit function theorem, we have been able to justify mathematically this result.

Assume here that $N = nI$ where n is constant and I denotes the identity matrix. Let us define $\mathcal{W}_0(D) := \mathcal{U}_0(D) \cap H_0(\operatorname{div} 0, D)$ where

$$H_0(\operatorname{div} 0, D) = \left\{ \mathbf{u} \in L^2(D)^3 / \operatorname{div} \mathbf{u} = 0 \text{ and } \nu \cdot \mathbf{u} = 0 \right\}.$$

$$\begin{aligned}
\lambda_1(n, \tau) &= \inf_{\mathbf{u} \in \mathcal{W}_0(D)} \frac{\frac{1}{n-1} ||\operatorname{curl} \operatorname{curl} \mathbf{u} - \tau \mathbf{u}||_D^2 + \tau^2 ||\mathbf{u}||_D^2}{||\operatorname{curl} \mathbf{u}||_D^2} \\
&= \frac{1}{n-1} \inf_{\mathbf{u} \in \mathcal{W}_0(D)} \frac{||\operatorname{curl} \operatorname{curl} \mathbf{u}||_D^2 + n\tau^2 ||\mathbf{u}||_D^2}{||\operatorname{curl} \mathbf{u}||_D^2} - 2\frac{\tau}{n-1}.
\end{aligned}$$

The first transmission eigenvalue $k > 0$ corresponds to the first zero $\tau := k^2$ of

$$f(n, \tau) := \mu_1(n\tau^2) - (n+1)\tau = 0$$

where

$$\mu_1(b) := \inf_{\mathbf{u} \in \mathcal{W}_0(D)} \frac{||\operatorname{curl} \operatorname{curl} \mathbf{u}||_D^2 + b||\mathbf{u}||_D^2}{||\operatorname{curl} \mathbf{u}||_D^2}.$$

To apply the implicit function theorem to the function f, we need to differentiate the function f with respect to τ. The following lemma gives the derivative of the function μ_1.

Lemma 2.4.5. *The function* $\mu_1 :]0, +\infty[\to]0, +\infty[$ *is differentiable and* $\mu_1'(b) = ||\mathbf{u}_b||^2$ *where* $\mathbf{u}_b \in \mathcal{W}_0(D)$ *satisfies*

$$||\operatorname{curl} \operatorname{curl} \mathbf{u}_b||_D^2 + b||\mathbf{u}_b||_D^2 = \mu_1(b) \text{ and } ||\operatorname{curl} \mathbf{u}_b||_D = 1.$$

Proof. Let $h \in [-\varepsilon, \varepsilon]$ with $\varepsilon > 0$. We have

$$\begin{aligned}
\mu_1(b+h) - \mu_1(b) &\leq \left(||\operatorname{curl} \operatorname{curl} \mathbf{u}_b||_D^2 + (b+h)||u_b||_D^2 \right) - \left(||\operatorname{curl} \operatorname{curl} \mathbf{u}_b||_D^2 + b||\mathbf{u}_b||_D^2 \right) \\
&\leq h||\mathbf{u}_b||_D^2
\end{aligned}$$

and

$$\begin{aligned}
\mu_1(b+h) - \mu_1(b) &\geq \left(||\operatorname{curl} \operatorname{curl} \mathbf{u}_{b+h}||_D^2 + (b+h)||u_{b+h}||_D^2 \right) \\
&\quad - \left(||\operatorname{curl} \operatorname{curl} \mathbf{u}_{b+h}||_D^2 + b||\mathbf{u}_{b+h}||_D^2 \right) \\
&\geq h||\mathbf{u}_{b+h}||_D^2.
\end{aligned}$$

Therefore

$$||\mathbf{u}_{b+h}||_D^2 \leq \frac{\mu_1(b+h) - \mu_1(b)}{h} \leq ||\mathbf{u}_b||_D^2. \tag{2.18}$$

Now, we want to show that $(\mathbf{u}_{b+h})_h$ converges to $\mathbf{u}_b$ in the L^2-norm as h tends to 0. From the previous inequality we have that

$$||\mathbf{u}_{b+h}||_D^2 \leq ||\mathbf{u}_0||_D^2.$$

This proves that $(\mathbf{u}_{b+h})_h$ is bounded in L^2. Furthermore

$$\begin{aligned}||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_{b+h}||_D^2 &= \left\|\mu_1(b+h) - (b+h)||\mathbf{u}_{b+h}||_D^2\right\| \\ &\leq \mu_1(b+h) + (b+\varepsilon)||\mathbf{u}_{b+h}||_D^2 \\ &\leq \mu_1(b) + \varepsilon||\mathbf{u}_b||_D^2 + (b+\varepsilon)||\mathbf{u}_b||_D^2.\end{aligned}$$

Hence, $(\mathbf{u}_{b+h})_h$ is bounded in $\mathcal{U}_0(D)$. Since $\mathcal{W}_0(D)$ is compactly embedded in $H_0(\mathrm{curl}\,, D)$ and in $L^2(D)$ we have that $(\mathbf{u}_{b+h})_h$ converges to $\tilde{\mathbf{u}}$ weakly in the $\mathcal{U}_0(D)$-norm and strongly in the $H_0(\mathrm{curl}\,, D)$-norm and in the $L^2(D)$-norm. For all $\boldsymbol{\psi} \in \mathcal{U}_0(D)$, we have

$$\int_D \mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_{b+h} \cdot \mathrm{curl}\,\mathrm{curl}\,\boldsymbol{\psi} dx + (b+h)\int_D \mathbf{u}_{b+h} \cdot \boldsymbol{\psi} dx = \mu_1(b+h)\int_D \mathrm{curl}\,\mathbf{u}_{b+h} \cdot \mathrm{curl}\,\boldsymbol{\psi} dx.$$

Letting $h \to 0$, $\tilde{\mathbf{u}}$ satisfies

$$\int_D \mathrm{curl}\,\mathrm{curl}\,\tilde{\mathbf{u}} \cdot \mathrm{curl}\,\mathrm{curl}\,\boldsymbol{\psi} dx + b\int_D \tilde{\mathbf{u}} \cdot \boldsymbol{\psi} dx = \mu_1(b)\int_D \mathrm{curl}\,\tilde{\mathbf{u}} \cdot \mathrm{curl}\,\boldsymbol{\psi} dx.$$

We deduce that $\tilde{\mathbf{u}} = \mathbf{u}_b$ and then $(\mathbf{u}_{b+h})_h$ converges to $\mathbf{u}_b$ in the L^2-norm as h tends to 0. Therefore from (2.18)

$$\mu_1'(b) = ||\mathbf{u}_b||_D^2.$$

□

We now have all the tools to prove the continuity of the first transmission eigenvalue with respect to n.

Theorem 2.4.6. *The first transmission eigenvalue is continuous with respect to n.*

Proof. We shall use the implicit function theorem on f to establish the continuity. By the previous lemma, f is continuous and differentiable for all $n > 0$ and $\tau > 0$. If we denote by τ_1 the first transmission eigenvalue for a fixed n we have $f(n, \tau_1) = 0$. Moreover

$$\frac{\partial f}{\partial \tau}(n, \tau_1) = 2n\tau_1||\mathbf{u}_{n\tau_1^2}||_D^2 - (n+1).$$

In particular, since the divergence of $\mathbf{u}_{n\tau_1^2}$ is zero, from the Poincaré inequality we have that

$$||\mathbf{u}_{n\tau_1^2}||^2 \leq 1/\lambda_0(D)$$

where $\lambda_0(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D. Hence $\frac{\partial f}{\partial \tau} < 0$ provided $\tau_1 < \frac{n+1}{2n}\lambda_0(D)$. Now we show that this inequality is satisfied if n is large enough. Remark that, by definition of τ_1, if there exist $\tilde{\tau}$ and $\mathbf{u} \in \mathcal{U}_0(D)$ satisfying

$$\frac{1}{n-1}||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} + \tilde{\tau}\mathbf{u}||_D^2 + \tilde{\tau}^2||\mathbf{u}||_D^2 \leq \tilde{\tau}||\mathrm{curl}\,\mathbf{u}||_D^2$$

i.e.

$$\frac{1}{n-1}\left(||\text{curl}\,\text{curl}\,\mathbf{u}||_D^2 - 2\tilde{\tau}||\text{curl}\,\mathbf{u}||_D^2 + \tilde{\tau}^2||\mathbf{u}||^2\right) + \tilde{\tau}^2||\mathbf{u}||_D^2 \leq \tilde{\tau}||\text{curl}\,\mathbf{u}||_D^2$$

then $\tau_1 \leq \tilde{\tau}$. Let $\mathbf{u} \in \mathcal{U}_0(D)$ be such that

$$\begin{cases} ||\text{curl}\,\text{curl}\,\mathbf{u}||_D^2 = \mu_1(0) \\ ||\text{curl}\,\mathbf{u}||_D^2 = 1. \end{cases}$$

Using the fact that $||\mathbf{u}||_D^2 \leq \dfrac{1}{\lambda_0(D)}$, it is sufficient that τ satisfies

$$\frac{1}{n-1}\left(\mu_1(0) - 2\tau + \frac{\tau^2}{\lambda_0(D)}\right) + \frac{\tau^2}{\lambda_0(D)} \leq \tau \Leftrightarrow \frac{n}{\lambda_0(D)}\tau^2 - (n+1)\tau + \mu_1(0) \leq 0.$$

The discriminant of the left-hand side should be positive

$$\Delta = (n+1)^2 - 4\frac{n\mu_1(0)}{\lambda_0(D)} \geq 0,$$

which is true when n is large enough.

Hence, $\dfrac{\partial f}{\partial \tau}(n,\tau_1) < 0$ provided that n is large enough. In this case the implicit function theorem implies that there exist a neighborhood U of n, a neighborhood V of τ_1 and a $\mathcal{C}^1$ function $k_1 : U \longrightarrow V$ such that

$$f(n,\tau_1) = 0 \Longleftrightarrow \tau_1 = k_1(n).$$

Therefore, locally, the first transmission eigenvalue is a continuous function of n. $\square$

2.5 Characterization of transmission eigenvalues from far field data

We have seen in the first chapter that Maxwell eigenvalues can be retrieved from the far field data in the case of an impenetrable scatterer. This section is devoted to show the same characterization from far field data in the case of a penetrable object where the role of Maxwell eigenvalues is replaced here by transmission eigenvalues.

2.5.1 Reminder of the notations

Let us first recall the hypothesis. The index of refraction N is supposed to be a 3×3 symmetric matrix whose entries are bounded real valued functions in $\mathbb{R}^3$ and such that $N = I$ in $\mathbb{R}^3 \setminus D$. We further assume that there exists a constant $\gamma > 0$ such that either

$$\text{Re}((N-I)^{-1}\xi,\xi) \geq \gamma|\xi|^2$$

or

$$\text{Re}((I-N)^{-1}\xi,\xi) \geq \gamma|\xi|^2$$

for all ξ in $\mathbb{C}^3$ and almost everywhere in D.

Given $\mathbf{E}^i$ an entire solution to Maxwell equations

$$\operatorname{curl}\operatorname{curl}\mathbf{E}^i - k^2\mathbf{E}^i = 0 \text{ in } \mathbb{R}^3,$$

the direct scattering problem can be formulated as the problem of finding an electric field $\mathbf{E} \in \mathcal{U}_{loc}(\mathbb{R}^3)$ such that

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 \text{ in } \mathbb{R}^3 \\ \mathbf{E} = \mathbf{E}^s + \mathbf{E}^i \\ \lim\limits_{|x|\to\infty} (\operatorname{curl}\mathbf{E}^s \times x - \mathrm{i}k|x|\mathbf{E}^s) = 0 \text{ uniformly in } \hat{x} = x/|x|. \end{cases}$$

We recall that the far field operator is given by

$$\mathcal{F}g(\hat{x}) := \int_\Omega \mathbf{E}_\infty(\hat{x}, d, g(d))ds(d).$$

Let us define the mapping $\mathcal{B}$ by

$$\begin{array}{rccc} \mathcal{B}: & H_{\text{inc}}(D) & \to & L^2_t(\Gamma) \\ & \mathbf{E}^i & \mapsto & \mathbf{E}_\infty \end{array}$$

where

$$H_{\text{inc}}(D) := \left\{\mathbf{u} \in L^2(D)^3 / \operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u} = 0 \text{ in } D\right\}$$

so that the far field operator can be expressed in terms of this new operator $\mathcal{B}$

$$\mathcal{F}g = \mathcal{B}(\mathbf{E}_g).$$

We can remark that $\mathbf{E}_{e,\infty}$ is in the range of $\mathcal{B}$ i.e. there exists $\mathbf{E}_0$ such that $\mathcal{B}\mathbf{E}_0 = \mathbf{E}_{e,\infty}$ if and only if $\mathbf{E}$ and $\mathbf{E}_0$ are solutions to the following interior transmission problem : find $\mathbf{E}$ and $\mathbf{E}_0$ in $L^2(D)^3$ such that $\mathbf{E} - \mathbf{E}_0 \in \mathcal{U}(D)$ and

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 & \text{in } D \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 & \text{in } D \\ \nu \times \mathbf{E} - \nu \times \mathbf{E}_0 = \nu \times \mathbf{E}_e(\cdot, z, q) & \text{on } \Gamma \\ \nu \times \operatorname{curl}\mathbf{E} - \nu \times \operatorname{curl}\mathbf{E}_0 = \nu \times \operatorname{curl}\mathbf{E}_e(\cdot, z, q) & \text{on } \Gamma. \end{cases} \tag{2.19}$$

This is a direct consequence to Theorem 1.2.8 and the fact that the set of Herglotz electric wave functions $\mathbf{E}_g$ with $g \in L^2_t(\Omega)$ is dense in $H_{inc}(D)$ with respect to the $L^2(D)$ norm.

Let $\mathcal{F}^\delta$ denote the noisy operator corresponding to noisy measurements $\mathbf{E}^\delta_\infty(\hat{x}, d, q)$. We assume that for all $g \in L^2_t(\Omega)$

$$\mathcal{F}^\delta g = -\mathcal{B}^\delta(\mathbf{E}_g), \text{ where } ||\mathcal{B}^\delta - \mathcal{B}|| \leq \delta$$

where $\delta > 0$ is a measure of the noise level and $\mathcal{B}^\delta$ denotes the noisy bounded operator associated with $\mathcal{B}$. In particular, $\mathcal{F}^\delta$ is a bounded and compact linear operator.

For each fixed z and q, the regularized solution $g_{z,q,\delta}$ is determined by minimizing the Tikhonov functional

$$||\mathcal{F}^\delta g_{z,q,\delta} - \mathbf{E}_{e,\infty}(.,z,q)||^2 + \varepsilon||g_{z,q,\delta}||^2 \tag{2.20}$$

where $\varepsilon := \varepsilon(\delta) \to 0$ as $\delta \to 0$ is the regularization parameter. We assume that $\varepsilon(\delta)$ is such that

$$\lim_{\delta\to 0} ||\mathcal{F}^\delta g_{z,q,\delta} - \mathbf{E}_{e,\infty}(\cdot,z,q)|| = 0. \tag{2.21}$$

Let χ be a cutoff function such that $\chi = 1$ in a neighborhood of Γ and $\chi = 0$ in a neighborhood of the point $z \in D$. We define the function $\mathbf{\Theta}_z \in H^2(D)^3$ by $\mathbf{\Theta}_z = \chi\mathbf{E}_e(\cdot,z,q)$. Then $\mathbf{\Theta}_z$ satisfies the boundary conditions

$$\begin{cases} \nu \times \mathbf{\Theta}_z = \nu \times \mathbf{E}_e(\cdot,z,q) & \text{on } \Gamma \\ \nu \times \operatorname{curl}\mathbf{\Theta}_z = \nu \times \operatorname{curl}\mathbf{E}_e(\cdot,z,q) & \text{on } \Gamma. \end{cases}$$

Consequently, the variational formulation of (2.19) amounts to finding

$$\mathbf{F}_z = \mathbf{E} - \mathbf{E}_0 - \mathbf{\Theta}_z \in \mathcal{U}_0(D)$$

such that

$$\int_D (N-I)^{-1}(\operatorname{curl}\operatorname{curl} - k^2N)(\mathbf{F}_z + \mathbf{\Theta}_z)\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx = 0 \tag{2.22}$$

for all $\mathbf{\Psi} \in \mathcal{U}_0(D)$.

2.5.2 Main theorem

Theorem 2.5.1. *Assume that k is a transmission eigenvalue and that* (2.21) *is verified. Then for almost every $z \in D$, there exists $q \in \mathbb{R}^3$ such that $||\mathbf{E}_{g_{z,q,\delta}}||_{H(\operatorname{curl},D)}$ cannot be bounded when $\delta \to 0$.*

Proof. Assume that for a set $\mathcal{A}$ of points $z \in D$ which has a positive measure, there exists a constant $M > 0$ such that for all $q \in \mathbb{R}^3$,

$$||\mathbf{E}_{g_{z,q,\delta}}||_{H(\operatorname{curl},D)} \leq M.$$

Then we have

$$||\mathcal{F}^\delta g_{z,q,\delta} - \mathcal{F}g_{z,q,\delta}||_{L^2_t(D)} \leq ||\mathcal{B}^\delta - \mathcal{B}||\,||\mathbf{E}_{g_{z,q,\delta}}||_{H_{inc}(D)} \leq M\delta.$$

Using (1.25) and the previous inequality, we deduce that

$$\lim_{\delta\to 0} ||\mathcal{F}g_{z,q,\delta} - \mathbf{E}_{e,\infty}(\cdot,z,q)||_{L^2_t(D)} = 0.$$

Since the set of Herglotz electric wave functions $\mathbf{E}_g$ with $g \in L^2_t(\Omega)$ is dense in $H_{inc}(D)$ with respect to the $L^2(D)$ norm, there exists a subsequence $\mathbf{E}_n := \mathbf{E}_{g_{z,q,\delta_n}}$ which weakly converges to $\mathbf{E} \in L^2(D)$ such that $\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = 0$ in D. We deduce that $\nu \times \mathbf{E}_n$

weakly converges to $\nu \times \mathbf{E}$ in $H^{-1/2}_{\mathrm{div}}(\Gamma)$, and by the compactness of $\mathcal{B}$ we conclude that $||\mathcal{B}\mathbf{E}_n - \mathcal{B}\mathbf{E}||_{L^2_t(D)} \to 0$ as $n \to \infty$ i.e.

$$\lim_{n\to\infty} ||\mathcal{F}g_{z,q,\delta_n} - \mathcal{B}\mathbf{E}||_{L^2_t(D)} = 0$$

Therefore, there exists $\mathbf{E} \in H_{inc}(D)$ such that $\mathbf{E}_{e,\infty}(\cdot, z, q) = \mathcal{B}\mathbf{E}$. As a consequence, there exist $\mathbf{E}$, $\mathbf{E}_0$ solution to the interior transmission problem (2.19). Then $\mathbf{u}_z := \mathbf{E} - \mathbf{E}_0 \in \mathcal{U}_0(D)$ satisfies

$$\int_D (N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, - k^2 N)(\mathbf{u}_z + \mathbf{\Theta}_z)\cdot(\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx = 0 \tag{2.23}$$

for all $\mathbf{\Psi} \in \mathcal{U}_0(D)$.

As seen in the previous sections of this chapter, (2.23) can also be written as $(I - k^2 T_k)\mathbf{u}_z = f_{\mathbf{\Theta}_z}$ where T_k is a compact self-adjoint operator. Since k is a transmission eigenvalue, the kernel of $I - k^2 T_k$ is non trivial. Using the Fredholm alternative and the fact that T_k is self-adjoint, we deduce that $((I - k^2 T_k)\mathbf{u}_z, \mathbf{u}_0) = 0$ i.e.

$$\int_D (N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\,\mathbf{\Theta}_z - k^2\mathbf{\Theta}_z)\cdot(\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_0 - k^2 N\mathbf{u}_0)dx = 0 \tag{2.24}$$

where $\mathbf{u}_0 \in \mathcal{U}_0(D)$ is an eigenvector associated to the transmission eigenvalue k. The Green's second vector theorem

$$\int_D \mathbf{u}\cdot\mathrm{curl}\,\mathrm{curl}\,\mathbf{v}dx = \int_D \mathbf{v}\cdot\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}dx + \int_\Gamma \nu\times\mathbf{u}\cdot\mathrm{curl}\,\mathbf{v} - \int_\Gamma \nu\times\mathbf{v}\cdot\mathrm{curl}\,\mathbf{u}$$

valid for all $\mathbf{u}$ and $\mathbf{v}$ such that $\mathrm{div}\,\mathbf{u} = \mathrm{div}\,\mathbf{v} = 0$ applied to (2.24) for $\mathbf{u} = (N - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, - k^2 N)\mathbf{u}_0$ and $\mathbf{v} = \mathbf{\Theta}_z$ implies that

$$\int_\Gamma \mathrm{curl}\,\Big((N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, - k^2N)\mathbf{u}_0\Big)\cdot\nu\times\mathbf{\Theta}_z$$
$$- \int_\Gamma \nu\times\Big((N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_0 - k^2N\mathbf{u}_0)\Big)\cdot\mathrm{curl}\,\mathbf{\Theta}_z = 0 \tag{2.25}$$

since $(\mathrm{curl}\,\mathrm{curl}\, - k^2)(N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_0 - k^2N\mathbf{u}_0) = 0$ in D.

Setting $\mathbf{F} := (N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, - k^2N)\mathbf{u}_0$ in D and using the definition of $\mathbf{\Theta}_z$, equation (2.25) becomes

$$\int_\Gamma \mathrm{curl}\,\mathbf{F}\cdot\nu\times\mathbf{E}_e(\cdot,z,q) - \int_\Gamma \nu\times\mathbf{F}\cdot\mathrm{curl}\,\mathbf{E}_e(\cdot,z,q) = 0 \tag{2.26}$$

for all $z \in \mathcal{A}$. Using the representation theorems for solutions to Maxwell equations we have

$$\mathbf{F}(z) = -\mathrm{curl}_z \int_\Gamma \nu(x)\times\mathbf{F}(x)\Phi_k(x,z)ds(x)$$
$$- \frac{1}{k^2}\mathrm{curl}_z\,\mathrm{curl}_z \int_\Gamma \nu(x)\times\mathrm{curl}\,\mathbf{F}(x)\Phi_k(x,z)ds(x).$$

Since (2.26) is equivalent to $ikq \cdot \mathbf{F}(z) = 0$ for all $z \in \mathcal{A}$ (see Lemma B.2.1 in Appendix A), then $\mathbf{F}(z) = 0$ for all $z \in \mathcal{A}$ and by the unique continuation principle for all z in D. We deduce that $\operatorname{curl}\operatorname{curl}\mathbf{u}_0 - k^2 N\mathbf{u}_0 = 0$ in D. Using the boundary conditions $\nu \times \mathbf{u}_0 = \nu \times \operatorname{curl}\mathbf{u}_0 = 0$ on Γ and the representation theorems for solutions to Maxwell equations, one concludes that $\mathbf{u}_0 = 0$ in D which contradicts the fact that $\mathbf{u}_0$ is an eigenvector. $\square$

2.5.3 Illustration of the theorem

We consider the far field equation in 2 dimensions for a circle of radius 1 and index of refraction $n = 4$. We have computed the regularized solution g_z of the far field equation for three different source point z inside the circle. We can remark that the norm of g_z explodes for some particuar wave numbers that correspond to transmission eigenvalues. Moreover, we can see the importance of taking multiple source points since the second peak has not been found for one source point z.

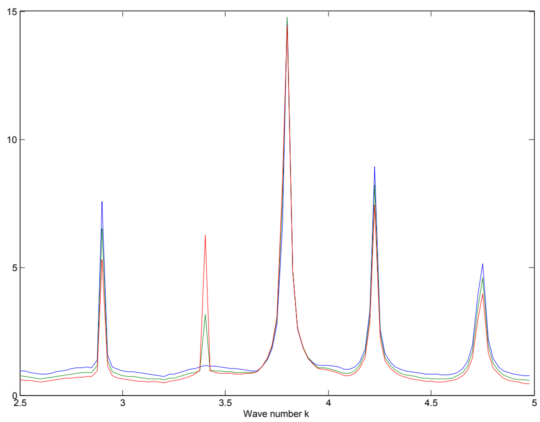


Figure 2.4: The norm of the regularized solution g_z for three different point $z \in D$.

2.6 Inverse problem : estimates of n from the knowledge of the first transmission eigenvalue

For this problem, we assume that the shape of the scatterer D is known and that according to the previous section, we have computed the transmission eigenvalues using the far field data.

We consider here a double-layer sphere of radius $R = 1$ containing a concentric sphere of radius r that we shall mke varying for the numerical tests.

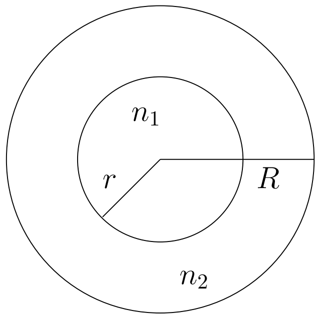


Figure 2.5: Spherical geometry

We assume that the index of refraction of the sphere is such that

$$n(x) = \begin{cases} 4 & \text{if } r < |x| < R \\ 2 & \text{if } |x| < r. \end{cases}$$

As we shall see in Appendix E, the transmission eigenvalues for spherical geometry are easy to compute using a separation of variables method. We are then able to compute the first transmission eigenvalue k_0 corresponding to the object D where the inside radius varies. Now, knowing the value of the first transmission eigenvalue, we are able to give an estimate on the value of the index of refraction of the object assuming that the index of refraction is constant.

r	0	0.1	0.3	0.5	0.7	0.9	1
k_0	3.141	3.142	3.238	3.971	5.152	5.582	7.358
n	4.00	3.996	3.826	3.052	2.498	2.326	2.00

Chapter 3

Electromagnetic interior transmission problem for dielectrics with cavities

In this chapter, we consider the interior transmission problem for anisotropic inclusions that may contain some cavities; i.e. regions where the index of refraction has the same value as the exterior medium. Mathematically, the cavity region corresponds with a degenerate form of the ITP (the two fields satisfy the same equation) and therefore causes difficulties in extending the techniques used for "regular forms" studied in Chapter 2. A first study of this configuration was initiated in [12] for the scalar problem. We provide here an extension of this work to the full electromagnetic problem by following a similar route. The ITP is reformulated as a fourth order PDE outside the cavity region and the latter is taken into account as a constrain in the variational space. Besides the technicality inherent to Maxwell's equations, the main difficulty here is in proving the equivalence between weak and variational solutions and is in finding the appropriate splitting of the variational form into coercive and compact parts. In a second step, and inspired by the recent works [18, 17], we use this formulation to prove the existence of an infinite discrete set of transmission eigenvalues for general cases and prove a monotonicity property with respect to the cavity size of the first eigenvalue. The main difficulty in this part lies in the fact that the variational space depends of the frequency. The introduction of a projection operator and continuity properties of the latter with respect to the frequency are used in order to solve the problem.

The chapter is organized as follows. The first section is dedicated to introducing the ITP and establishing the appropriate variational formulation of the problem. The Fredholm property of the obtained problem is then proved. The second section is devoted to the study of transmission eigenvalues. We prove the existence of an infinite discrete set of these special frequencies in the case without absorption and establish monotonicity properties with respect to the cavity size and the medium index. These parts are extracted from the published article [30]. We complement these results with the theorem characterizing the transmission eigenvalues from far field data in section 3.3.

3.1 Interior transmission problem

Let $D \subset \mathbb{R}^3$ be a simply connected and bounded region with a piece wise smooth boundary $\Gamma := \partial D$. We denote by $(\cdot,\cdot)_D$ the $L^2(D)^3$ scalar product and consider the Hilbert spaces

$$H(\mathrm{curl}\,, D) := \{\mathbf{u} \in L^2(D)^3/\mathrm{curl}\,\mathbf{u} \in L^2(D)^3\}$$

$$H_0(\mathrm{curl}\,, D) := \{\mathbf{u} \in H(\mathrm{curl}\,, D)/\nu \times \mathbf{u} = 0 \text{ on } \Gamma\}$$

equipped with the scalar product $(\mathbf{u},\mathbf{v})_{\mathrm{curl}} = (\mathbf{u},\mathbf{v})_D + (\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v})_D$ and the corresponding norm $||\cdot||_{\mathrm{curl}}$. We also define

$$\mathcal{U}(D) := \{\mathbf{u} \in H(\mathrm{curl}\,, D)/\mathrm{curl}\,\mathbf{u} \in H(\mathrm{curl}\,, D)\}$$

$$\mathcal{U}_0(D) := \{\mathbf{u} \in H_0(\mathrm{curl}\,, D)/\mathrm{curl}\,\mathbf{u} \in H_0(\mathrm{curl}\,, D)\}$$

equipped with the scalar product $(\mathbf{u},\mathbf{v})_{\mathcal{U}} = (\mathbf{u},\mathbf{v})_{\mathrm{curl}} + (\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{v})_{\mathrm{curl}}$ and the corresponding norm $||\cdot||_{\mathcal{U}}$.

We assume that D contains a region $D_0 \subset D$ which can possibly be multiply connected and with a piece wise smooth boundary $\Sigma := \partial D_0$ and such that $D\backslash\overline{D}_0$ is connected (see Fig. 3.1). Let ν denote the unit outward normal to Γ and Σ. Let N be a 3×3 symmetric matrix whose entries are bounded complex-valued functions in $\mathbb{R}^3$ and such that $N = I$ in D_0. This matrix will represent the medium index inside D. Similarly to the case without cavity (see chapter 2), we shall use a fourth order formulation and consequently we need the same restriction on the index of refraction. We assume that there exists a constant $\gamma > 0$ such that either

$$\mathrm{Re}((N-I)^{-1}\xi,\xi) \geq \gamma|\xi|^2 \quad \text{or} \quad \mathrm{Re}((I-N)^{-1}\xi,\xi) \geq \gamma|\xi|^2$$

for all ξ in $\mathbb{C}^3$ and almost everywhere in $D\backslash\overline{D}_0$.

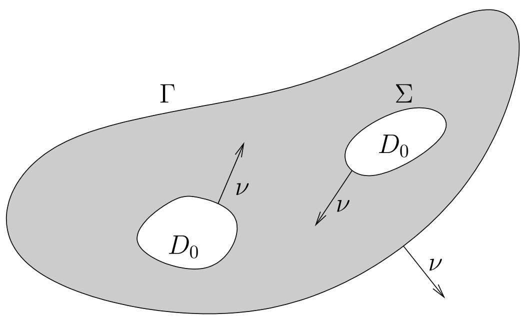


Figure 3.1: Geometry and notation

Let $s \geq 0$ be a given real number and $H^s(\Gamma)$ be the usual Sobolev space. We define

$$TH^s(\Gamma) := \left\{\boldsymbol{\varphi} \in H^s(\Gamma)^3/\boldsymbol{\varphi}\cdot\nu = 0\right\}.$$

We consider the following interior transmission problem

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\,\mathbf{E} - k^2 N\mathbf{E} = 0 & \text{in } D \\ \mathrm{curl}\,\mathrm{curl}\,\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 & \text{in } D \\ \nu\times\mathbf{E} - \nu\times\mathbf{E}_0 = \mathbf{G} & \text{on } \Gamma \\ \nu\times\mathrm{curl}\,\mathbf{E} - \nu\times\mathrm{curl}\,\mathbf{E}_0 = \mathbf{H} & \text{on } \Gamma \end{cases} \tag{ITP3.1}$$

with $\mathbf{G} \in TH^{3/2}(\Gamma)$ and $\mathbf{H} \in TH^{1/2}(\Gamma)$ some given boundary data.

Definition 3.1.1. *A weak solution to* (ITP3.1) *is a pair of functions* $\mathbf{E}$ *and* $\mathbf{E}_0$ *in* $L^2(D)^3$ *solution to the two first equations of* (ITP3.1) *in the distributional sense such that* $\mathbf{F} = \mathbf{E} - \mathbf{E}_0 \in \mathcal{U}(D)$ *satisfies the boundary conditions of* (ITP3.1).

3.1.1 Variational formulation

Similarly to the case without cavities, we shall use a fourth order formulation. However, since $\mathbf{E}$ and $\mathbf{E}_0$ satisfies the same equation in the cavity D_0, this is only valid in the complementary domain $D\backslash\overline{D}_0$. A second equation satisfied by $\mathbf{E}$ or $\mathbf{E}_0$ is necessary to keep the equivalence between (ITP3.1) and the new one.

Let us consider consider a weak solution $\mathbf{E}$ and $\mathbf{E}_0$ to (ITP3.1). Then $\mathbf{F} := \mathbf{E} - \mathbf{E}_0$ satisfies

$$\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2 N\mathbf{F} = k^2(N-I)\mathbf{E}_0 \quad \text{in } D\backslash\overline{D}_0$$

or

$$\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F} = k^2(N-I)\mathbf{E} \quad \text{in } D\backslash\overline{D}_0. \tag{3.1}$$

Multiplying (3.1) by $(N-I)^{-1}$ and applying $(\operatorname{curl}\operatorname{curl} - k^2N)$ $\mathbf{F}$ satisfies the fourth order equation

$$(\operatorname{curl}\operatorname{curl} - k^2N)(N-I)^{-1}(\operatorname{curl}\operatorname{curl} - k^2)\mathbf{F} = 0 \quad \text{in } D\backslash\overline{D}_0 \tag{3.2}$$

together with the boundary conditions

$$\nu\times\mathbf{F} = \mathbf{G} \quad ; \quad \nu\times\operatorname{curl}\mathbf{F} = \mathbf{H} \quad \text{on } \Gamma. \tag{3.3}$$

Moreover, inside D_0, $\mathbf{F}$ satisfies

$$(\operatorname{curl}\operatorname{curl} - k^2)\mathbf{F} = 0 \tag{3.4}$$

with the continuity of the Cauchy data across Σ

$$\nu\times\mathbf{F}^- = \nu\times\mathbf{F}^+ \quad ; \quad \nu\times\operatorname{curl}\mathbf{F}^- = \nu\times\operatorname{curl}\mathbf{F}^+ \tag{3.5}$$

where for a regular function $\mathbf{F}$, $\mathbf{F}^\pm(x) := \lim_{h\to 0^+}\mathbf{F}(x\pm h\nu_x)$ for $x\in\Sigma$ and ν_x the outward unit normal to Σ.

However, the latter equations are not sufficient to define $\mathbf{E}$ and $\mathbf{E}_0$ inside Σ, so one needs to add an additional unknown inside D_0, for instance, the function $\mathbf{E}$ that satisfies

$$(\operatorname{curl}\operatorname{curl} - k^2)\mathbf{E} = 0 \text{ in } D_0. \tag{3.6}$$

From the continuity of the Cauchy data of (3.1) we also get two more conditions on Σ

$$\nu\times\left(\frac{1}{k^2}(N-I)^{-1}\left(\operatorname{curl}\operatorname{curl} - k^2\right)\mathbf{F}\right)^+ = \nu\times\mathbf{E}^-, \tag{3.7}$$

$$\nu\times\operatorname{curl}\left(\frac{1}{k^2}(N-I)^{-1}(\operatorname{curl}\operatorname{curl} - k^2)\mathbf{F}\right)^+ = \nu\times\operatorname{curl}\mathbf{E}^-. \tag{3.8}$$

One can now easily verifies the following theorem :

Theorem 3.1.1. *$\mathbf{F} \in \mathcal{U}(D)$ and $\mathbf{E} \in L^2(D_0)^3$ is a solution to* (3.2) − (3.8) *if and only if $\mathbf{E}$ and $\mathbf{E}_0$ are weak solutions to* (ITP3.1) *with*

$$\mathbf{E} := \frac{1}{k^2}(N-I)^{-1}(\operatorname{curl}\operatorname{curl} - k^2)\mathbf{F} \ \text{ in } D\backslash\overline{D}_0 \quad \text{ and } \quad \mathbf{E}_0 := \mathbf{E} - \mathbf{F} \text{ in } D.$$

In order to study the existence and uniqueness of the solution, we shall use a variational approach. The main difficulty is to define an appropriate variational space which will guarantee the well-posedness and the equivalence between the variational formulation and the system (3.2)-(3.8).

Let us define the Hilbert space

$$V(D, D_0, k) := \{\mathbf{E} \in \mathcal{U}(D)/\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = 0 \text{ in } D_0\},$$

equipped with the scalar product $(\cdot,\cdot)_{\mathcal{U}}$, and the closed subspace

$$V_0(D, D_0, k) := \{\mathbf{E} \in \mathcal{U}_0(D)/\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = 0 \text{ in } D_0\}.$$

Let $\mathbf{F} \in V(D, D_0, k)$ and consider a test function $\boldsymbol{\psi} \in V_0(D, D_0, k)$. We assume that $\mathbf{F}$ and $\boldsymbol{\psi}$ are regular enough to justify the various integrating by parts. Multiplying (3.2) by $\overline{\boldsymbol{\psi}}$ and integrating by parts, we obtain

$$\begin{aligned}
0 &= \int_{D\backslash\overline{D}_0} \left(\operatorname{curl}\operatorname{curl} - k^2 N\right)(N-I)^{-1}\left(\operatorname{curl}\operatorname{curl} - k^2\right)\mathbf{F}\cdot\overline{\boldsymbol{\psi}}dx \\
&= \int_{D\backslash\overline{D}_0} \left(\left(\operatorname{curl}\operatorname{curl} - k^2\right)(N-I)^{-1}\left(\operatorname{curl}\operatorname{curl} - k^2\right)\mathbf{F} - k^2\left(\operatorname{curl}\operatorname{curl} - k^2\right)\mathbf{F}\right)\cdot\overline{\boldsymbol{\psi}}dx \\
&= \int_{D\backslash\overline{D}_0} (N-I)^{-1}(\operatorname{curl}\operatorname{curl} - k^2)\mathbf{F}\cdot(\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\psi}} - k^2\overline{\boldsymbol{\psi}})dx \\
&\quad - k^2\int_{D\backslash\overline{D}_0} (\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F})\cdot\overline{\boldsymbol{\psi}}dx \\
&\quad + \int_{\partial D_0} \nu\times\left((N-I)^{-1}\left(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}\right)\right)\cdot\operatorname{curl}\overline{\boldsymbol{\psi}}ds \\
&\quad - \int_{\partial D_0} \nu\times\overline{\boldsymbol{\psi}}\cdot\operatorname{curl}\left((N-I)^{-1}\left(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}\right)\right)ds
\end{aligned}$$

Using the fact that $\overline{\boldsymbol{\psi}} \in V_0(D, D_0, k)$, Green's second vector theorem implies that

$$\begin{aligned}
0 &= \int_{D_0} (\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E})\cdot\overline{\boldsymbol{\psi}}dx - \int_{D_0} \mathbf{E}\cdot(\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\psi}} - k^2\overline{\boldsymbol{\psi}})dx \\
&= \int_{\Sigma} \nu\times\mathbf{E}\cdot\operatorname{curl}\overline{\boldsymbol{\psi}}ds - \int_{\Sigma} \nu\times\overline{\boldsymbol{\psi}}\cdot\operatorname{curl}\mathbf{E}ds.
\end{aligned}$$

Using (3.7) and (3.8), we obtain

$$\begin{aligned}
&\int_{\Sigma} \nu\times\left((N-I)^{-1}\left(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}\right)\right)\cdot\operatorname{curl}\overline{\boldsymbol{\psi}}ds \\
&\qquad - \int_{\Sigma} \nu\times\overline{\boldsymbol{\psi}}\cdot\operatorname{curl}\left((N-I)^{-1}\left(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}\right)\right)ds = 0. \quad (3.9)
\end{aligned}$$

Therefore, we finally have that

$$\int_{D\setminus\overline{D}_0} (N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}) \cdot (\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\psi}} - k^2\overline{\boldsymbol{\psi}})dx \\ - k^2 \int_{D\setminus\overline{D}_0} (\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}) \cdot \overline{\boldsymbol{\psi}} dx = 0, \quad (3.10)$$

which is required to be valid for all $\boldsymbol{\psi} \in V_0(D, D_0, k)$.

In order to have the variational formulation for functions in $V_0(D, D_0, k)$, we need a lifting result. To this end, we shall assume that there exists $\boldsymbol{\Theta} \in H^2(D)^3$ such that

$$\nu \times \boldsymbol{\Theta} = \mathbf{G} \quad \text{and} \quad \nu \times \operatorname{curl}\boldsymbol{\Theta} = \mathbf{H} \text{ on } \Gamma, \quad (3.11)$$

(see Remark 3.1.1 below). Using a cutoff function one can also guarantee that $\boldsymbol{\Theta} = 0$ in $D_{\boldsymbol{\Theta}}$ where $D_0 \subset D_{\boldsymbol{\Theta}} \subset D$, which will be assumed in the sequel.

Remark 3.1.1. *It is proven in [34] that if Γ is sufficiently regular (e.g. C^3), then for any boundary data $\mathbf{G} \in TH^{3/2}(\Gamma)$ and $\mathbf{H} \in TH^{1/2}(\Gamma)$, there exists $\boldsymbol{\Theta} \in H^2(D)^3$ such that* (3.11) *holds and such that*

$$||\boldsymbol{\Theta}||_{H^2(D)} \leq c\,(||\mathbf{G}||_{H^{3/2}} + ||\mathbf{H}||_{H^{1/2}})$$

where c is a constant independent of $\mathbf{G}$ and $\mathbf{H}$.

Consequently the variational formulation amounts to finding $\mathbf{F}_0 := \mathbf{F} - \boldsymbol{\Theta} \in V_0(D, D_0, k)$ such that

$$\int_{D\setminus\overline{D}_0} (N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F}_0 - k^2\mathbf{F}_0) \cdot (\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\psi}} - k^2\overline{\boldsymbol{\psi}})dx \\ - k^2 \int_{D\setminus\overline{D}_0} (\operatorname{curl}\operatorname{curl}\mathbf{F}_0 - k^2\mathbf{F}_0) \cdot \overline{\boldsymbol{\psi}} dx \\ = - \int_{D\setminus\overline{D}_0} (N-I)^{-1}(\operatorname{curl}\operatorname{curl}\boldsymbol{\Theta} - k^2\boldsymbol{\Theta}) \cdot (\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\psi}} - k^2\overline{\boldsymbol{\psi}})dx \\ + k^2 \int_{D\setminus\overline{D}_0} (\operatorname{curl}\operatorname{curl}\boldsymbol{\Theta} - k^2\boldsymbol{\Theta}) \cdot \overline{\boldsymbol{\psi}} dx \quad (3.12)$$

for all $\boldsymbol{\psi} \in V_0(D, D_0, k)$. In the following, this variational formulation will be used in the case where $(N-I)^{-1}$ is bounded positive definite. In the other case where $(I-N)^{-1}$ is bounded positive definite, we shall use the equivalent variational formulation

$$\int_{D\setminus\overline{D}_0} N(I-N)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F}_0 - k^2\mathbf{F}_0) \cdot (\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\psi}} - k^2\overline{\boldsymbol{\psi}})dx \\ + \int_{D\setminus\overline{D}_0} (\operatorname{curl}\operatorname{curl}\mathbf{F}_0 - k^2\mathbf{F}_0) \cdot \operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\psi}} dx \\ = - \int_{D\setminus\overline{D}_0} N(I-N)^{-1}(\operatorname{curl}\operatorname{curl}\boldsymbol{\Theta} - k^2\boldsymbol{\Theta}) \cdot (\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\psi}} - k^2\overline{\boldsymbol{\psi}})dx \\ - \int_{D\setminus\overline{D}_0} (\operatorname{curl}\operatorname{curl}\boldsymbol{\Theta} - k^2\boldsymbol{\Theta}) \cdot \operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\psi}} dx. \quad (3.13)$$

One can remark that the above variational formulations (3.12) and (3.13) involve only $\mathbf{F}$. To show the equivalence between this variational formulation and (ITP3.1), we need to show that the existence of $\mathbf{E}$ is implicitly contained in the variational formulation.an additional result which leads to the existence of a solution in $L^2(D_0)$ to Maxwell's equations in D_0.

3.1.2 Solutions in L^2 to Maxwell's equations

We assume that Λ is a bounded simply connected domain of class $\mathcal{C}^2$.

Definition 3.1.2. *A real λ is a called a Maxwell eigenvalue in Λ if there exists $\mathbf{v} \in H(\operatorname{curl}, \Lambda)$, a non trivial solution to*

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{v} - \lambda\mathbf{v} = 0 & \textit{in } \Lambda \\ \mathbf{v} \times \nu = 0 & \textit{on } \partial\Lambda. \end{cases} \tag{3.14}$$

Remark 3.1.2. *We remark that if $\lambda \neq 0$, then existence of a non trivial solution $\mathbf{v} \in H(\operatorname{curl}, \Lambda)$ to* (3.14) *is equivalent to existence of a non trivial solution $\mathbf{w} = \operatorname{curl}\mathbf{v} \in H(\operatorname{curl}, \Lambda)$ to*

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{w} - \lambda\mathbf{w} = 0 & \textit{in } \Lambda \\ \operatorname{curl}\mathbf{w} \times \nu = 0 & \textit{on } \partial\Lambda. \end{cases}$$

The next theorem shows how we can construct a solution in $L^2(\Lambda)$ to

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = 0 & \text{in } \Lambda \\ \mathbf{E} \times \nu = \alpha \times \nu & \text{on } \partial\Lambda \end{cases}$$

with $\alpha \in TH^{-1/2}(\partial\Lambda)$ from a solution in $H(\operatorname{curl}, \Lambda)$.

Theorem 3.1.2. *We assume that k^2 is not a Maxwell eigenvalue for Λ. For every $\alpha \in TH^{-1/2}(\partial\Lambda)$, there exists a solution $\mathbf{E} \in L^2(\Lambda)^3$ to*

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = 0 & \textit{in } \Lambda \\ \mathbf{E} \times \nu = \alpha \times \nu & \textit{on } \partial\Lambda. \end{cases}$$

Proof. We will use the following properties of Sobolev spaces for Λ of class $\mathcal{C}^{k+1}$ (see [31])

$$H^{k+1}(\Lambda)^3 = \left\{\mathbf{u} \in L^2(\Lambda)^3, \operatorname{curl}\mathbf{u} \in H^k(\Lambda)^3, \operatorname{div}\mathbf{u} \in H^k(\Lambda), \mathbf{u} \times \nu \in H^{k+1/2}(\partial\Lambda)^3\right\}$$

or

$$H^{k+1}(\Lambda)^3 = \left\{\mathbf{u} \in L^2(\Lambda)^3, \operatorname{curl}\mathbf{u} \in H^k(\Lambda)^3, \operatorname{div}\mathbf{u} \in H^k(\Lambda), \mathbf{u} \cdot \nu \in H^{k+1/2}(\partial\Lambda)^3\right\}.$$

Let $\alpha \in TH^{1/2}(\partial\Lambda)$ and $\mathbf{E} \in H(\operatorname{curl}, \Lambda)$ satisfy

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = 0 & \text{in } \Lambda \\ \mathbf{E} \times \nu = \alpha \times \nu & \text{on } \partial\Lambda. \end{cases}$$

Since $\operatorname{div}\mathbf{E} = 0$ and $\mathbf{E}\times\nu \in TH^{1/2}(\partial\Lambda)$, we have that $\mathbf{E}$ is in $H^1(\Lambda)^3$. Now let $\mathbf{F} \in H(\operatorname{curl},\Lambda)$ be a solution to

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F} = \mathbf{E} & \text{in } \Lambda \\ \mathbf{F}\times\nu = 0 & \text{on } \partial\Lambda. \end{cases} \tag{3.15}$$

We show that $\mathbf{F} \in H^2(\Lambda)^3$: since $\mathbf{F} \in H(\operatorname{curl},\Lambda)$, $\operatorname{div}\mathbf{F} = 0$ and $\mathbf{F}\times\nu = 0$ we have that $\mathbf{F}$ is in $L^2(\Lambda)^3$, $\operatorname{div}\mathbf{F}$ in $H^1(\Lambda)$ and $\mathbf{F}\times\nu$ in $H^{3/2}(\partial\Lambda)^3$. Then it only remains to show that $\tilde{\mathbf{F}} := \operatorname{curl}\mathbf{F} \in H^1(\Lambda)^3$. The fact that $\mathbf{F} \in H(\operatorname{curl},\Lambda)$ implies that $\tilde{\mathbf{F}} \in L^2(\Lambda)^3$. The equalities

$$\operatorname{div}\tilde{\mathbf{F}} = 0$$

and

$$\operatorname{curl}\tilde{\mathbf{F}} = \mathbf{E} + k^2\mathbf{F}$$

show that $\operatorname{div}\tilde{\mathbf{F}} \in L^2(\Lambda)$ and $\operatorname{curl}\tilde{\mathbf{F}} \in L^2(\Lambda)$. Moreover,

$$\tilde{\mathbf{F}}\cdot\nu = \operatorname{curl}\mathbf{F}\cdot\nu = \operatorname{div}_{\partial\Lambda}(\mathbf{F}\times\nu) = 0$$

where $\operatorname{div}_{\partial\Lambda}$ denotes the surface divergence operator, implies also that $\tilde{\mathbf{F}}\cdot\nu \in H^{1/2}(\partial\Lambda)$. Then $\tilde{\mathbf{F}} \in H^1(\Lambda)^3$ and

$$||\tilde{\mathbf{F}}||_{H^1(\Lambda)} \leq C\left(||\tilde{\mathbf{F}}||^2_{L^2(\Lambda)} + ||\operatorname{curl}\tilde{\mathbf{F}}||^2_{L^2(\Lambda)}\right)^{1/2} \leq C\left(||\mathbf{F}||_{H(\operatorname{curl},\Lambda)} + ||\mathbf{E}||_{L^2(\Lambda)}\right).$$

We deduce that $\mathbf{F} \in H^2(\Lambda)^3$ and

$$||\mathbf{F}||_{H^2(\Lambda)} \leq C\left(||\mathbf{F}||_{H(\operatorname{curl},\Lambda)} + ||\mathbf{E}||_{L^2(\Lambda)}\right) \leq C||\mathbf{E}||_{L^2(\Lambda)}.$$

From (3.15) and using the Stokes formula one easily obtains

$$\begin{aligned} ||\mathbf{E}||^2_{L^2(\Lambda)} &= \left|\int_{\partial\Lambda} \alpha\cdot(\nu\times\operatorname{curl}\mathbf{F})\right| \\ &\leq ||\alpha||_{H^{-1/2}(\partial\Lambda)}||\nu\times\operatorname{curl}\mathbf{F}||_{H^{1/2}(\partial\Lambda)} \\ &\leq C||\alpha||_{H^{-1/2}(\partial\Lambda)}||\mathbf{E}||_{L^2(\Lambda)} \end{aligned}$$

and therefore the solution operator $\alpha \longmapsto \mathbf{E}$ is continuous from $TH^{-1/2}(\partial\Lambda)$ into $L^2(\Lambda)$. Similar arguments also show that if k^2 is not an eigenvalue with the boundary condition $\operatorname{curl}\mathbf{E}\times\nu = 0$ then the solution operator $\operatorname{curl}\alpha \longmapsto \mathbf{E}$ is continuous from $TH^{-3/2}(\partial\Lambda)$ into $L^2(\Lambda)$. □

The following lemma shows the existence of a L^2-solution to Maxwell's equations satisfying given boundary data in the inclusion D_0 of our domain D.

Lemma 3.1.3. *We assume that $\Sigma = \partial D_0$ is a C^2 boundary and that $k^2 > 0$ is not a Maxwell eigenvalue in D_0. Let $(\alpha,\beta) \in TH^{-3/2}(\Sigma)\times TH^{-1/2}(\Sigma)$ such that for all $\boldsymbol{\psi} \in V_0(D, D_0, k)\cap H^2(D)^3$,*

$$\langle \nu\times\operatorname{curl}\boldsymbol{\psi}, \beta\rangle_{TH^{1/2}(\Sigma),TH^{-1/2}(\Sigma)} + \langle \alpha, \nu\times\boldsymbol{\psi}\rangle_{TH^{-3/2}(\Sigma),TH^{3/2}(\Sigma)} = 0. \tag{3.16}$$

Then there exists $\mathbf{E} \in L^2(D_0)$ *such that*

$$\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = 0 \ \ \textit{in } D_0$$

and

$$(\nu \times \mathbf{E}, \nu \times \operatorname{curl}\mathbf{E}) = (\nu \times \beta, \nu \times \alpha) \ \ \textit{on } \Sigma.$$

Proof. From the previous theorem with $\Lambda = D_0$, we know that there exists a solution $\mathbf{E} \in L^2(D_0)^3$ to

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = 0 & \text{in } D_0 \\ \nu \times \mathbf{E} = \nu \times \beta & \text{on } \Sigma. \end{cases}$$

Let $\boldsymbol{\psi} \in V_0(D, D_0, k)$. Integrating by parts we obtain

$$\int_{D_0} \left(\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E}\right) \cdot \overline{\boldsymbol{\psi}} dx = \int_{D_0} \left(\operatorname{curl}\operatorname{curl}\overline{\boldsymbol{\psi}} - k^2\overline{\boldsymbol{\psi}}\right) \cdot \mathbf{E} dx$$
$$+ \langle \operatorname{curl}\boldsymbol{\psi} \times \nu, \mathbf{E}\rangle_{TH^{1/2}(\Sigma),TH^{-1/2}(\Sigma)} - \langle \nu \times \boldsymbol{\psi}, \operatorname{curl}\mathbf{E}\rangle_{TH^{3/2}(\Sigma),TH^{-3/2}(\Sigma)}$$

Using the fact that $\operatorname{curl}\operatorname{curl}\boldsymbol{\psi} - k^2\boldsymbol{\psi} = 0$ in D_0, we obtain

$$\langle \operatorname{curl}\boldsymbol{\psi} \times \nu, \mathbf{E}\rangle_{TH^{1/2}(\Sigma),TH^{-1/2}(\Sigma)} - \langle \nu \times \boldsymbol{\psi}, \operatorname{curl}\mathbf{E}\rangle_{TH^{3/2}(\Sigma),TH^{-3/2}(\Sigma)} = 0$$

and from (3.16), we obtain

$$\langle \nu \times \boldsymbol{\psi}, \operatorname{curl}\mathbf{E} - \alpha\rangle_{TH^{3/2}(\Sigma),TH^{-3/2}(\Sigma)} = 0$$

for all $\boldsymbol{\psi} \in V_0(D, D_0, k) \cap H^2(D)$. Using the density theorem 3 in [52] since $V_0(D, D_0, k)$ contains the set of Herglotz functions, we can conclude that the traces of functions in $V_0(D, D_0, k)$ are dense in $TH^{3/2}(\Sigma)$. Then $\alpha \times \nu = \operatorname{curl}\mathbf{E} \times \nu$ in $TH^{-3/2}(\Sigma)$. □

3.1.3 Well-posedness of the ITP

Using the previous lemma, we now can state the equivalence between weak solutions to (ITP3.1) and solutions to the variational formulations (3.12) and (3.13).

Theorem 3.1.4. *Assume that* Σ *is a* C^2 *boundary and that* $k^2 > 0$ *is not a Maxwell eigenvalue in* D_0*. Then the existence and uniqueness of a weak solution* $\mathbf{E}$ *and* $\mathbf{E}_0$ *to the interior transmission problem* (ITP3.1) *is equivalent to the existence and uniqueness of a solution* $\mathbf{F}_0$ *to the variational problem* (3.12) *or* (3.13).

Proof. It remains only to verify that a solution of (3.12) or (3.13) defines a weak solution w and v to the interior transmission problem (ITP3.1). Let $\boldsymbol{\psi}$ be a $\mathcal{C}^\infty$ function with compact support in $D\backslash\overline{D}_0$. From (3.10), we can show that $\mathbf{F}$ satisfies (3.3). In particular the function

$$\mathbf{E}^+ := \left(-\frac{1}{k^2}(N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F})\right)_{|D\backslash\overline{D}_0}$$

satisfies $\mathbf{E}^+ \in L^2(D\backslash\overline{D}_0)$ and $\operatorname{curl}\operatorname{curl}\mathbf{E}^+ - k^2N\mathbf{E}^+ = 0$ in $D\backslash\overline{D}_0$. For an arbitrary test function $\boldsymbol{\psi} \in V_0(D, D_0, k) \cap H^2(D)$, integrating by parts (3.10) we obtain

$$\langle \nu \times \operatorname{curl}\boldsymbol{\psi}, \mathbf{E}^+\rangle_{TH^{1/2}(\Sigma),TH^{-1/2}(\Sigma)} - \langle \operatorname{curl}\mathbf{E}^+, \nu \times \boldsymbol{\psi}\rangle_{TH^{-3/2}(\Sigma),TH^{3/2}(\Sigma)} = 0.$$

Applying Lemma 3.1.3 we now obtain the existence of $\mathbf{E}^- \in L^2(D_0)$ satisfying (3.6)-(3.8). □

The following theorem concludes this section by proving the existence and uniqueness of a solution of (ITP3.1) using the Fredholm alternative. In the following we exclude the values of k for which the uniqueness does not hold, namely the so-called transmission eigenvalues.

Definition 3.1.3. *Values of $k > 0$ that are not Maxwell eigenvalues and for which the homogeneous variational problem (i.e. for $\boldsymbol{\Theta} = 0$) has nontrivial solutions $\mathbf{F}_0$ are called transmission eigenvalues.*

Theorem 3.1.5. *Assume that $N \in L^\infty(D, \mathbb{R}^{3\times 3})$. Then the interior transmission problem has a unique solution provided that k is not a transmission eigenvalue. This solution depends continuously on the data $\boldsymbol{\Theta} \in H^2(D)^3$.*

Proof. We first assume that $(N - I)^{-1}$ is bounded positive definite. Let us define the following bounded sesquilinear forms on $V_0(D, D_0, k) \times V_0(D, D_0, k)$:

$$\mathcal{A}_k(\mathbf{u}, \mathbf{v}) = \int_{D\backslash\overline{D}_0} (N - I)^{-1} \left(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}\right) \cdot \left(\operatorname{curl}\operatorname{curl}\overline{\mathbf{v}} - k^2\overline{\mathbf{v}}\right) dx + k^4 \int_D \mathbf{u}\cdot\overline{\mathbf{v}}dx$$

and

$$\begin{aligned}\mathcal{B}(\mathbf{u}, \mathbf{v}) &= \int_{D\backslash\overline{D}_0} \overline{\mathbf{v}} \cdot \operatorname{curl}\operatorname{curl}\mathbf{u}dx - k^4 \int_{D_0} \mathbf{u}\cdot\overline{\mathbf{v}}dx \\ &= \int_D \operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\overline{\mathbf{v}}dx.\end{aligned}$$

The variational equation now becomes

$$\mathcal{A}_k(\mathbf{F}_0, \boldsymbol{\psi}) - k^2\mathcal{B}(\mathbf{F}_0, \boldsymbol{\psi}) = \ell_k(\boldsymbol{\psi})$$

where

$$\ell_k(\boldsymbol{\psi}) := -\mathcal{A}_k(\boldsymbol{\Theta}, \boldsymbol{\psi}) + k^2\mathcal{B}(\boldsymbol{\Theta}, \boldsymbol{\psi}).$$

Since $(N - I)^{-1}$ is bounded positive definite, there exists a constant γ such that

$$\mathcal{A}_k(\mathbf{u}, \mathbf{u}) \geq \gamma||\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}||^2_{D\backslash\overline{D}_0} + k^4||\mathbf{u}||^2_D.$$

Using the inequality

$$||\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}||^2_{D\backslash\overline{D}_0} \geq ||\operatorname{curl}\operatorname{curl}\mathbf{u}||^2_{D\backslash\overline{D}_0} - 2k^2||\operatorname{curl}\operatorname{curl}\mathbf{u}||_{D\backslash\overline{D}_0}||\mathbf{u}||_{D\backslash\overline{D}_0} + k^4||\mathbf{u}||^2_{D\backslash\overline{D}_0},$$

we obtain

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}) \geq \gamma||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_{D\setminus\overline{D}_0} - 2\gamma k^2||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_{D\setminus\overline{D}_0}||\mathbf{u}||_{D\setminus\overline{D}_0} + k^4(1+\gamma)||\mathbf{u}||^2_{D\setminus\overline{D}_0} + k^4||\mathbf{u}||^2_{D_0}.$$

Using for example the identity

$$\gamma X^2 - 2\gamma XY + (1+\gamma)Y^2 = (\gamma+\frac{1}{2})\left(Y - \frac{\gamma}{\gamma+\frac{1}{2}}X\right)^2 + \frac{1}{2}Y^2 + \frac{\gamma}{1+2\gamma}X^2 \qquad (3.17)$$

and setting $\tilde{\gamma} = \dfrac{\gamma}{1+2\gamma}$ we deduce that

$$\begin{aligned}\mathcal{A}_k(\mathbf{u},\mathbf{u}) &\geq \tilde{\gamma}\left(||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_{D\setminus\overline{D}_0} + k^4||\mathbf{u}||^2_{D\setminus\overline{D}_0}\right) + k^4||\mathbf{u}||^2_{D_0}\\ &\geq \tilde{\gamma}\left(||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_{D\setminus\overline{D}_0} + k^4||\mathbf{u}||^2_{D\setminus\overline{D}_0} + 2k^4||\mathbf{u}||^2_{D_0}\right).\end{aligned}$$

Since $\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} = k^2\mathbf{u}$ in D_0, one also has that

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}) \geq \tilde{\gamma}\left(||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_D + k^4||\mathbf{u}||^2_D\right). \qquad (3.18)$$

Integrating by parts, one has the following identity valid for all $\mathbf{u} \in V_0(D, D_0, k)$

$$||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u}||^2_{D\setminus\overline{D}_0} = ||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_D + k^4||\mathbf{u}||^2_D - 2k^2||\mathrm{curl}\,\mathbf{u}||^2_D.$$

Hence,

$$2k^2||\mathrm{curl}\,\mathbf{u}||^2_D \leq ||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_D + k^4||\mathbf{u}||^2_D. \qquad (3.19)$$

Let $c_k = \min\left(\frac{k^2}{1+k^2}, \frac{k^4}{1+k^2}\right)$. Using (3.18) and (3.19) we obtain that

$$\begin{aligned}\mathcal{A}_k(\mathbf{u},\mathbf{u}) &\geq \tilde{\gamma}\left(c_k\frac{1+k^2}{k^2}||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_D + c_k(1+k^2)||\mathbf{u}||^2_D\right)\\ &\geq c_k\tilde{\gamma}\left(||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_D + ||\mathbf{u}||^2_D + \frac{1}{k^2}\left(||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_D + k^4||\mathbf{u}||^2_D\right)\right)\\ &\geq c_k\tilde{\gamma}||\mathbf{u}||^2_{\mathcal{U}(D)}.\end{aligned}$$

Therefore $\mathcal{A}_k$ is a coercive sesquilinear form on $V_0(D, D_0, k) \times V_0(D, D_0, k)$. Moreover, from Lemma 2.3.3 in Chapter 2, $\mathcal{B}$ is compact in $\mathcal{U}_0(D)$. Thus, it is easy to see that $\mathcal{B}$ defines a compact perturbation of $\mathcal{A}_k$. The result now follows from the application of the Fredholm alternative.

Now assume that $(I-N)^{-1}$ is bounded positive definite. The variational formulation can also be written as

$$\tilde{\mathcal{A}}_k(\mathbf{F}_0, \boldsymbol{\psi}) + \mathcal{B}_k(\mathbf{F}_0, \boldsymbol{\psi}) = \tilde{\ell}_k(\boldsymbol{\psi})$$

with

$$\tilde{\mathcal{A}}_k(\mathbf{u},\mathbf{v}) := \int_{D\setminus\overline{D}_0} N(I-N)^{-1}\left(\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u}\right)\cdot\left(\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{v}} - k^2\overline{\mathbf{v}}\right)dx + \int_{D\setminus\overline{D}_0}\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{v}}dx + k^4\int_{D_0}\mathbf{u}\cdot\overline{\mathbf{v}}dx$$

and

$$\tilde{\ell}_k(\boldsymbol{\psi}) = -\tilde{\mathcal{A}}_k(\boldsymbol{\Theta}, \boldsymbol{\psi}) + k^2\mathcal{B}(\boldsymbol{\Theta}, \boldsymbol{\psi}).$$

Since $(I - N)^{-1}$ is bounded positive definite, there exists a constant γ such that

$$\tilde{\mathcal{A}}_k(\mathbf{u}, \mathbf{u}) \geq (\gamma + 1)||\operatorname{curl}\operatorname{curl}\mathbf{u}||^2_{D\setminus\overline{D}_0} - 2\gamma k^2||\operatorname{curl}\operatorname{curl}\mathbf{u}||_{D\setminus\overline{D}_0}||\mathbf{u}||_{D\setminus\overline{D}_0} + \gamma k^4||\mathbf{u}||^2_{D\setminus\overline{D}_0} + k^4||\mathbf{u}||^2_{D_0}. \quad (3.20)$$

Then, the same method used in the previous case from (3.17) shows that $\tilde{\mathcal{A}}_k$ is a coercive sesquilinear form on $V_0(D, D_0, k) \times V_0(D, D_0, k)$ and the result follows from the Fredholm alternative. □

3.2 Transmission eigenvalues

We now study the homogeneous interior transmission problem with $\mathbf{G} = \mathbf{H} = 0$

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 & \text{in } D \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 & \text{in } D \\ \nu \times \mathbf{E} = \nu \times \mathbf{E}_0 & \text{on } \Gamma \\ \nu \times \operatorname{curl}\mathbf{E} = \nu \times \operatorname{curl}\mathbf{E}_0 & \text{on } \Gamma. \end{cases}$$

3.2.1 Case of complex index of refraction

We give here a similar result as in the case without cavity for index of refraction with definite positive imaginary part. The next theorem shows that, in this case, transmission eigenvalues do not exist.

We first note that k is a transmission eigenvalue if and only if the homogeneous problem

$$\mathcal{A}_k(\mathbf{F}_0, \boldsymbol{\psi}) + \mathcal{B}_k(\mathbf{F}_0, \boldsymbol{\psi}) = 0 \text{ for all } \boldsymbol{\psi} \in V_0(D, D_0, k)$$

has a nontrivial solution $\mathbf{F}_0 \in V_0(D, D_0, k)$. Taking $\boldsymbol{\psi} = \mathbf{F}_0$ we obtain

$$\int_{D\setminus\overline{D}_0} (N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F}_0 - k^2\mathbf{F}_0) \cdot (\operatorname{curl}\operatorname{curl}\overline{\mathbf{F}}_0 - k^2\overline{\mathbf{F}}_0)dx + k^4 \int_{D\setminus\overline{D}_0} |\mathbf{F}_0|^2 dx - k^2 \int_{D\setminus\overline{D}_0} |\operatorname{curl}\mathbf{F}_0|^2 dx + k^2 \int_\Sigma \nu \times \overline{\mathbf{F}}_0 \cdot \operatorname{curl}\mathbf{F}_0 ds = 0. \quad (3.21)$$

In order to study transmission eigenvalues it suffices to study (3.21).

Theorem 3.2.1. *If $N \in L^\infty(D, \mathbb{R}^{3\times 3})$ is such that $N = I$ in D_0 and $\operatorname{Im}(N)$ is positive almost everywhere in $D\setminus\overline{D}_0$, then there are no transmission eigenvalues.*

Proof. Using the fact that $\mathbf{F}_0 \in V_0(D, D_0, k)$ and in particular $\operatorname{curl}\operatorname{curl}\mathbf{F}_0 - k^2\mathbf{F}_0 = 0$ in D_0, we can rewrite (3.21) as

$$\int_{D\setminus\overline{D}_0} (N-I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F}_0 - k^2\mathbf{F}_0)\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{F}}_0 - k^2\overline{\mathbf{F}}_0)dx$$

$$+ k^4\int_D |\mathbf{F}_0|^2 dx - k^2\int_D |\operatorname{curl}\mathbf{F}_0|^2 dx = 0. \quad (3.22)$$

Since $\operatorname{Im}((N-I)^{-1})$ is negative definite in $D\setminus\overline{D}_0$ and all the terms in the above equation are real except for the first one, by taking the imaginary part we deduce that $\mathbf{F}_0$ satisfies Maxwell's equations in $D\setminus\overline{D}_0$ and then in all D. Since $\mathbf{F}_0$ has zero Cauchy data on Γ, we obtain $\mathbf{F}_0 = 0$ in D, and therefore k is not a transmission eigenvalue.

□

Remark 3.2.1. *Similarly to the case without cavities (see Remark 2.4.2), we can extend this result to the case where* $\operatorname{Im}(N)$ *is positive, only in a subset of* $D\setminus\overline{D}_0$.

In the following, we shall assume that $\operatorname{Im}(N) = 0$.

3.2.2 Estimates on the first transmission eigenvalue

To show discreteness of the set of transmission eigenvalues, we follow the same method as in the case without cavity. Again we want to show that the problem is Fredholm. The next theorem shows that the operators $\mathcal{A}_k - k^2\mathcal{B}$ or $\tilde{\mathcal{A}}_k - k^2\mathcal{B}$ are coercive provided k is small enough. This leads to estimates on the first transmission eigenvalue that are the same as in the previous case. It can be compared with the first Dirichlet eigenvalue of the whole domain D and eventually to the upper bound of the index of refraction.

Theorem 3.2.2. *Let* $N \in L^\infty(D, \mathbb{R}^{3\times 3})$. *Let* $0 < \eta_1(x) < \eta_2(x) < \eta_3(x)$ *be the eigenvalues of the positive definite matrix* N. *We denote by* $N^* = \sup_{D\setminus\overline{D}_0}\eta_3(x)$ *and* $N_* = \inf_{D\setminus\overline{D}_0}\eta_1(x)$. *If* k *is a transmission eigenvalue then*

$$k^2 > \frac{\lambda_0(D)}{N^*} \text{ if } (N-I)^{-1} \text{ is bounded positive definite }, \quad (3.23)$$

or

$$k^2 > \lambda_0(D) \text{ if } (I-N)^{-1} \text{ is bounded positive definite}, \quad (3.24)$$

where $\lambda_0(D)$ *is the first Dirichlet eigenvalue of* $-\Delta$ *in* D.

Proof. We want to show that if $k > 0$ is sufficiently small, then k is not a transmission eigenvalue. It suffices to show that for $k > 0$ sufficiently small, if $\mathbf{u} \in V_0(D, D_0, k)$ satisfies (3.22), then $\mathbf{u}$ is zero.

We first assume that $(N-I)^{-1}$ is bounded positive definite. In order to find the lower bound for the first transmission eigenvalue we study (3.22). For $\gamma = \dfrac{1}{N^*-1}$ we have

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}) \geq \gamma||\operatorname{curl}\operatorname{curl}\mathbf{u}||^2_{D\setminus\overline{D}_0} - 2\gamma k^2||\operatorname{curl}\operatorname{curl}\mathbf{u}||_{D\setminus\overline{D}_0}||\mathbf{u}||_{D\setminus\overline{D}_0}$$

$$+ k^4(1+\gamma)||\mathbf{u}||^2_{D\setminus\overline{D}_0} + k^4||\mathbf{u}||^2_{D_0}.$$

From the identity

$$\gamma X^2 - 2\gamma XY + (1+\gamma)Y^2 = \varepsilon\left(Y - \frac{\gamma}{\varepsilon}X\right)^2 + \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)X^2 + (1+\gamma-\varepsilon)Y^2 \qquad (3.25)$$

with $X = ||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_{D\setminus\overline{D}_0}$ and $Y = ||\mathbf{u}||_{D\setminus\overline{D}_0}$ we deduce that

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}) \geq \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_{D\setminus\overline{D}_0} + k^4(1+\gamma-\varepsilon)||\mathbf{u}||^2_{D\setminus\overline{D}_0} + k^4||\mathbf{u}||^2_{D_0}.$$

Using $k^2\mathbf{u} = \mathrm{curl}\,\mathrm{curl}\,\mathbf{u}$ in D_0 and $1 \geq \left(\gamma - \frac{\gamma^2}{\varepsilon}\right) + (1+\gamma-\varepsilon)$ we obtain

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}) - k^2\mathcal{B}(\mathbf{u},\mathbf{u}) \geq \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_D + k^4(1+\gamma-\varepsilon)||\mathbf{u}||^2_D - k^2||\mathrm{curl}\,\mathbf{u}||^2_D.$$

Moreover, for all $\mathbf{u} \in \mathcal{U}_0(D)$ we have the following inequality (see the proof of Lemma 2.4.1 in Chapter 2)

$$||\mathrm{curl}\,\mathbf{u}||^2_{L^2(D)} \leq \frac{1}{\lambda_0(D)}||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_{L^2(D)}. \qquad (3.26)$$

Then, using (3.26), we obtain

$$\mathcal{A}_k(\mathbf{u},\mathbf{u}) - k^2\mathcal{B}(\mathbf{u},\mathbf{u}) \geq \left(\gamma - \frac{\gamma^2}{\varepsilon} - \frac{k^2}{\lambda_0(D)}\right)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_D + k^4(1+\gamma-\varepsilon)||\mathbf{u}||^2_D.$$

Hence letting ε arbitrarily close to $\gamma+1$, k is not a transmission eigenvalue if

$$k^2 \leq \frac{\lambda_0(D)}{N^*}.$$

Now we prove the result when $(I-N)^{-1}$ is bounded positive definite. We have the following equality

$$\tilde{\mathcal{A}}_k(\mathbf{u},\mathbf{v}) - k^2\mathcal{B}(\mathbf{u},\mathbf{v}) = \int_{D\setminus\overline{D}_0} N(I-N)^{-1}\left(\mathrm{curl}\,\mathrm{curl}\,\mathbf{u} - k^2\mathbf{u}\right)\cdot\left(\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{v}} - k^2\overline{\mathbf{v}}\right)dx$$
$$+ k^4\int_{D_0}\mathbf{u}\cdot\overline{\mathbf{v}}dx + \int_{D\setminus\overline{D}_0}\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{v}}dx - k^2\int_D \mathrm{curl}\,\mathbf{u}\cdot\mathrm{curl}\,\overline{\mathbf{v}}dx. \qquad (3.27)$$

For $\gamma = \dfrac{N_*}{1-N_*}$, using (3.25) and (3.26) we have

$$\begin{aligned}\mathcal{A}_k(\mathbf{u},\mathbf{u}) - k^2\mathcal{B}(\mathbf{u},\mathbf{u}) &\geq (\gamma+1)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_{D\setminus\overline{D}_0} - 2\gamma k^2||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||_{D\setminus\overline{D}_0}||\mathbf{u}||_{D\setminus\overline{D}_0}\\ &\quad + k^4\gamma||\mathbf{u}||^2_{D\setminus\overline{D}_0} + k^4||\mathbf{u}||^2_{D_0} - k^2||\mathrm{curl}\,\mathbf{u}||^2_D\\ &\geq (1+\gamma-\varepsilon)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}|^2|_D + \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)||\mathbf{u}||^2_D - k^2||\mathrm{curl}\,\mathbf{u}||^2_D\\ &\geq \left(1+\gamma-\varepsilon - \frac{k^2}{\lambda_0(D)}\right)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}||^2_D + \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)||\mathbf{u}||^2_D.\end{aligned}$$

Hence letting ε arbitrarily close to γ, we deduce that k cannot be a transmission eigenvalue if

$$k^2 \leq \lambda_0(D).$$

□

3.2.3 Discreteness of the set of transmission eigenvalues

The proof of the discreteness of the set of transmission eigenvalues relies on the analytic Fredholm theory. This theory will guarantee that the operator associated with the resolution of the interior transmission problem is injective except for at most a discrete set of values of k. Thus, we first show that this operator is analytic with respect to $k \in \mathbb{C}$ in some neighborhood of the real axis. The difficulty here relies on the fact that the variational space in a function of the parameter k. In order to apply the analytic Fredholm theory, we first need to give an equivalent formulation of the problem in a space that does not depend on k. In particular, we use a projection operator from $\mathcal{U}_0(D)$ into $V_0(D, D_0, k)$.

Finding transmission eigenvalues is equivalent to finding $k > 0$ such that the problem

$$\mathcal{A}_k(\mathbf{E}, \boldsymbol{\psi}) - k^2\mathcal{B}(\mathbf{E}, \boldsymbol{\psi}) = 0 \text{ for all } \boldsymbol{\psi} \in V_0(D, D_0, k)$$

has non trivial solutions $\mathbf{E} \in V_0(D, D_0, k)$. This is equivalent to finding the values of k for which

$$A_k - k^2 B : V_0(D, D_0, k) \longrightarrow V_0(D, D_0, k)$$

has a nontrivial kernel, where A_k is the positive definite self-adjoint operator associated with the coercive bilinear form $\mathcal{A}_k(\cdot, \cdot)$ and B is the compact operator associated with the bilinear form $\mathcal{B}(\cdot, \cdot)$.

To avoid dealing with function spaces depending on k, we shall make use of an analytic operator $\tilde{P}_k$ from $\mathcal{U}_0(D)$ into $V_0(D, D_0, k)$ in order to build an analytic extension of A_k and B with operators acting on $\mathcal{U}_0(D)$.

Let $k \in \mathbb{C}$. For $\mathbf{E} \in \mathcal{U}(D)$, we define $\theta_k \mathbf{E}$ by

$$(\theta_k \mathbf{E})(x) = \int_{D_0} (\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E})(y)\mathbb{G}(x, y)dy$$

with $\mathbb{G}(x, y) = y_0(k|x - y|)I + \dfrac{1}{k^2}\nabla_x \operatorname{div}_x (y_0(k|x - y|)I)$, where $y_0(t) = k\dfrac{\cos(t)}{4\pi t}$. One has in particular

$$\operatorname{curl}_x \operatorname{curl}_x \mathbb{G}(x, y) - k^2\mathbb{G}(x, y) = \delta_y I.$$

Using the regularity of the volume potential, we have that $\theta_k \mathbf{E} \in L^2(D)^3$. Moreover, since

$$\operatorname{curl}_x \mathbb{G}(x, y) = \operatorname{curl}_x (y_0(k|x - y|)I),$$

we have

$$\operatorname{curl}(\theta_k \mathbf{E})(x) = \operatorname{curl}_x \int_{D_0} (\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E})(y) y_0(k|x - y|) I dy, \quad x \in D.$$

Once again using regularity of volume potential, we obtain that $\operatorname{curl}(\theta_k \mathbf{E}) \in H^1(D)$. Therefore, $\theta_k \mathbf{E} \in \mathcal{U}(D)$, and there exists a constant $C(k)$ such that

$$||\theta_k \mathbf{E}||_{\mathcal{U}(D)} \leq C(k)||\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E}||_{L^2(D_0)}.$$

$\theta_k : \mathcal{U}(D) \longrightarrow \mathcal{U}(D)$ depends analytically with respect to $k \in \mathbb{C}$.

Let χ be a cutoff function that equals 1 in D_0 and 0 outside D. We define the continuous operator $\tilde{P}_k : \mathcal{U}_0(D) \longrightarrow \mathcal{U}_0(D)$ by

$$\tilde{P}_k \mathbf{E} := \mathbf{E} - \chi \theta_k \mathbf{E}. \tag{3.28}$$

We first observe that for $\mathbf{E} \in V_0(D, D_0, k)$ we have

$$\theta_k \mathbf{E} = 0 \quad \text{ and } \quad \tilde{P}_k \mathbf{E} = \mathbf{E}. \tag{3.29}$$

Moreover, since

$$\operatorname{curl}\operatorname{curl} \theta_k \mathbf{E} - k^2 \theta_k \mathbf{E} = \operatorname{curl}\operatorname{curl} \mathbf{E} - k^2 \mathbf{E} \text{ in } D_0,$$

we also have $\tilde{P}_k \mathbf{E} \in V_0(D, D_0, k)$ for all $\mathbf{E} \in \mathcal{U}_0(D)$. Finally, from the analyticity of θ_k, $\tilde{P}_k$ also depends analytically on complex k with positive real part.

Using the Riesz representation theorem, we now introduce the operators $\tilde{A}_k$ and $\tilde{B}$ from $\mathcal{U}_0(D)$ into $\mathcal{U}_0(D)$ defined by

$$\begin{aligned}
(\tilde{A}_k \mathbf{E}, \mathbf{F})_{\mathcal{U}(D)} &= \mathcal{A}_k \left(\tilde{P}_k \mathbf{E}, \overline{\tilde{P}_k \overline{\mathbf{F}}} \right) + \alpha \left(\theta_k \mathbf{E}, \overline{\theta_k \overline{\mathbf{F}}} \right)_{\mathcal{U}(D)}, \\
(\tilde{B} \mathbf{E}, \mathbf{F})_{\mathcal{U}(D)} &= \mathcal{B} \left(\tilde{P}_k \mathbf{E}, \overline{\tilde{P}_k \overline{\mathbf{F}}} \right)
\end{aligned}$$

for all $\mathbf{E}$, $\mathbf{F}$ in $\mathcal{U}_0(D)$, where α is a positive constant that will be fixed later (and is independent of k). The analyticity of $\tilde{P}_k$ and θ_k as well as the expression of $\mathcal{A}_k$ and $\mathcal{B}$ show that $\tilde{A}_k$ and $\tilde{B}$ depend analytically on $k \in \mathbb{C}$ with $\operatorname{Re}(k) > 0$. Moreover, the operator $\tilde{B}$ is compact.

Observe that if k is real, for $\mathbf{E} \in V_0(D, D_0, k)$, then we have $\overline{\mathbf{E}} \in V_0(D, D_0, k)$, and hence from (3.29), we have that

$$\tilde{A}_k \mathbf{E} = A_k \mathbf{E}, \qquad \tilde{B} \mathbf{E} = B \mathbf{E} \qquad \forall \mathbf{E} \in V_0(D, D_0, k) \text{ and } \forall k \in \mathbb{R}.$$

Hence we conclude that for real k, if $A_k - k^2 B$ is not injective, then $\tilde{A}_k - k^2 \tilde{B}$ is not injective. In order to show that the set of transmission eigenvalues is at most discrete it is sufficient to prove that the set of k for which $\tilde{A}_k - k^2 \tilde{B}$ is not injective is at most discrete. For that purpose we shall prove the following lemma

Lemma 3.2.3. *Let k be positive and real. Then*

(a) *there exists α_0 independent of k such that for all $\alpha \geq \alpha_0$ the operator $\tilde{A}_k$ is strictly coercive for all $k > 0$;*

(b) *there exists k_0 such that for all $0 < k \leq k_0$ the operator $\tilde{A}_k - k^2 \tilde{B}$ is injective.*

Proof. Assume that k is real. Therefore

$$(\tilde{A}_k \mathbf{E}, \mathbf{E})_{\mathcal{U}(D)} = \mathcal{A}_k \left(\tilde{P}_k \mathbf{E}, \overline{\tilde{P}_k \overline{\mathbf{E}}} \right) + \alpha ||\theta_k \mathbf{E}||^2_{\mathcal{U}(D)}.$$

From the coercivity of $\mathcal{A}_k$ on $V_0(D, D_0, k)$ we have that

$$(\tilde{A}_k \mathbf{E}, \mathbf{E})_{\mathcal{U}(D)} \geq \gamma_k ||\tilde{P}_k \mathbf{E}||^2_{\mathcal{U}(D)} + \alpha ||\theta_k \mathbf{E}||^2_{\mathcal{U}(D)},$$

where $\gamma_k = c_k\tilde{\gamma}$ with $c_k < 1$ and $\tilde{\gamma}$ depending only on N. From the expression of $\tilde{P}_k\mathbf{E}$ one sees that there exists a constant c that depends only on χ such that

$$||\tilde{P}_k\mathbf{E}||^2_{\mathcal{U}(D)} \geq ||\mathbf{E}||^2_{\mathcal{U}(D)} - 2c||\mathbf{E}||_{\mathcal{U}(D)}||\theta_k\mathbf{E}||_{\mathcal{U}(D)} + ||\chi\theta_k\mathbf{E}||^2_{\mathcal{U}(D)},$$

$$(\tilde{A}_k\mathbf{E},\mathbf{E})_{\mathcal{U}(D)} \geq \gamma_k||\mathbf{E}||^2_{\mathcal{U}(D)} - 2c\gamma_k||\mathbf{E}||_{\mathcal{U}(D)}||\theta_k\mathbf{E}||_{\mathcal{U}(D)} + \alpha||\theta_k\mathbf{E}||^2_{\mathcal{U}(D)}.$$

Let $\alpha_0 = \tilde{\gamma}c^2$. Since $\gamma_k < \tilde{\gamma}$, we observe that $(\gamma_k c)^2 < \gamma_k\alpha$ for all k and $\alpha \geq \alpha_0$. Therefore we can conclude that the operator $\tilde{A}_k$ is strictly coercive for $\alpha \geq \alpha_0$.

We now prove the second assertion. We observe that

$$(\tilde{A}_k\mathbf{E},\mathbf{E})_{\mathcal{U}(D)} - k^2(\tilde{B}\mathbf{E},\mathbf{E})_{\mathcal{U}(D)} = \mathcal{A}_k\left(\tilde{P}_k\mathbf{E},\overline{\tilde{P}_k\mathbf{E}}\right) + \alpha||\theta_k\mathbf{E}||^2_{\mathcal{U}(D)} - k^2\mathcal{B}\left(\tilde{P}_k\mathbf{E},\overline{\tilde{P}_k\mathbf{E}}\right).$$

Therefore $(\tilde{A}_k\mathbf{E},\mathbf{E})_{\mathcal{U}(D)} - k^2(\tilde{B}\mathbf{E},\mathbf{E})_{\mathcal{U}(D)} = 0$ implies that $\theta_k\mathbf{E} = 0$ and

$$\mathcal{A}_k\left(\tilde{P}_k\mathbf{E},\overline{\tilde{P}_k\mathbf{E}}\right) - k^2\mathcal{B}\left(\tilde{P}_k\mathbf{E},\overline{\tilde{P}_k\mathbf{E}}\right) = 0. \tag{3.30}$$

According to Theorem 3.2.2, if (3.30) holds for k small enough then $\tilde{P}_k\mathbf{E} = 0$. We conclude that $\mathbf{E} = \tilde{P}_k\mathbf{E} + \chi\theta_k\mathbf{E} = 0$. Then for k small enough, $\tilde{A}_k - k^2\tilde{B}$ is injective. □

Theorem 3.2.4. *Let $N \in L^\infty(D,\mathbb{R}^{3\times 3})$. Then the set of transmission eigenvalues is discrete.*

Proof. The previous lemma shows in particular that for α sufficiently large $\tilde{A}_k$ is coercive in a neighborhood of the real axis (since $\tilde{A}_k$ is continuous with respect to k) and therefore invertible. In this neighborhood $\tilde{A}_k^{-1}$ is analytic and hence the operator $I - k^2\tilde{A}_k^{-1}\tilde{B}$ depends analytically on k and is injective for k sufficiently small. The analytic Fredholm theory now shows that this operator is injective for all values of k in this neighborhood except for at most a discrete set of values. □

3.2.4 Existence of transmission eigenvalues

We observe that $k > 0$ is a transmission eigenvalue if and only if the operator

$$A_k - k^2 B : V_0(D, D_0, k) \longrightarrow V_0(D, D_0, k)$$

has a non trivial kernel, where A_k is the positive definite self-adjoint operator associated with the coercive bilinear form $\mathcal{A}_k(\cdot,\cdot)$, and B is the compact operator associated with the bilinear form $\mathcal{B}(\cdot,\cdot)$. Define the operator $A_k^{-1/2}$ by $A_k^{-1/2} = \int_0^\infty \lambda^{-1/2} dE_\lambda$ where dE_λ is the spectral measure associated with the positive self-adjoint operator A_k. In particular, $A_k^{-1/2}$ is also bounded, positive definite and self-adjoint. Hence it is obvious that k is a transmission eigenvalue if and only if the operator

$$I_k - k^2 A_k^{-1/2} B A_k^{-1/2} : V_0(D, D_0, k) \longrightarrow V_0(D, D_0, k) \tag{3.31}$$

has a nontrivial kernel. Note that $A_k^{-1/2} B A_k^{-1/2}$ is a compact self-adjoint operator. To avoid dealing with k-dependent function space $V_0(D, D_0, k)$ we consider

$$I - k^2 R_k A_k^{-1/2} B A_k^{-1/2} P_k : \mathcal{U}_0(D) \longrightarrow \mathcal{U}_0(D) \tag{3.32}$$

where $P_k : \mathcal{U}_0(D) \longrightarrow V_0(D, D_0, k)$ and $R_k : V_0(D, D_0, k) \longrightarrow \mathcal{U}_0(D)$ are respectively the orthogonal projection and the injection operator. We first have to show that P_k is continuous from $\mathcal{U}_0(D)$ into $V_0(D, D_0, k)$. For this purpose we need the following lemma

Lemma 3.2.5. *Assume $|k| < k_0$ for $k_0 > 0$. Then there exists a constant $C(k_0)$ such that*

$$||\mathbf{E} - P_k\mathbf{E}||_{\mathcal{U}_0(D)} \leq C(k_0)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{E} - k^2\mathbf{E}||_{L^2(D_0)}$$

for all $\mathbf{E} \in \mathcal{U}_0(D)$.

Proof. Let $\mathbf{E} \in \mathcal{U}_0(D)$ and let $\tilde{P}_k$ be the operator defined by (3.28). Then

$$\begin{aligned} ||P_k\mathbf{E} - \mathbf{E}||_{\mathcal{U}_0(D)} &\leq ||\tilde{P}_k\mathbf{E} - \mathbf{E}||_{\mathcal{U}_0(D)} = ||\chi\theta_k\mathbf{E}||_{\mathcal{U}_0(D)} \\ &\leq C||\theta_k\mathbf{E}||_{\mathcal{U}_0(D)} \leq CC(k)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{E} - k^2\mathbf{E}||_{L^2(D_0)}. \end{aligned}$$

Since θ_k depends continuously on k, one can bound $CC(k)$ by a constant that depends only on k_0 for all $k \leq k_0$. □

Theorem 3.2.6. *The projection operator $P_k : \mathcal{U}_0(D) \longrightarrow V_0(D, D_0, k)$ is continuous with respect to $k > 0$.*

Proof. Let k and k' be real positive less than k_0, and let $\mathbf{E}$ be in $\mathcal{U}_0(D)$. Set $\mathbf{E}_k := P_k\mathbf{E}$ and $\mathbf{E}_{k'} := P_{k'}\mathbf{E}$. Then

$$||\mathbf{E}_k - \mathbf{E}_{k'}||^2_{\mathcal{U}_0(D)} = ||P_k(\mathbf{E}_k - \mathbf{E}_{k'})||^2_{\mathcal{U}_0(D)} + ||(I - P_k)(\mathbf{E}_k - \mathbf{E}_{k'})||^2_{\mathcal{U}_0(D)}.$$

On the one hand, using Lemma 3.2.5,

$$\begin{aligned} ||(I - P_k)(\mathbf{E}_k - \mathbf{E}_{k'})||_{\mathcal{U}_0(D)} &= ||(I - P_k)\mathbf{E}_{k'}||_{\mathcal{U}_0(D)} \\ &\leq C(k_0)||\mathrm{curl}\,\mathrm{curl}\,\mathbf{E}_{k'} - k^2\mathbf{E}_{k'}||_{L^2(D_0)} \\ &= C(k_0)|k'^2 - k^2|||\mathbf{E}_{k'}||_{L^2(D_0)} \\ &\leq C(k_0)|k'^2 - k^2|||\mathbf{E}||_{\mathcal{U}_0(D)}, \end{aligned}$$

and on the other hand,

$$\begin{aligned} ||P_k(\mathbf{E}_k - \mathbf{E}_{k'})||^2_{\mathcal{U}_0(D)} &= (P_k(\mathbf{E}_k - \mathbf{E}_{k'}), P_k(\mathbf{E}_k - \mathbf{E}_{k'}))_{\mathcal{U}_0(D)} \\ &= (P_k(\mathbf{E}_k - \mathbf{E}_{k'}), \mathbf{E}_k - \mathbf{E}_{k'})_{\mathcal{U}_0(D)} \\ &= (P_k(\mathbf{E}_k - \mathbf{E}_{k'}), \mathbf{E}_k - \mathbf{E} + \mathbf{E} - \mathbf{E}_{k'})_{\mathcal{U}_0(D)} \qquad (3.33) \\ &= (P_k(\mathbf{E}_k - \mathbf{E}_{k'}), \mathbf{E} - \mathbf{E}_{k'})_{\mathcal{U}_0(D)} \\ &= ((I - P_{k'})P_k(\mathbf{E}_k - \mathbf{E}_{k'}), \mathbf{E})_{\mathcal{U}_0(D)}\,. \end{aligned}$$

Applying Lemma 3.2.5, we have

$$\begin{aligned} ||(I - P_{k'})P_k(\mathbf{E}_k - \mathbf{E}_{k'})||_{\mathcal{U}_0(D)} &\leq C(k_0)||(\mathrm{curl}\,\mathrm{curl}\, - k'^2)P_k(\mathbf{E}_k - \mathbf{E}_{k'})||_{L^2(D_0)} \\ &= C(k_0)|k'^2 - k^2|||P_k(\mathbf{E}_k - \mathbf{E}_{k'})||_{L^2(D_0)} \qquad (3.34) \\ &\leq C(k_0)|k'^2 - k^2|||\mathbf{E}_k - \mathbf{E}_{k'}||_{\mathcal{U}_0(D)}. \end{aligned}$$

Therefore, from (3.34) and (3.33), we have

$$||P_k(\mathbf{E}_k - \mathbf{E}_{k'})||^2_{\mathcal{U}_0(D)} \leq C(k_0)|k'^2 - k^2|||\mathbf{E}_k - \mathbf{E}_{k'}||_{\mathcal{U}_0(D)}||\mathbf{E}||_{\mathcal{U}_0(D)}.$$

Using the previous estimates in the first equality yields

$$||\mathbf{E}_k - \mathbf{E}_{k'}||_{\mathcal{U}_0(D)} \leq \frac{\sqrt{5}+1}{2} C(k_0)|k'^2 - k^2|||\mathbf{E}||_{\mathcal{U}_0(D)}$$

which proves in particular that $k \longmapsto P_k\mathbf{E}$ is continuous. □

We can show that the mapping $k \longmapsto R_k A_k^{-1/2} B A_k^{-1/2} P_k$ is continuous for $k > 0$. Therefore, we can apply Theorem 2.2.4 to $I - k^2 T_k$ to show the existence of transmission eigenvalues.

Let $r > 0$. We denote by $k_0(r, n)$ the first transmission eigenvalue for a ball of radius r and $N = nI$ (see [26] for the existence of such eigenvalues). Let $M(r)$ be the maximum number of two-by-two disjoint balls of radius r that can be inserted in $D\backslash\overline{D}_0$.

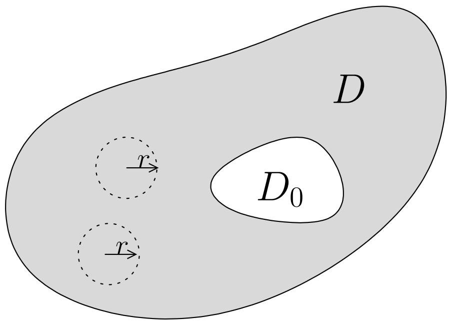


Figure 3.2: Balls of radius r included in $D\backslash\overline{D}_0$.

Theorem 3.2.7. *For all $r > 0$, there exist at least $M(r)$ transmission eigenvalues in*

1. $\left[\frac{\lambda_0(D)}{N_*}, k_0(r, N_*)\right]$ *if $(N - I)^{-1}$ is bounded positive definite,*

2. $[\lambda_0(D), k_0(r, N^*)]$ *if $(I - N)^{-1}$ is bounded positive definite.*

Proof. Assume first that $(N - I)^{-1}$ is bounded positive definite. From Theorem 3.2.2, we know that $A_{k_1} - k_1^2 B$ is positive for $k_1 := \dfrac{\lambda_0(D)}{N^*}$.

Next, we look for $k_2 > k_1$ such that $I - k_2^2 T_{k_2}$ is non positive on a p-dimensional subspace of $\mathcal{U}_0(D)$. We first notice that for all $k > 0$,

$$\begin{aligned}(A_k\mathbf{u} - k^2 B\mathbf{u}, \mathbf{u})_{\mathcal{U}(D)} &= \int_{D\backslash\overline{D}_0} (N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}) \cdot (\operatorname{curl}\operatorname{curl}\overline{\mathbf{u}} - k^2\overline{\mathbf{u}})dx \\ &\quad - k^2 \int_D |\operatorname{curl}\mathbf{u}|^2 dx + k^4 \int_D |\mathbf{u}|^2 dx \\ &\leq \frac{1}{N_* - 1} \int_{D\backslash\overline{D}_0} |\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}|^2 dx \\ &\quad - k^2 \int_D |\operatorname{curl}\mathbf{u}|^2 dx + k^4 \int_D |\mathbf{u}|^2 dx.\end{aligned}$$

Let B_j^r, $j = 1, \ldots, M(r)$ be $M(r)$ two-by-two disjoint balls of radius r included in $D \backslash \overline{D}_0$. For each ball B_j^r, there exists an eigenvector $\mathbf{u}_j \in \mathcal{U}_0(B_j^r)$ corresponding to the transmission eigenvalue $k_2 := k_0(r, N_*)$ which satisfies the variational formulation of the corresponding interior transmission problem :

$$\frac{1}{N_* - 1} \int_{B_j^r} \left|\operatorname{curl}\operatorname{curl}\mathbf{u}_j - k_2^2 \mathbf{u}_j\right|^2 dx - k_2^2 \int_{B_j^r} |\operatorname{curl}\mathbf{u}_j|^2 \, dx + k_2^4 \int_{B_j^r} |\mathbf{u}_j|^2 \, dx = 0.$$

We denote by $\tilde{\mathbf{u}}_j$ the extension of $\mathbf{u}_j$ by 0 to all D. Then $\tilde{\mathbf{u}}_j \in V_0(D, D_0, k)$ and

$$\frac{1}{N_* - 1} \int_{D \backslash \overline{D}_0} \left|\operatorname{curl}\operatorname{curl}\tilde{\mathbf{u}}_j - k_2^2 \tilde{\mathbf{u}}_j\right|^2 dx - k_2 \int_D |\operatorname{curl}\tilde{\mathbf{u}}_j|^2 \, dx + k_2^4 \int_D |\tilde{\mathbf{u}}_j|^2 \, dx = 0.$$

Define $\mathcal{V} := \operatorname{Vect}\{\tilde{\mathbf{u}}_j, 1 \leq j \leq M(r)\}$ a $M(r)$-dimensional subspace of $V_0(D, D_0, k)$. Let $\mathbf{u} \in \mathcal{V}$, $\mathbf{u} = \sum_{j=1}^{M(r)} \alpha_j \tilde{\mathbf{u}}_j$. Since $\tilde{\mathbf{u}}_j$ and $\tilde{\mathbf{u}}_m$ have disjoint supports if $j \neq m$, we get

$$\begin{aligned}
\left((A_{k_2} - k_2^2 B)\mathbf{u}, \mathbf{u}\right)_{\mathcal{U}(D)} &= \sum_{j=1}^{M(r)} \sum_{m=1}^{M(r)} \alpha_j \overline{\alpha}_m \left((A_{k_2} - k_2^2 B)\tilde{\mathbf{u}}_j, \tilde{\mathbf{u}}_m\right)_{\mathcal{U}(D)} \\
&= \sum_{j=1}^{M(r)} |\alpha_j|^2 \left((A_{k_2} - k_2^2 B)\tilde{\mathbf{u}}_j, \tilde{\mathbf{u}}_j\right)_{\mathcal{U}(D)} \\
&\leq \sum_{j=1}^{M(r)} |\alpha_j|^2 \left(\frac{1}{N_* - 1} \int_{D \backslash \overline{D}_0} \left|\operatorname{curl}\operatorname{curl}\tilde{\mathbf{u}}_j - k_2^2 \tilde{\mathbf{u}}_j\right|^2 dx \right. \\
&\quad \left. - k_2^2 \int_D |\operatorname{curl}\tilde{\mathbf{u}}_j|^2 \, dx + k_2^4 \int_D |\tilde{\mathbf{u}}_j|^2 \, dx\right) \\
&\leq 0.
\end{aligned}$$

From Theorem 2.2.4, we deduce that there exist $M(r)$ transmission eigenvalues between $\dfrac{\lambda_0(D)}{N^*}$ and $k_0(r, N_*)$ counting multiplicity.

The same method shows the result in the case where $(I - N)^{-1}$ is bounded positive definite using $\tilde{A}_k$ instead of A_k. □

By letting $r \to 0$ in the previous theorem we have the following corollary.

Corollary 3.2.8. *There exist infinitely many transmission eigenvalues having $+\infty$ as the only accumulation point.*

We denote by $k_0(D_0, N)$ the first transmission eigenvalue for the domain D containing the cavity D_0 and with index N in $D \backslash \overline{D}_0$. We have the following monotonicity result with respect to the index of refraction but also to the size of the cavity

Theorem 3.2.9. *If $D_0 \subseteq D_0'$ and $N_2 \leq N_1$ then*

1. $k_0(D_0, N_1) \leq k_0(D_0', N_2)$ *if $(N_i - I)^{-1}$ is bounded positive definite for $i = 1, 2$,*

2. $k_0(D_0, N_2) \leq k_0(D_0', N_1)$ *if* $(I - N_i)^{-1}$ *is bounded positive definite for* $i = 1, 2$.

Proof. Consider the case where $(N_1 - I)^{-1}$ and $(N_2 - I)^{-1}$ are both bounded positive definite. First, Theorem 2.2.4 for the problem with $N = N_1$ with $k_1 := \frac{\lambda_0(D)}{N_1^*}$ and $k_2 := k_0(D_0, N_2)$ shows that $k_0(D_0, N_1) \leq k_0(D_0, N_2)$. We can show in the same way that when $(I - N_1)^{-1}$ and $(I - N_1)^{-1}$ are bounded positive definite, $k_0(D_0, N_2) \leq k_0(D_0, N_1)$.

It only remains to show that for all N we have that $k_0(D_0, N) \leq k_0(D_0', N)$. The proof is similar to the proof of Theorem 3.2.7. We consider the interior transmission problem for a domain D containing a void D_0. First, from Theorem 3.2.2, we know that $A_{k_1} + B_{k_1}$ is positive for $k_1 = \dfrac{\lambda_0(D)}{N}$. We define $k_2 := k_0(D_0', N)$ and let $\mathbf{v} \in V_0(D, D_0', k_2)$ be an eigenvector corresponding to the eigenvalue k_2. Since $\operatorname{curl}\operatorname{curl}\mathbf{v} - k_2^2\mathbf{v} = 0$ in D_0',

$$\begin{aligned}
(A_{k_2}\mathbf{v} + B_{k_2}\mathbf{v}, \mathbf{v})_{\mathcal{U}(D)} &= \int_{D\setminus\overline{D}_0} (N-1)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{v} - k_2^2\mathbf{v}) \cdot \overline{(\operatorname{curl}\operatorname{curl}\mathbf{v} - k_2^2\mathbf{v})}dx \\
&\quad - k_2^2 \int_D |\operatorname{curl}\mathbf{v}|^2 dx + k_2^4 \int_D |\mathbf{v}|^2 dx \\
&= \int_{D\setminus\overline{D}_0'} (N-1)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{v} - k_2^2\mathbf{v}) \cdot \overline{(\operatorname{curl}\operatorname{curl}\mathbf{v} - k_2^2\mathbf{v})}dx \\
&\quad - k_2^2 \int_D |\operatorname{curl}\mathbf{v}|^2 dx + k_2^4 \int_D |\mathbf{v}|^2 dx \\
&= 0.
\end{aligned}$$

We deduce that there exists an eigenvalue in $\left[\dfrac{\lambda_0(D)}{N}, k_0(D_0', N)\right]$, and consequently, for all N, we have $k_0(D_0, N) \leq k_0(D_0', N)$.

□

One can remark that the bigger the cavity is, the bigger is the first transmission eigenvalue when $N - I$ is positive definite but the first transmission eigenvalue becomes decreasing when $N - I$ is negative definite.

3.3 Characterization of transmission eigenvalues from far field data

In this section, we give the equivalent theorem to Theorem 2.5.1 for the case where the obstacle contains a cavity. The proof differs from the first one with the variational formulation of the interior transmission problem and the fact that the corresponding operator involves only $D\setminus\overline{D}_0$.

Given $\mathbf{E}^i$ an entire solution to Maxwell's equations

$$\operatorname{curl}\operatorname{curl}\mathbf{E}^i - k^2\mathbf{E}^i = 0 \text{ in } \mathbb{R}^3,$$

the direct scattering problem can be formulated as the problem of finding an electric field $\mathbf{E} \in \mathcal{U}_{loc}(\mathbb{R}^3)$ such that

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 & \text{in } \mathbb{R}^3 \\ \mathbf{E} = \mathbf{E}^s + \mathbf{E}^i & \\ \lim\limits_{|x|\to\infty} (\operatorname{curl}\mathbf{E}^s \times x - \mathrm{i}k|x|\mathbf{E}^s) = 0 & \text{uniformly in } \hat{x} = x/|x|. \end{cases}$$

We recall that the far field operator can be written

$$\mathcal{F}g = \mathcal{B}(\mathbf{E}_g)$$

where

$$\begin{array}{rccc} \mathcal{B}: & H_{\text{inc}}(D) & \to & L^2_t(\Gamma) \\ & \mathbf{E}^i & \mapsto & \mathbf{E}_\infty. \end{array}$$

In particular $\mathcal{B}\mathbf{E}_0 = \mathbf{E}_{e,\infty}$ if and only if $\mathbf{E}$ and $\mathbf{E}_0$ are solutions to the following interior transmission problem : find $\mathbf{E}$ and $\mathbf{E}_0$ in $L^2(D)^3$ such that $\mathbf{E} - \mathbf{E}_0 \in \mathcal{U}(D)$ and

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 & \text{in } D \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 & \text{in } D \\ \nu\times\mathbf{E} - \nu\times\mathbf{E}_0 = \nu\times\mathbf{E}_e(\cdot,z,q) & \text{on } \Gamma \\ \nu\times\operatorname{curl}\mathbf{E} - \nu\times\operatorname{curl}\mathbf{E}_0 = \nu\times\operatorname{curl}\mathbf{E}_e(\cdot,z,q) & \text{on } \Gamma. \end{cases} \tag{3.35}$$

Let χ be a cutoff function such that $\chi = 1$ in a neighborhood of Γ and $\chi = 0$ in a neighborhood of the point $z \in D$. We define the function $\mathbf{\Theta}_z \in H^2(D)^3$ by $\mathbf{\Theta}_z = \chi\mathbf{E}_e(\cdot,z,q)$. Then $\mathbf{\Theta}_z$ satisfies the boundary conditions

$$\begin{cases} \nu\times\mathbf{\Theta}_z = \nu\times\mathbf{E}_e(\cdot,z,q) & \text{on } \Gamma \\ \nu\times\operatorname{curl}\mathbf{\Theta}_z = \nu\times\operatorname{curl}\mathbf{E}_e(\cdot,z,q) & \text{on } \Gamma. \end{cases}$$

Using another cutoff function, we can guarantee that $\mathbf{\Theta}_z = 0$ in $D_{\mathbf{\Theta}_z}$ such that $D_0 \subset D_{\mathbf{\Theta}} \subset D$. We recall that $\mathbf{E}$, $\mathbf{E}_0$ in $L^2(D)^3$ is a solution to the interior transmsision problem (3.35) if and only if $\mathbf{F} := \mathbf{E} - \mathbf{E}_0 \in V_0(D, D_0, k)$ satisfies

$$\int_{D\setminus\overline{D}_0} (N-I)^{-1}(\operatorname{curl}\operatorname{curl} - k^2N)(\mathbf{F}_z + \mathbf{\Theta}_z)\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx = 0 \tag{3.36}$$

for all $\mathbf{\Psi} \in V_0(D, D_0, k)$.

Let $\mathcal{F}^\delta$ be the perturbed linear operator corresponding to the noisy measurements $\mathbf{E}^\delta_\infty(\hat{x}, d, q)$. We assume that for all $g \in L^2_t(\Omega)$

$$\mathcal{F}^\delta g = -\mathcal{B}^\delta(\mathbf{E}_g), \text{ where } ||\mathcal{B}^\delta - \mathcal{B}|| \leq \delta$$

where $\delta > 0$ is a measure of the noise level and $\mathcal{B}^\delta$ denotes the noisy bounded operator associated with $\mathcal{B}$.

For each fixed z and q, the regularized solution $g_{z,q,\delta}$ is defined by minimizing the Tikhonov functional

$$||\mathcal{F}^\delta g_{z,q,\delta} - \mathbf{E}_{e,\infty}(.,z,q)||^2 + \varepsilon||g_{z,q,\delta}||^2 \tag{3.37}$$

where $\varepsilon := \varepsilon(\delta) \to 0$ as $\delta \to 0$ is the regularization parameter. We assume that $\varepsilon(\delta)$ is such that

$$\lim_{\delta\to 0} ||\mathcal{F}^\delta g_{z,q,\delta} - \mathbf{E}_{e,\infty}(\cdot, z, q)|| = 0. \tag{3.38}$$

Theorem 3.3.1. *Assume that k is a transmission eigenvalue and that (3.38) is verified. We also assume that k^2 is not a Maxwell eigenvalue in D_0. Then for almost every $z \in D$, there exists $q \in \mathbb{R}^3$ such that $||\mathbf{E}_{g_{z,q,\delta}}||_{H(\mathrm{curl}, D)}$ cannot be bounded when $\delta \to 0$.*

Proof. The beginning of the proof is the same as in the case without cavity. We assume that for a set $\mathcal{A}$ of points $z \in D$ which has a positive measure, there exists a constant $M > 0$ such that for all $q \in \mathbb{R}^3$,

$$||\mathbf{E}_{g_{z,q,\delta}}||_{H(\mathrm{curl}, D)} \leq M.$$

This leads to the existence of $\mathbf{E}_z \in H_{inc}(D)$ such that $\mathcal{B}(\mathbf{E}_z) = \mathbf{E}_{e,\infty}(\cdot, z)$. We deduce that there exists $\mathbf{u}_z \in V_0(D, D_0, k)$ such that

$$\int_{D\setminus\overline{D}_0} (N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, - k^2 N)(\mathbf{u}_z + \mathbf{\Theta}_z) \cdot (\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx = 0 \tag{3.39}$$

for all $\mathbf{\Psi} \in V_0(D, D_0, k)$.

The previous equation (3.39) can also be written as $(I - k^2 T_k)\mathbf{u}_z = f_{\mathbf{\Theta}_z}$ where A_k is a compact self-adjoint operator. Since k is a transmission eigenvalue, the kernel of $I - k^2 T_k$ is non trivial. Using the Fredholm alternative and the fact that T_k is self-adjoint, we deduce that

$$((I - k^2 T_k)\mathbf{u}_z, \mathbf{u}_0) = 0$$

i.e.

$$\int_{D\setminus\overline{D}_0} (N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, - k^2)\mathbf{\Theta}_z \cdot (\mathrm{curl}\,\mathrm{curl}\, - k^2 N)\mathbf{u}_0 dx = 0$$

where $\mathbf{u}_0 \in V_0(D, D_0, k)$ is an eigenvector associated to the transmission eigenvalue k. Using Green's theorem and the definition of $\mathbf{\Theta}_z$, we get

$$\int_\Gamma \mathrm{curl}\,\big((N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, - k^2 N)\mathbf{u}_0\big) \cdot \nu \times \mathbf{\Theta}_z$$
$$- \int_\Gamma \nu \times \big((N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_0 - k^2 N\mathbf{u}_0)\big) \cdot \mathrm{curl}\,\mathbf{\Theta}_z = 0.$$

Set $\mathbf{F} := (N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, - k^2 N)\mathbf{u}_0$ in $D\setminus\overline{D}_0$. The main difference with the case without cavities appears here. $\mathbf{F}$ is only defined in $D\setminus\overline{D}_0$. However from Theorem 3.1.2, we can extend $\mathbf{F}$ in a function of $L^2(D)^3$ such that $\mathrm{curl}\,\mathrm{curl}\,\mathbf{F} - k^2\mathbf{F} = 0$ in D. Using the definition of $\mathbf{\Theta}_z$ we have the following equality

$$\int_\Gamma \mathrm{curl}\,\mathbf{F} \cdot \nu \times \mathbf{E}_e(\cdot, z, q) - \int_\Gamma \nu \times \mathbf{F} \cdot \mathrm{curl}\,\mathbf{E}_e(\cdot, z, q) = 0 \tag{3.40}$$

for all $z \in \mathcal{A}$. Since $\mathbf{F}$ solves Maxwell's equations in D, it can be Using the representation theorems for solutions to Maxwell's equations we have

$$\mathbf{F}(z) = -\mathrm{curl}_z \int_\Gamma \nu(x) \times \mathbf{F}(x)\Phi_k(x,z)ds(x) \\ - \frac{1}{k^2}\mathrm{curl}_z\,\mathrm{curl}_z \int_\Gamma \nu(x) \times \mathrm{curl}\,\mathbf{F}(x)\Phi_k(x,z)ds(x).$$

Since (3.40) is equivalent to $ikq \cdot \mathbf{F}(z) = 0$ for all $z \in \mathcal{A}$ (see Appendix A), then $\mathbf{F}(z) = 0$ for all $z \in \mathcal{A}$ and by the unique continuation principle for all z in D. We deduce that $\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_0 - k^2 N\mathbf{u}_0 = 0$ in $D\backslash\overline{D}_0$ and then $\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_0 - k^2 N\mathbf{u}_0 = 0$ in D since $\mathbf{u}_0 \in H(\mathrm{curl}\,, D)$ and $\mathrm{curl}\,\mathrm{curl}\,\mathbf{u}_0 - k^2\mathbf{u}_0 = 0$ in D_0. Using the boundary conditions $\nu \times \mathbf{u}_0 = \nu \times \mathrm{curl}\,\mathbf{u}_0 = 0$ on Γ and the representation theorems for solutions to Maxwell's equations , one concludes that $\mathbf{u}_0 = 0$ in D which contradicts the fact that $\mathbf{u}_0$ is an eigenvector. □

We consider a circle of radius 1 and index of refraction $n = 4$ containing a concentric cavity of radius 0.5. We have computed the regularized solution g_z of the far field equation for different source point z inside the circle. The solid line represents the sum of the norm of the regularized solutions g_z for the different points z.

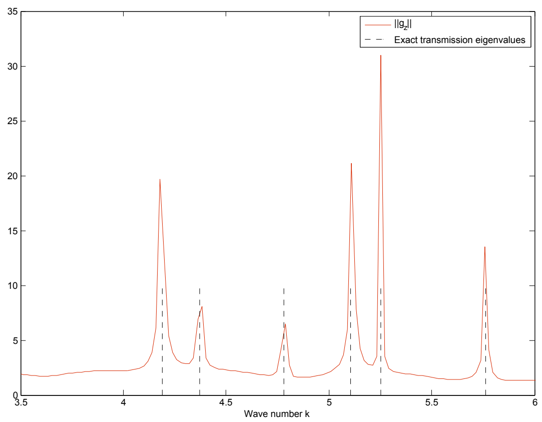


Figure 3.3: The sum of the norm of g_z for several source points z with the values of the exact transmission eigenvalues.

Chapter 4

Scalar interior transmission problem for dielectrics with perfectly conducting inclusions

This chapter is devoted to the study of the interior transmission problem corresponding to the scattering of an inhomogeneous (possibly anisotropic) medium D of $\mathbb{R}^d$ ($d = 2$ or $d = 3$) containing a perfect conductor D_0 assuming that the contrast in the medium is given by two different functions. From practical point of view, the importance of this problem lies in the possibility of using transmission eigenvalues to detect anomalies inside inhomogeneous media in non-destructive testing. This type of problem is considered in [41] where the authors recover the obstacle embedded in an inhomogeneous medium.

Since this is a new type of interior transmission problem that has not been studied yet in any articles, we only focus on the scalar case. The corresponding scattering problem is, find $u \in H^1_{loc}(\mathbb{R}^d)$ such that

$$\begin{cases} \nabla \cdot A\nabla u + k^2 n u = 0 & \text{in } \mathbb{R}^d \backslash \overline{D}_0 \\ u = u^i + u^s & \text{in } \mathbb{R}^d \backslash \overline{D_0} \\ u = 0 & \text{on } \partial D_0 \end{cases}$$

where the incident wave u^i is an entire solution to Helmholtz equation and the scattered field u^s satisfies the Sommerfeld condition. We assume that A and n are defined in $\mathbb{R}^d \backslash \overline{D}_0$ and that the support of $A - I$ and $n - 1$ is a connected region $D \backslash \overline{D}_0$ with $D_0 \subset D$. Note that across the boundary ∂D, u and $\nu \cdot A\nabla u$ are continuous. The corresponding interior transmission eigenvalue problem is defined by

$$\begin{cases} \nabla \cdot A\nabla w + k^2 n w = 0 & \text{in } D \backslash \overline{D}_0 \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \partial D \\ \nu \cdot A\nabla w = \nu \cdot \nabla v & \text{on } \partial D \\ w = 0 & \text{on } \partial D_0. \end{cases}$$

In this chapter, we focus our attention in the study of the existence and discreteness of the set of transmission eigenvalues, which again are the values of $k \in \mathbb{C}$ for which the interior transmission problem has a non trivial solution. We divide the study into two parts: first we study the isotropic case i.e. $A = I$ and then we consider the anisotropic case ($A \neq I$). For each case, we are confronted with several difficulties due to the fact that the field w is only defined in $D\backslash\overline{D}_0$.

In the isotropic case, the first difficulty is to define the space in which the problem is well-posed. For the isotropic medium, solutions w and v are usually defined in $L^2(D)$ such that the difference $u := w - v$ is in $H^2(D)$. However, the difference u here can only be defined in $D\backslash\overline{D}_0$ and since we have no information on its normal derivative on ∂D_0, u cannot belong to $H^2(D\backslash\overline{D}_0)$. Thus we are forced to define u in a weaker space. Having found the right space, we reformulate the problem as a fourth order equation for u in $D\backslash\overline{D}_0$, paired with the Helmholtz equation satisfied by v inside D_0. In order to obtain the existence and discreteness of the set of eigenvalues, we split the variational operator into a coercive operator and a compact one. However, on the contrary to the previous studied cases, the weak space in which u is defined only ensures the compactness of the lowest order terms in the formulation. Thus, we will see that the only case we can treat is when n is less than one.

For the anisotropic case, the difficulty does not lie in the definition of solutions spaces but in the reformulation of the interior transmission problem as a Fredholm problem. Our first approach following [11, 18] could have been formulating the problem in terms of the new variables

$$\mathbf{w} := A\nabla w \quad \text{and} \quad \mathbf{v} = \nabla v$$

which leads to a fourth order formulation similar to the previous case of isotropic media which unfortunately provides results under restrictive hypothesis on the contrasts. Instead here, we adapt an approach developed in [38] and [19] that can treat both existence and discreteness. However, for the discreteness we choose to expose an alternative method inspired from the study of metamaterials that uses the T-coercivity. For the case $A - I$ positive definite, we show that there exists an infinite discrete set of transmission eigenvalues when n is less than one but we only show the existence of a finite number of transmission eigenvalues when n is greater than one. For the case $I - A$ positive definite, we only show that the set of transmission eigenvalues is discrete (if they exist) for n less than one.

4.1 The scalar isotropic case

We start our discussion by considering the case of the interior transmission problem for an isotropic inhomogeneous medium with a Dirichlet obstacle inside. Let $D \subset \mathbb{R}^d$, $d = 2, 3$ be a simply connected and bounded region with piece-wise smooth boundary $\Gamma := \partial D$. Inside D, we consider a region $D_0 \subset D$ possibly be multiply connected with piece-wise smooth boundary $\Sigma := \partial D_0$ such that $\mathbb{R}^d\backslash\overline{D}_0$ is connected. We assume that D_0 is an impenetrable obstacle satisfying the Dirichlet boundary condition, whereas $D\backslash\overline{D}_0$ is an inhomogenous medium with index of refraction n where $n \in L^\infty(D\backslash\overline{D}_0)$ is such that $n \geq c > 0$. Let ν denote the unit outward normal to Γ and Σ.

The interior transmission problem corresponding to the scattering problem for the scatterer D reads

$$\begin{cases} \Delta w + k^2 n w = 0 & \text{in } D\backslash\overline{D}_0 \\ \Delta v + k^2 v = 0 & \text{in } D \\ w - v = g & \text{on } \Gamma \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = h & \text{on } \Gamma \\ w = 0 & \text{on } \Sigma. \end{cases} \tag{ITPH}$$

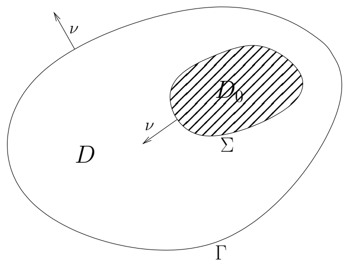


Figure 4.1: Geometry and notations.

Due to the fact that the function w is only defined in $D\backslash\overline{D}_0$, the first difficulty that we meet is to correctly define a solution to this problem in appropriate function spaces. Indeed, the difference u between w and v can only be considered in the set $D\backslash\overline{D}_0$ and we do not have enough information about u and in particular about its normal derivative $\dfrac{\partial u}{\partial \nu}$ on the boundary Σ to conclude the H^2-regularity for u. In particular, u is not necessarily in $H^2(D\backslash\overline{D}_0)$ and the only thing we can say is that $\Delta u \in L^2(D\backslash\overline{D}_0)$. Thus we introduce the Hilbert space

$$H^1_\Delta(D\backslash\overline{D}_0) := \left\{ u \in H^1(D\backslash\overline{D}_0) \text{ such that } \Delta u \in L^2(D\backslash\overline{D}_0) \right\}$$

and we define a weak solution to (ITP4.1) as follows:

Definition 4.1.1. *For given $g \in H^{3/2}(\Gamma)$ and $h \in H^{1/2}(\Gamma)$, a weak solution to* (ITPH) *is a pair of functions $w \in L^2(D\backslash\overline{D}_0)$ and $v \in L^2(D)$ satisfying the first two equations of* (ITPH) *in the distributional sense such that $w = 0$ on Σ and $u = w - v \in H^1_\Delta(D\backslash\overline{D}_0)$ satisfies the boundary conditions on Γ, $u = g$ and $\dfrac{\partial u}{\partial \nu} = h$.*

4.1.1 Variational formulation

In order to analyze (ITPH) we first write the problem as a forth order partial differential equation. To this end, let us assume that $1/|n-1| \in L^\infty(D\backslash\overline{D}_0)$ and let w and v be a weak solution to (ITPH). Then $u := w - v$ satisfies

$$\Delta u + k^2 n u = -k^2(n-1)v \quad \text{in } D\backslash\overline{D}_0. \tag{4.1}$$

Dividing both sides of (4.1) by $(n-1)$ and applying the operator $(\Delta + k^2)$ we get a fourth order equation for u in $D\backslash\overline{D}_0$

$$(\Delta + k^2)\frac{1}{n-1}(\Delta + k^2 n)u = 0 \quad \text{in } D\backslash\overline{D}_0 \tag{4.2}$$

together with the boundary conditions on Γ

$$u = g \quad ; \quad \frac{\partial u}{\partial \nu} = h \quad \text{ on } \Gamma \tag{4.3}$$

and on Σ, we have that

$$u = -v \quad \text{ on } \Sigma. \tag{4.4}$$

Furthermore v satisfies Helmholtz equation in D_0

$$\Delta v + k^2 v = 0 \text{ in } D_0 \tag{4.5}$$

with continuity of the Cauchy data across Σ that can be written using (4.1) as

$$\left(\frac{1}{k^2(n-1)}(\Delta + k^2 n)u\right)^+ = v^- \text{ and } \frac{\partial}{\partial \nu}\left(\frac{1}{k^2(n-1)}(\Delta + k^2 n)u\right)^+ = \frac{\partial v^-}{\partial \nu}. \tag{4.6}$$

Conversely, it is easily verified that a solution $u \in H^1_\Delta(D \backslash \overline{D}_0)$ and $v \in L^2(D_0)$ of (4.2)-(4.6) defines a weak solution w and v to (ITPH) by

$$v := \frac{-1}{k^2(n-1)}(\Delta + k^2 n)u \text{ in } D\backslash\overline{D}_0 \text{ and } w := u + v \text{ in } D\backslash\overline{D}_0.$$

Thus (4.2)-(4.6) and the interior transmission problem are equivalent. Now, we are ready to write the interior the interior transmission problem in a variational formulation. Indeed for a solution (v, w) of (ITPH) we define u in D by $u = w - v$ in $D\backslash\overline{D}_0$ and $u = -v$ in D_0. Then clearly u is in $H^1(D) \cap H^1_\Delta(D\backslash\overline{D}_0)$, satisfies (4.2)-(4.3),

$$u^+ = u^- \text{ on } \Sigma,$$

$$\left(\frac{-1}{k^2(n-1)}(\Delta u + k^2 n u)\right)^+ = -u^- \text{ and } \frac{\partial}{\partial \nu}\left(\frac{-1}{k^2(n-1)}(\Delta u + k^2 n u)\right)^+ = -\frac{\partial u^-}{\partial \nu} \text{ on } \Sigma$$

and

$$\Delta u + k^2 u = 0 \text{ in } D_0.$$

Taking a test function φ such that $\varphi = 0$ and $\dfrac{\partial \varphi}{\partial \nu} = 0$ on Γ, multiplying (4.2) by φ and integrating by parts and using the boundary conditions, we obtain

$$
\begin{aligned}
0 &= \int_{D\backslash\overline{D}_0} (\Delta + k^2)\frac{1}{n-1}(\Delta u + k^2 n u)\overline{\varphi} dx \\
&= \int_{D\backslash\overline{D}_0} (\Delta + k^2)\frac{1}{n-1}(\Delta u + k^2 u)\overline{\varphi} dx + k^2 \int_{D\backslash\overline{D}_0} (\Delta u + k^2 u)\overline{\varphi} dx \\
&= \int_{D\backslash\overline{D}_0} \frac{1}{n-1}(\Delta u + k^2 u)(\Delta\overline{\varphi} + k^2\overline{\varphi}) dx + k^2 \int_{D\backslash\overline{D}_0} (\Delta u + k^2 u)\overline{\varphi} dx \\
&\quad + \int_{\Sigma} \left(\frac{1}{n-1}(\Delta u + k^2 u)\right)^{+} \frac{\partial \varphi^+}{\partial \nu} ds - \int_{\Sigma} \frac{\partial}{\partial \nu}\left(\frac{1}{n-1}(\Delta u + k^2 u)\right)^{+} \overline{\varphi}^+ ds \\
&= \int_{D\backslash\overline{D}_0} \frac{1}{n-1}(\Delta u + k^2 u)(\Delta\overline{\varphi} + k^2\overline{\varphi}) dx + k^2 \int_{D\backslash\overline{D}_0} (\Delta u + k^2 u)\overline{\varphi} dx \\
&\quad + k^2 \int_{\Sigma} \frac{\partial u^+}{\partial \nu} \overline{\varphi}^+ ds - k^2 \int_{\Sigma} \frac{\partial u^-}{\partial \nu} \overline{\varphi}^- ds \\
&= \int_{D\backslash\overline{D}_0} \frac{1}{n-1}(\Delta u + k^2 u)(\Delta\overline{\varphi} + k^2\overline{\varphi}) dx + k^4 \int_{D\backslash\overline{D}_0} u\overline{\varphi} dx - k^2 \int_{D\backslash\overline{D}_0} \nabla u \cdot \nabla\overline{\varphi} dx \\
&\quad + k^4 \int_{D_0} u\overline{\varphi} dx - k^2 \int_{D_0} \nabla u \cdot \nabla\overline{\varphi} dx \\
&= \int_{D\backslash\overline{D}_0} \frac{1}{n-1}(\Delta u + k^2 u)(\Delta\overline{\varphi} + k^2\overline{\varphi}) dx + k^4 \int_{D} u\overline{\varphi} dx - k^2 \int_{D} \nabla u \cdot \nabla\overline{\varphi} dx.
\end{aligned}
$$

Now, let θ be a lifting function in $H^2(D)$ such that $\theta = g$ and $\frac{\partial \theta}{\partial \nu} = h$ on Γ. Then $u_0 := u - \theta \in H_0^1(D) \cap H_\Delta^1(D\backslash\overline{D}_0)$ and the natural variational space for the above variational problem is the Hilbert space given by

$$
W := \left\{ u \in H_0^1(D) \cap H_\Delta^1(D\backslash\overline{D}_0) \, \text{such that} \, \frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma \right\}
$$

equipped with the norm

$$
||u||_W^2 = ||u||_{H^1(D)}^2 + ||\Delta u||_{L(D\backslash\overline{D}_0)}^2.
$$

Therefore, the variational formulation of the interior transmission problem becomes: find $u_0 \in W$ such that

$$
\begin{aligned}
&\int_{D\backslash\overline{D}_0} \frac{1}{n-1}(\Delta u_0 + k^2 u_0)(\Delta\overline{\varphi} + k^2\overline{\varphi}) dx + k^4 \int_{D} u_0\overline{\varphi} dx - k^2 \int_{D} \nabla u_0 \cdot \nabla\overline{\varphi} dx \\
&\quad = -\int_{D\backslash\overline{D}_0} \frac{1}{n-1}(\Delta \theta + k^2 \theta)(\Delta\overline{\varphi} + k^2\overline{\varphi}) dx - k^4 \int_{D} \theta\overline{\varphi} dx + k^2 \int_{D} \nabla \theta \cdot \nabla\overline{\varphi} dx \qquad (4.7)
\end{aligned}
$$

for all $\varphi \in W$. By taking appropriate test functions it is easy to see that a solution of the variational problem (4.7) defines a week solution to (4.2)-(4.6) and therefore to the interior transmission problem.

Remark 4.1.1. *One can remark that on the contrary to the previously studied cases [17], since u is less regular, only the zero order term on the left hand side of* (4.7) *defines a compact operator whereas the last term does not. Furthermore for n greater than one, the operator defined by the following bilinear from*

$$\tilde{\mathcal{A}}_k(u,\varphi) := \int_{D\setminus\overline{D}_0} \frac{1}{n-1}(\Delta u + k^2 u)(\Delta\overline{\varphi} + k^2\overline{\varphi})dx - k^2\int_D \nabla u\cdot\nabla\overline{\varphi}dx$$

has no chance to be coercive because of the negative sign in front of the last term of the operator. For this reason, using this variational formulation, we are only able to treat the problem for n less than one, since in this case we can show that $-\tilde{\mathcal{A}}_k$ is indeed coercive.

Next, we denote by $n_* = \inf_{D\setminus\overline{D}_0} n(x)$ and $n^* = \sup_{D\setminus\overline{D}_0} n(x)$ and from now on we assume that $n_* < n(x) < n^* < 1$.

Let us define the following sesquilinear forms

$$\mathcal{A}_k(u,\varphi) := \int_{D\setminus\overline{D}_0} \frac{1}{1-n}(\Delta u + k^2 u)(\Delta\overline{\varphi} + k^2\overline{\varphi})dx + k^4\int_D u\overline{\varphi}dx + k^2\int_D \nabla u\cdot\nabla\overline{\varphi}dx$$

and

$$\mathcal{B}(u,\varphi) := 2\int_D u\overline{\varphi}dx$$

and the bounded linear functional

$$\ell(\varphi) := -\int_{D\setminus\overline{D}_0} \frac{1}{n-1}(\Delta\theta + k^2\theta)(\Delta\overline{\varphi} + k^2\overline{\varphi})dx - k^4\int_D \theta\overline{\varphi}dx + k^2\int_D \nabla\theta\cdot\nabla\overline{\varphi}dx$$

Then the interior transmission problem in the variational form now consists of finding $u_0 \in W$ such that

$$\mathcal{A}_k(u_0,\varphi) - k^4\mathcal{B}(u_0,\varphi) = \ell(\varphi) \quad \text{for all } \varphi \in W.$$

Using the Riesz representation theorem we define two bounded linear operators $A_k : W \to W$ and $B : W \to W$ by

$$(A_k u,\varphi)_W := \mathcal{A}_k(u,\varphi) \text{ and } (Bu,\varphi)_W := \mathcal{B}(u,\varphi).$$

Theorem 4.1.1. *Assume that $n_* < n(x) < n^* < 1$. Then*

(i) *The operator $B : W \to W$ is compact.*

(ii) *The operator $A_k : W \to W$ is coercive.*

Proof. (i) The compactly embedding of $H^1(D)$ into $L^2(D)$ implies that B is compact operator on W.

(ii) Now we show that A_k is coercive. Setting $\gamma = \dfrac{1}{1-n_*}$ and using the equality

$$\gamma X^2 - 2\gamma XY + (1+\gamma)Y^2 = \varepsilon\left(Y - \frac{\gamma}{\varepsilon}X\right)^2 + \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)X^2 + (1+\gamma-\varepsilon)Y^2, \quad (4.8)$$

for $X = ||\Delta u||^2_{D\setminus\overline{D}_0}$ and $Y = k^2||u||_{D\setminus\overline{D}_0}$, where for a generic region $\mathcal{O} \in \mathbb{R}^d$, $\|\cdot\|_{\mathcal{O}}$ denotes the $L^2(\mathcal{O})$, we have

$$\begin{aligned}
(A_k u, u)_W &= \int_{D\setminus\overline{D}_0} \frac{1}{1-n}|\Delta u + k^2 u|^2 dx + k^4||u||^2_{D\setminus\overline{D}_0} + k^2||\nabla u||^2_D + k^4||u||^2_{D_0} \\
&\geq \gamma||\Delta u||^2_{D\setminus\overline{D}_0} - 2k^2\gamma||\Delta u||_{D\setminus\overline{D}_0}||u||_{D\setminus\overline{D}_0} + k^4\left(1+\gamma\right)||u||^2_{D\setminus\overline{D}_0} \\
&\quad + k^2||\nabla u||^2_D + k^4||u||^2_{D_0} \\
&\geq \left(\gamma - \frac{\gamma^2}{\varepsilon}\right)||\Delta u||^2_{D\setminus\overline{D}_0} + k^4(\gamma+1-\varepsilon)||u||^2_{D\setminus\overline{D}_0} + k^2||\nabla u||^2_D + k^4||u||^2_{D_0}
\end{aligned}$$

where $\gamma < \varepsilon < \gamma + 1$. For such an ε, we conclude that there exists a constant $C > 0$ such that

$$(A_k u, u) \geq C||u||^2_W$$

for all $u \in W$ which proves that $A_k : W \to W$ is coercive.

□

The above theorem shows that the operator $A_k - k^4 B$ is Fredholm with index zero, whence a solution exists if the uniqueness holds. In the following with be concerned with the injectivity $A_k - k^4 B$ which leads to the study of the transmission eigenvalues which are in fact the of main interest in this paper.

4.1.2 Transmission eigenvalues

The interior transmission eigenvalue problem in the considered case is

$$\begin{cases} \Delta w + k^2 n w = 0 & \text{in } D\setminus\overline{D}_0 \\ \Delta v + k^2 v = 0 & \text{in } D \\ w - v = 0 & \text{on } \Gamma \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma \\ w = 0 & \text{on } \Sigma. \end{cases} \tag{TEP}$$

As already known from the literature [9], [43], [28] this eigenvalue problem is non self-adjoint and therefore it may have complex transmission eigenvalues. However for this study we are limited to the case of real eigenvalues corresponding to (TEP).

Definition 4.1.2. *The values of $k > 0$ for which* (TEP) *has a nontrivial solution are called the* transmission eigenvalues*.*

In term of the operators defined above $k > 0$ is a transmission eigenvalue if the kernel of the operator $A_k - k^4 B$ is nontrivial. In the following we are concerned with the existence and discreteness of transmission eigenvalues.

Theorem 4.1.2. *Assume that $n_* < n(x) < n^* < 1$. Then the set of transmission eigenvalues is discrete and $+\infty$ is the only possible accumulation point.*

Proof. To prove the discreteness of transmission eigenvalues we use the analytic Fredholm theory [26]. We have seen earlier that thanks to the coercivity of $\mathcal{A}_k(\cdot,\cdot)$, A_k^{-1} exists as a bounded operator on W. Thus, the transmission eigenvalues are the values of $k>0$ for which $I-k^4A_k^{-1}B$ has a nontrivial kernel. Furthermore, the operator A_k is obviously analytic with respect to $k\in\mathbb{C}$ and hence the mapping $k\mapsto A_k^{-1}$ is analytic in a neighbourhood of the real axis. To apply the analytic Fredholm theorem, it remains to show that $I-k^4A_k^{-1}B$ or A_k-k^4B is injective for at least one k. To this end, we recall the Poincaré inequality which is valid for all $u\in H_0^1(D)$

$$||u||_D^2 \le \frac{1}{\lambda_0(D)}||\nabla u||_D^2$$

where $\lambda_0(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D. Then, for all $u\in W$ we have that

$$\begin{aligned}\mathcal{A}_k(u,u)-k^4\mathcal{B}(u,u) &= \int_{D\setminus\overline{D}_0}\frac{1}{1-n}|\Delta u+k^2u|^2dx-k^4||u||_D^2+k^2||\nabla u||_D^2\\ &\ge k^2\left(||\nabla u||_D^2-k^2||u||_D^2\right)\\ &\ge k^2||\nabla u||_D^2\left(1-\frac{k^2}{\lambda_0(D)}\right).\end{aligned}$$

We deduce that $\mathcal{A}_k(u,u)-k^4\mathcal{B}(u,u)>0$ for all $k>0$ such that $k^2<\lambda_0(D)$ and hence A_k-k^4B is injective for such k. Hence, the analytical Fredholm theory implies that the set of transmission eigenvalues is discrete and from the analyticity with $+\infty$ and the only possible accumulation point. □

Remark 4.1.2. *From the previous theorem, we deduce a lower bound for the first transmission eigenvalue. Indeed, if $k>0$ is a transmission eigenvalue, then*

$$k\ge\lambda_0(D).$$

Next we want to prove the existence of transmission eigenvalues following [17]. If we consider the generalized eigenvalue problem

$$A_k-\lambda(k)Bu=0 \qquad u\in W$$

which is known to have an infinite sequence of eigenvalues $\lambda_j(k)$, $j\in\mathbb{N}$, then the transmission eigenvalues are the solutions $\lambda_j(k)=k^4$. The proof of the existence of transmission eigenvalues makes use of the following theorem shown in [18] and given in Chapter 2.

Theorem 4.1.3. *Let $k\longmapsto A_k$ be a continuous mapping from $]0,\infty[$ to the set of self-adjoint and positive definite bounded linear operators on W and let B be a self-adjoint and non negative compact bounded linear operator on W. We assume that there exists two positive constant $k_0>0$ and $k_1>0$ such that*

1. *$A_{k_0}-k_0^4B$ is positive on W,*
2. *$A_{k_1}-k_1^4B$ is non positive on a m dimensional subspace of W.*

Then each of the equations $\lambda_j(k) = k^4$ *for* $j = 1, \dots, m$, *has at least one solution in* $[k_0, k_1]$ *where* $\lambda_j(k)$ *is the* j^{th} *eigenvalue (counting multiplicity) of* A_k *with respect to* B, *i.e.* $\ker(A_k - \lambda_j(k)B) \neq \{0\}$.

Theorem 4.1.4. *Assume that* $n_* < n(x) < n^* < 1$. *There exists an infinite discrete set of transmission eigenvalues.*

Proof. We have already seen that for $k_0 < \lambda_0(D)$, then $A_{k_0} - k_0^4 B$ is positive in W. Now let us find k_1 such that $A_{k_1} - k_1^4 B$ is non positive in a subspace of W. Let B_r^j, $j = 1...M(r)$, be $M(r)$ balls of radius r included in $D \backslash \overline{D}_0$.

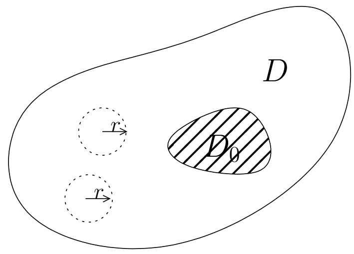


Figure 4.2: Balls of radius r included in $D \backslash \overline{D}_0$.

We denote by k_1 the first transmission eigenvalue corresponding to the interior transmission problem for B_r^j for all $j = 1..M(r)$ with index of refraction n^* which is know to exist [26], and let $u_j \in H_0^2(B_r^j)$, $1 \leq j \leq M(r)$, be the corresponding eigenvector which satisfy

$$\int_{B_r^j} \frac{1}{1-n^*}(\Delta u_j + k_1^2 n^* u_j)(\Delta\overline{\varphi} + k_1^2 \overline{\varphi})dx = 0$$

for all $\varphi \in H_0^2(B_r^j)$. We denote by $\tilde{u}_j \in H_0^2(D)$ the extension of u_j by zero to the whole of D and we define a $M(r)$-dimensional subspace of W by $\mathcal{V} := \text{Vect}\ \{\tilde{u}_j, 1 \leq j \leq M(r)\}$. Since for $j \neq m$, $\tilde{u}_j$ and $\tilde{u}_m$ have disjoint support, for $u = \sum_{j=1}^{M(r)} \alpha_j \tilde{u}_j \in \mathcal{V}$, we have

$$\begin{aligned}
&\mathcal{A}_{k_1}(u,u) - k_1^4 \mathcal{B}(u,u) \\
&\quad = \sum_{j=1}^{M(r)} |\alpha_j|^2 \left(\int_{D\backslash\overline{D}_0} \frac{1}{1-n} |\Delta\tilde{u}_j + k_1^2 \tilde{u}_j|^2 dx - k_1^4 \int_D |\tilde{u}_j|^2 dx + k_1^2 \int_D |\nabla\tilde{u}_j|^2 dx \right) \\
&\quad = \sum_{j=1}^{M(r)} |\alpha_j|^2 \left(\int_{B_r^j} \frac{1}{1-n} |(\Delta u_j + k_1^2 u_j)|^2 dx - k_1^4 \int_{B_r^j} |u_j|^2 dx + k_1^2 \int_{B_r^j} |\nabla u_j|^2 dx \right) \\
&\quad \leq \sum_{j=1}^{M(r)} |\alpha_j|^2 \left(\frac{1}{1-n^*} \int_{B_r^j} |(\Delta u_j + k_1^2 u_j)|^2 dx - k_1^4 \int_{B_r^j} |u|^2 dx + k_1^2 \int_{B_r^j} |\nabla u_j|^2 dx \right) \\
&\quad = \sum_{j=1}^{M(r)} |\alpha_j|^2 \left(\int_{B_r^j} \frac{1}{1-n^*} (\Delta u_j + k_1^2 n^* u_j)(\Delta\overline{u}_j + k_1^2 \overline{u}_j) dx \right) = 0.
\end{aligned}$$

Thus, we conclude that there exist $M(r)$ transmission eigenvalues in $]\lambda_0(D), k_1]$. Letting $r \to 0$, we have that $M(r) \to \infty$ and thus we can now deduce that there exists an infinite set of transmission eigenvalues.

$\square$

We close this section with a monotonicity result for the first transmission eigenvalue with respect to the size of D_0, which can be useful in non-destructive testing. We denote by $k_1(D_0, n)$ the first transmission eigenvalue corresponding to (ITPH) with a perfect conductor D_0 and index of refraction n inside $D \backslash \overline{D}_0$.

Theorem 4.1.5. *Let $D_0 \subset D_0'$ and $n < 1$. Then*

$$k_1(D_0', n) \leq k_1(D_0, n).$$

Proof. Let $\tilde{u} \in W$ be the eigenvector corresponding to $k_1(D_0, n)$. Then $\tilde{u}$ satisfies

$$\int_{D\backslash\overline{D}_0} \frac{1}{1-n} |\Delta \tilde{u} + k_1(D_0, n)^2 \tilde{u}|^2 dx - k_1(D_0, n)^4 \int_D |\tilde{u}|^2 dx + k_1(D_0, n)^2 \int_D |\nabla \tilde{u}|^2 dx = 0.$$

Since $D \backslash D_0' \subset D \backslash \overline{D}_0$, we have $\tilde{u} \in W(D_0) \subset W(D_0')$ and

$$\begin{aligned}
\mathcal{A}_{k_1(D_0,n)}(\tilde{u}, \tilde{u}) - k_1(D_0, n)^4 \mathcal{B}(\tilde{u}, \tilde{u}) = & \int_{D\backslash\overline{D}_0'} \frac{1}{1-n} |\Delta \tilde{u} + k_1(D_0, n)^2 \tilde{u}|^2 dx \\
& - k_1(D_0, n)^4 \int_D |\tilde{u}|^2 dx + k_1(D_0, n)^2 \int_D |\nabla \tilde{u}|^2 dx \\
\leq & \int_{D\backslash\overline{D}_0} \frac{1}{1-n} |\Delta \tilde{u} + k_1(D_0, n)^2 \tilde{u}|^2 dx - k_1(D_0, n)^4 \int_D |\tilde{u}|^2 dx \\
& + k_1(D_0, n)^2 \int_D |\nabla \tilde{u}|^2 dx = 0.
\end{aligned}$$

Hence $(A_{k_1(D_0,n)} - k_1(D_0, n)^4 B)\tilde{u} < 0$, where $A_{k_1(D_0,n)}$ and B are the operators corresponding to $D \backslash \overline{D}_0'$ and thus can deduce that $k_1(D_0', n) \leq k_1(D_0, n)$. $\square$

Remark 4.1.3. *The Fredholm property of the interior transmission problem and the discreteness of transmission eigenvalues can be proven also for complex valued index of refraction n such that $1 > \Re(n) \geq c > 0$ and $\Im(n) \geq 0$. It merely suffices to take the real part of $\mathcal{A}(\cdot, \cdot)$ when proving the coercivity property in part (ii) Theorem 4.1.1. However, it is easy to show by taking the $\Im(\mathcal{A}_k(u, u) - k^4 \mathcal{B}(u, u))$ that there are no transmission eigenvalues if $\Im(n) > 0$ almost everywhere in $D \backslash \overline{D}_0$.*

4.2 The anisotropic scalar case

In this section, we consider that the medium inside $D \backslash \overline{D}_0$ is anisotropic. In particular, let A be a $d \times d$, $d = 2, 3$ matrix-real valued function whose entries are in $L^\infty(D \backslash \overline{D}_0)$ such that A is symmetric and $(\overline{\xi} \cdot A(x)\xi) \geq c > 0$, $(\overline{\xi} \cdot A(x)\xi) \geq c' > 0$, for all $\xi \in \mathbb{C}^d$. Again, we take $n \in L^\infty(D \backslash \overline{D}_0)$ to be a real valued function such that $n \geq c > 0$. We focus here only in the study of interior transmission eigenvalue problem which in this case reads: find $v \in H^1(D)$ and $w \in H^1(D\backslash\overline{D}_0)$ such that

$$\begin{cases} \nabla \cdot A\nabla w + k^2 nw = 0 & \text{in } D\backslash\overline{D}_0 \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \Gamma \\ \nu \cdot A\nabla w = \nu \cdot \nabla v & \text{on } \Gamma \\ w = 0 & \text{on } \Sigma. \end{cases} \tag{TEPA}$$

As it will become clear later on, if one is interested in the resolvability of the interior transmission problem with nonzero boundary data, our analysis proves the Fredholm structure of the problem. Again we focus on real values of k and define transmission eigenvalues as follows:

Definition 4.2.1. *The values of $k > 0$ for which* (TEPA) *has a nontrivial solution are called* transmission eigenvalues.

Due to the nature of the problem we employ different techniques for proving the discreteness and the existence of transmission eigenvalues. We start with the discreteness question.

In the following, we denote by

$$\gamma^* := \sup_{D\backslash\overline{D}_0} \sup_{||\xi||=1} (\overline{\xi} \cdot A(x)\xi) \quad \text{and} \quad \gamma_* := \inf_{D\backslash\overline{D}_0} \inf_{||\xi||=1} (\overline{\xi} \cdot A(x)\xi).$$

4.2.1 The discreteness of transmission eigenvalues

To find a variational formulation for the system (TEPA), we multiply the first and second equations by w' and v' respectively, where v' and w' are two test functions such that $w' = 0$ on Σ and integrate by parts to obtain

$$\int_{D\backslash\overline{D}_0} A\nabla w \cdot \nabla\overline{w}' dx - k^2 \int_{D\backslash\overline{D}_0} nw\overline{w}' dx - \int_\Gamma \overline{w}' \frac{\partial w}{\partial \nu_A} ds = 0 \tag{4.9}$$

and

$$-\int_D \nabla v \cdot \nabla\overline{v}' dx + k^2 \int_D v\overline{v}' dx + \int_\Gamma \overline{v}' \frac{\partial v}{\partial \nu} ds = 0. \tag{4.10}$$

Adding both (4.9) and (4.10) and using the boundary conditions, we have that

$$\int_{D\backslash\overline{D}_0} A\nabla w \cdot \nabla\overline{w}' dx - \int_D \nabla v \cdot \nabla\overline{v}' dx + k^2 \int_D v\overline{v}' dx - k^2 \int_{D\backslash\overline{D}_0} nw\overline{w}' dx = 0$$

Setting

$$\mathbb{H} := \left\{(v,w) \in H^1(D) \times H^1(D\backslash\overline{D}_0)/w = 0 \text{ on } \Sigma, \text{ such that } v = w \text{ on } \Gamma\right\},$$

the variational formulation of (TEPA) becomes: find (v,w) in $\mathbb{H}$ such that for all (v',w') in $\mathbb{H}$,

$$a_k((v,w),(v',w')) = 0 \tag{4.11}$$

where

$$a_k((v,w),(v',w')) = \int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}'dx - \int_D \nabla v\cdot\nabla\overline{v}'dx$$
$$+ k^2\int_D v\overline{v}'dx - k^2\int_{D\setminus\overline{D}_0} nw\overline{w}'dx.$$

One can easily verify that finding a solution to (4.11) is equivalent to finding a solution to (TEPA).

Obviously, due to the negative sign in front of the term $\int_D \nabla v\cdot\nabla\overline{v}'dx$, it is not possible to show directly that the variational formulation leads to a Fredholm type. To get around this difficulty, we use the concept of T-coercivity which has been initially used for the study of metamaterials in [6] and [5]. To this end let us recall the T-coercivity concept.

Definition 4.2.2. *Let T be a bijective bounded linear operator on a Hilbert space V. A bilinear form $b(\cdot,\cdot)$ is T-coercive on $V\times V$ if*

$$\exists\gamma>0,\ \forall v\in V,\ |b(v,Tv)|\geq\gamma||v||_V^2.$$

The proof of the following theorem can be found in [6].

Theorem 4.2.1. *Let $\ell(\cdot)$ be a continuous linear form on V and let $a(\cdot,\cdot)$ be a continuous bilinear form on $V\times V$. Assume that a can be splitted as $a(\cdot,\cdot)=b(\cdot,\cdot)+c(\cdot,\cdot)$ where the bilinear forms $b(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are both continuous and linear on $V\times V$, and that the bounded linear operator $C\in\mathcal{L}(V)$ associated with $c(\cdot,\cdot)$ is compact. Assume moreover that there exists a bijective bounded linear $T\in\mathcal{L}(V)$ such that $b(\cdot,\cdot)$ is T-coercive on $V\times V$. Then the variational problem of finding $u\in V$ such that*

$$\forall v\in V, a(u,v)=\ell(v) \tag{4.12}$$

has a solution if and only if the uniqueness holds (i.e. the only solution of (4.12) *with $\ell=0$ is $u=0$).*

The case of $(A-I)$ positive

In this section, we assume that $1<\gamma_*<\gamma^*$. Our goal is now to apply Theorem 4.2.1 to (4.11), and the key is to be able to construct an appropriate bijection $T\in\mathcal{L}(\mathbb{H})$. An obvious first idea would be to consider the linear operator of the form $T(v,w):=(-v,w)$ in order to to change the sign of $\int_D \nabla v\cdot\nabla\overline{v}'dx$ in the variational formulation (4.11). Unfortunately, $(-v,w)$ is not in $\mathbb{H}$ since $-v\neq w$ on Γ. Thus, we need to modify this operator so that it satisfies all the properties of $\mathbb{H}$. To this end, we introduce the step function χ such that $\chi=1$ in $D\setminus\overline{D}_0$ and $\chi=0$ in D_0. We now define the bijective bounded linear operator $T:\mathbb{H}\to\mathbb{H}$ ($T^2=I$) by

$$\begin{array}{cccc} T: & \mathbb{H} & \to & \mathbb{H} \\ & (v,w) & \mapsto & (-v+2\chi w,w). \end{array}$$

Since $w = 0$ on Σ, the function $-v + 2\chi w$ is continuous across Σ which implies that the function $-v + 2\chi w$ is in $H^1(D)$ and consequently the operator T is well defined on $\mathbb{H}$. Now, with the help of T we can define a new bilinear form

$$\begin{aligned}\tilde{a}_k((v,w),(v',w')) &= a_k((v,w),T(v',\overline{w}')) \\ &= \int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}'dx + \int_D \nabla v\cdot\nabla\overline{v}'dx - k^2\int_D v\overline{v}'dx \\ &\quad - k^2\int_{D\setminus\overline{D}_0} nw\overline{w}'dx - 2\int_D \nabla v\cdot\nabla(\chi\overline{w}')dx + 2k^2\int_D v\chi\overline{w'}dx\end{aligned}$$

and we show in the following that it satisfies the Fredholm property.

Lemma 4.2.2. *The bilinear form* $\tilde{a}_k(\cdot,\cdot) : \mathbb{H}\times\mathbb{H}\to\mathbb{C}$ *satisfies the Fredholm property.*

Proof. We can write $\tilde{a}_k((v,w),(v',w')) = b((v,w),(v',w')) + c_k((v,w),(v',w'))$ where

$$\begin{aligned}b((v,w),(v',w')) = \int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}'dx + \int_D \nabla v\cdot\nabla\overline{v}'dx \\ - 2\int_{D\setminus\overline{D}_0}\nabla v\cdot\nabla\overline{w}'dx + \int_D v\overline{v}'dx + \int_{D\setminus\overline{D}_0} w\overline{w}'dx\end{aligned}$$

and

$$c_k((v,w),(v',w')) = -(k^2+1)\int_D v\overline{v}'dx - \int_{D\setminus\overline{D}_0}(k^2n+1)w\overline{w}'dx + 2k^2\int_{D\setminus\overline{D}_0} v\overline{w}'dx.$$

From Riesz's representation theorem, we define the bounded linear operator C_k from $\mathbb{H}$ into $\mathbb{H}$ by

$$c_k((v,w),(v',w')) = (C_k(v,w),(v',w')).$$

The compact embedding of $H^1(D)$ into $L^2(D)$ implies that C_k is a compact operator for all $k > 0$. We now show that $b(\cdot,\cdot)$ is coercive.

$$\begin{aligned}b((v,w),(v,w)) &= \int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}dx + \int_D |\nabla v|^2dx + ||v||_D^2 + ||w||_{D\setminus\overline{D}_0}^2 \\ &\quad - 2\int_{D\setminus\overline{D}_0}\nabla v\cdot\nabla\overline{w}dx \\ &\geq \gamma_*||\nabla w||_{D\setminus\overline{D}_0}^2 + ||\nabla v||_D^2 + ||v||_D^2 + ||w||_{D\setminus\overline{D}_0}^2 - 2\int_{D\setminus\overline{D}_0}\nabla v\cdot\nabla\overline{w}dx.\end{aligned}$$

Using the following inequality

$$\begin{aligned}\left|-2\int_{D\setminus\overline{D}_0}\nabla v\cdot\nabla\overline{w}dx\right| &\leq \int_{D\setminus\overline{D}_0}|\nabla v\cdot\nabla\overline{w}|dx \\ &\leq \frac{1}{\eta}||\nabla v||_{D\setminus\overline{D}_0}^2 + \eta||\nabla w||_{D\setminus\overline{D}_0}^2\end{aligned}$$

with $\eta > 0$, we then obtain

$$\begin{aligned} b((v,w),(v,w)) &\geq (\gamma_* - \eta)||\nabla w||^2_{D\backslash\overline{D}_0} + \left(1 - \frac{1}{\eta}\right)||\nabla v||^2_D + ||v||^2_D + ||w||^2_{D\backslash\overline{D}_0} \\ &\geq C\left(||v||^2_{H^1(D)} + ||w||^2_{H^1(D\backslash\overline{D}_0)}\right) \end{aligned}$$

with $C > 0$ if $1 < \eta < \gamma_*$. We can finally conclude from (a slightly modified version of) Theorem 4.2.1 that $\tilde{a}_k(\cdot,\cdot)$ satisfies the Fredholm property. □

From the above theorem the bounded linear operator $B : \mathbb{H} \to \mathbb{H}$ defined by mean of Riesz's representation theorem as

$$\begin{aligned} (B(v,w),(v',w')) &:= b((v,w),(v',w')) \\ &= \int_{D\backslash\overline{D}_0} A\nabla w \cdot \nabla\overline{w}'dx + \int_D \nabla v \cdot \nabla\overline{v}'dx \\ &\quad - 2\int_{D\backslash\overline{D}_0} \nabla v \cdot \nabla\overline{w}'dx + \int_D v\overline{v}'dx + \int_{D\backslash\overline{D}_0} w\overline{w}'dx \end{aligned}$$

is invertible.

Remark 4.2.1. *Note that the operator $C_k : \mathbb{H} \to \mathbb{H}$ depends analytically on $k \in \mathbb{C}$. Also note that the operator B does not depend on k. Thus the eigenvalue problem becomes $(I + B^{-1}C_k)(v,w) = 0$ where $B^{-1}C_k : \mathbb{H} \to \mathbb{H}$ is compact and the mapping $k \to B^{-1}C_k$ is analytic in $\mathbb{C}$.*

Theorem 4.2.3. *Assume that $1 < \gamma_* < \gamma^* < \infty$ and $0 < n_* \leq n(x) \leq n^* < \infty$ where where $\gamma^* := \sup_{D\backslash\overline{D}_0}\sup_{||\xi||=1}(\overline{\xi} \cdot A(x)\xi)$, $\gamma_* := \inf_{D\backslash\overline{D}_0}\inf_{||\xi||=1}(\overline{\xi} \cdot A(x)\xi)$, $n_* = \inf_{D\backslash\overline{D}_0} n(x)$ and $n^* = \sup_{D\backslash\overline{D}_0} n(x)$. Then the set of transmission eigenvalues is discrete.*

Proof. To apply the analytic Fredholm theory, from Remark 4.2.1 it remains to show that there exists a $k \in \mathbb{C}$ for which $B + C_k$ is injective. We set $k = \mathrm{i}\kappa$.

$$\begin{aligned} \tilde{a}_{\mathrm{i}\kappa}((v,w),(v,w)) &= \int_{D\backslash\overline{D}_0} A\nabla w \cdot \nabla\overline{w}dx + \int_D |\nabla v|^2 dx + \kappa^2\int_D |v|^2dx + \kappa^2\int_{D\backslash\overline{D}_0} n|w|^2dx \\ &\quad - 2\int_{D\backslash\overline{D}_0} \nabla v \cdot \nabla\overline{w}dx - 2\kappa^2\int_{D\backslash\overline{D}_0} v\overline{w}dx \\ &\geq \gamma_*||\nabla w||^2_{D\backslash\overline{D}_0} + ||\nabla v||^2_D + \kappa^2||v||^2_D + \kappa^2 n_*||w||^2_{D\backslash\overline{D}_0} \\ &\quad - \frac{1}{\eta}||\nabla v||^2_D - \eta||\nabla w||^2_{D\backslash\overline{D}_0} - \frac{\kappa^2}{\alpha}||v||^2_D - \kappa^2\alpha||w||^2_{D\backslash\overline{D}_0} \\ &\geq (\gamma_* - \eta)||\nabla w||^2_{D\backslash\overline{D}_0} + \left(1 - \frac{1}{\eta}\right)||\nabla v||^2_D + \kappa^2\left(1 - \frac{1}{\alpha}\right)||v||^2_D \\ &\quad + \kappa^2(n_* - \alpha)||w||^2_{D\backslash\overline{D}_0} \end{aligned}$$

where $n_* = \inf\limits_{D\backslash\overline{D}_0} n(x)$. Furthermore, $w \in H^1(D\backslash\overline{D}_0)$ and it vanishes on the boundary Σ which implies the Poincaré inequality

$$||w||^2_{D\backslash\overline{D}_0} \leq \lambda||\nabla w||^2_{D\backslash\overline{D}_0},$$

and consequently

$$\tilde{a}_{i\kappa}((v,w),(v,w)) \geq \left((\gamma_* - \eta) - \kappa^2 \lambda |n_* - \alpha|\right) ||\nabla w||^2_{D\setminus\overline{D}_0} + \kappa^2 \left(1 - \frac{1}{\alpha}\right) ||v||^2_D + \left(1 - \frac{1}{\eta}\right) ||\nabla v||^2_D.$$

Then, for κ^2 small enough, $1 < \eta < \gamma_*$ and $\alpha > 1$, we deduce that $\tilde{a}_{i\kappa}$ is coercive and $B + C_{i\kappa}$ is injective. The analytic Fredholm theory now ensures the discreteness of the set of transmission eigenvalues. □

Note that the discreteness of transmission eigenvalues for the case of $A - I > 0$ is proven without any sign requirement on the contrast $n - 1$.

The case of $(I - A)$ positive

In this section, we assume that $0 < \gamma_* < \gamma^* < 1$ We again use the T-coercivity to show discreteness of transmission eigenvalues. As it will become clear later on, for this case we can prove the discreteness under the additional assumption that $n < 1$ only.

We recall that (v, w) is a solution to the interior transmission problem (TEPA) if and only if $u \in \mathbb{H}$ is the solution of the variational problem (4.11). Now, we use the cutoff function $\chi \in C^\infty(D)$ satisfying $0 \leq \chi \leq 1$ in $D\setminus\overline{D}_0$ and $\text{supp}(\chi) \cap D_0 = \emptyset$. Similarly to the approach in Section 4.2.1, we define a bijective bounded linear operator T from $\mathbb{H}$ to $\mathbb{H}$ by

$$\begin{array}{cccc} T: & \mathbb{H} & \to & \mathbb{H} \\ & (v,w) & \mapsto & (-v, w - 2\chi v). \end{array}$$

Again we consider the new bilinear form $\tilde{a}_k$ given by

$$\begin{aligned} \tilde{a}_k((v,w),(v',w')) &= a_k((v,w),T(v',\overline{w}')) \\ &= \int_{D\setminus\overline{D}_0} A\nabla w \cdot \nabla\overline{w}' dx + \int_D \nabla v \cdot \nabla\overline{v}' dx - k^2 \int_D v\overline{v}' dx \\ &- k^2 \int_{D\setminus\overline{D}_0} nw\overline{w}' dx - 2\int_{D\setminus\overline{D}_0} A\nabla w \cdot \nabla(\chi\overline{v}') dx + 2k^2 \int_{D\setminus\overline{D}_0} nw\chi\overline{v}' dx. \end{aligned}$$

Lemma 4.2.4. *The bilinear form $\tilde{a}_k(\cdot,\cdot)$ satisfies the Fredholm property.*

Proof. We can write $\tilde{a}_k((v,w),(v',w')) = b((v,w),(v',w')) + c_k((v,w),(v',w'))$ where

$$\begin{aligned} b((v,w),(v',w')) = &\int_{D\setminus\overline{D}_0} A\nabla w \cdot \nabla\overline{w}' dx + \int_D \nabla v \cdot \nabla\overline{v}' dx \\ &- 2\int_{D\setminus\overline{D}_0} \chi A\nabla w \cdot \nabla\overline{v}' dx + \int_D v\overline{v}' dx + \int_{D\setminus\overline{D}_0} w\overline{w}' dx \end{aligned}$$

and

$$\begin{aligned} c_k((v,w),(v',w')) = &-(k^2+1)\int_D v\overline{v}' dx - \int_{D\setminus\overline{D}_0} (k^2 n + 1) w\overline{w}' dx \\ &- 2\int_{D\setminus\overline{D}_0} \overline{v}' A\nabla w \cdot \nabla\chi dx + 2k^2 \int_{D\setminus\overline{D}_0} nw\chi\overline{v}' dx. \end{aligned}$$

From Riesz's representation theorem, we define the bounded operator C_k from $\mathbb{H}$ into $\mathbb{H}$ by

$$c_k((v,w),(v',w')) = (C_k(v,w),(v',w'))_{\mathbb{H}}.$$

The compact embedding of $H^1(D)$ into $L^2(D)$ implies that C_k is a compact operator for all $k > 0$. Next we show that $b(\cdot,\cdot)$ is coercive. To this end, let (v,w) be in $\mathbb{H}$.

$$\begin{aligned} b((v,w),(v,w)) &= \int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}dx + \int_D |\nabla v|^2 dx + ||v||_D^2 + ||w||^2_{D\setminus\overline{D}_0} \\ &\quad - 2\int_{D\setminus\overline{D}_0} \chi A\nabla w\cdot\nabla\overline{v}dx \\ &\geq \frac{1}{\gamma^*}||A\nabla w||^2_{D\setminus\overline{D}_0} + ||\nabla v||_D^2 + ||v||_D^2 + ||w||^2_{D\setminus\overline{D}_0} \\ &\quad - 2\int_{D\setminus\overline{D}_0} \chi A\nabla w\cdot\nabla\overline{v}dx. \end{aligned}$$

Using the following inequality

$$\begin{aligned} \left|-2\int_{D\setminus\overline{D}_0} \chi A\nabla w\cdot\nabla\overline{v}dx\right| &\leq 2\int_{\operatorname{supp}(\chi)} |A\nabla w\cdot\nabla\overline{v}|dx \\ &\leq \eta||\nabla v||_D^2 + \frac{1}{\eta}||A\nabla w||^2_{D\setminus\overline{D}_0} \end{aligned}$$

with $\eta > 0$ to be chosen later. Then

$$\begin{aligned} b((v,w),(v,w)) &\geq \frac{1}{\gamma^*}||A\nabla w||^2_{D\setminus\overline{D}_0} + ||\nabla v||_D^2 + ||v||_D^2 + ||w||^2_{D\setminus\overline{D}_0} \\ &\quad - \eta||\nabla v||^2_{D\setminus\overline{D}_0} - \frac{1}{\eta}||A\nabla w||^2_{D\setminus\overline{D}_0} \\ &\geq \left(\frac{1}{\gamma^*} - \frac{1}{\eta}\right)||\nabla w||^2_{D\setminus\overline{D}_0} + (1-\eta)||\nabla v||_D^2 + ||v||_D^2 + ||w||^2_{D\setminus\overline{D}_0} \\ &\geq C\left(||v||^2_{H^1(D)} + ||w||^2_{H^1(D\setminus\overline{D}_0)}\right) \end{aligned}$$

with $C > 0$ if $\gamma^* < \eta < 1$. We can conclude that $\tilde{a}_k$ satisfies the Fredholm property. □

Again we define the invertible bounded linear operator $B : \mathbb{H} \to \mathbb{H}$ associated with the coercive bilinear form $b(\cdot,\cdot)$ as follows $b((v,w),(v',w')) = (B(v,w),(v',w'))_{\mathbb{H}}$. The the transmission eigenvalue problem is equivalent to

$$(B + C_k)u = 0 \qquad \text{or} \qquad (I + B^{-1}C_k)u = 0 \qquad \text{in } \mathbb{H}. \tag{4.13}$$

Furthermore the mapping $k \to C_k$ is analytic in $\mathbb{C}$.

Remark 4.2.2. *One can remark that the Fredholm property of $\tilde{a}_k(\cdot,\cdot)$ holds true for any $n \geq c > 0$. The restriction on the sign of $n-1$ appears in the next theorem, and is needed to show that there exists at least one k for which $B + C_k$ is injective.*

Theorem 4.2.5. *Assume that* $0 < \gamma_* < \gamma^* < 1$ *and* $0 < n_* \le n(x) \le n^* < 1$ *where* $\gamma^* := \sup_{D\setminus\overline{D}_0}\sup_{||\xi||=1}(\overline{\xi}\cdot A(x)\xi)$, $\gamma_* := \inf_{D\setminus\overline{D}_0}\inf_{||\xi||=1}(\overline{\xi}\cdot A(x)\xi)$, $n_* = \inf_{D\setminus\overline{D}_0} n(x)$ *and* $n^* = \sup_{D\setminus\overline{D}_0} n(x)$. *Then the set of transmission eigenvalues is discrete.*

Proof. To apply the analytic Fredholm theory to (4.13), it remains to show that there exists a k for which $B + C_k$ is injective. To this end

$$
\begin{aligned}
\tilde{a}_{i\kappa}((v,w),(v,w)) = &\int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla\overline{w}dx + \int_D |\nabla v|^2dx + \kappa^2\int_D |v|^2dx \\
&+ \kappa^2\int_{D\setminus\overline{D}_0} n|w|^2dx - 2\int_{D\setminus\overline{D}_0} A\nabla w\cdot\nabla(\chi\overline{v})dx - 2\kappa^2\int_{\mathrm{supp}(\chi)} nw\overline{v} \\
&\ge \frac{1}{\gamma^*}||A\nabla w||^2_{D\setminus\overline{D}_0} + ||\nabla v||^2_D + \kappa^2||v||^2_D + \frac{\kappa^2}{n^*}||nw||^2_{D\setminus\overline{D}_0} - \frac{1}{\eta}||A\nabla w||^2_{D\setminus\overline{D}_0} \\
&- \eta||\nabla v||^2_D - \frac{1}{\alpha}||A\nabla w||^2_{D\setminus\overline{D}_0} - \alpha C||v||^2_D - \frac{\kappa^2}{\beta}||nw||^2_{D\setminus\overline{D}_0} - \kappa^2\beta||v||^2_D \\
&\left(\frac{1}{\gamma^*} - \frac{1}{\eta} - \frac{1}{\alpha}\right)||A\nabla w||^2_{D\setminus\overline{D}_0} + \left(\kappa^2(1-\beta) - \alpha C\right)||v||^2_D \\
&+ (1-\eta)||\nabla v||^2_D + \kappa^2\left(\frac{1}{n^*} - \frac{1}{\beta}\right)||nw||^2_{D\setminus\overline{D}_0}
\end{aligned}
$$

where $C = ||\nabla\chi||^2$.

Let $\gamma^* < \eta < 1$, $n^* < \beta < 1$ and α be such that $\frac{1}{\gamma^*} - \frac{1}{\eta} - \frac{1}{\alpha} > 0$. Then for κ large enough we have that $\kappa^2(1-\beta) - \alpha C > 0$, and thus $\tilde{a}_{i\kappa}$ is coercive which means $B + C_{i\kappa}$ is injective. Then the analytic Fredholm theory now ensures the discreteness of the set of transmission eigenvalues. □

4.2.2 The existence of transmission eigenvalues

The T-coercivity approach does not provide any framework for proving the existence of transmission eigenvalues. For this question we adapt the approach introduced in [19], [36] to treat the case $A - I >)$ and $n > 1$ or $n < 1$. Unfortunately, due to the presence of the Dirichlet obstacle D_0 this approach provides only the existence of a finite set of transmission eigenvalues provided that the area of D_0 is small enough. In the case when $n > 1$ we also require n to be small enough. The existence of transmission eigenvalues for $I - A > 0$ is still open.

Throughout this section we assume that $1 < \gamma_* < \gamma^* < +\infty$ where

$$\gamma^* := \sup_{D\setminus\overline{D}_0}\sup_{||\xi||=1}(\overline{\xi}\cdot A(x)\xi)$$

and

$$\gamma_* := \inf_{D\setminus\overline{D}_0}\inf_{||\xi||=1}(\overline{\xi}\cdot A(x)\xi).$$

Recall that $n_* = \inf_{D\setminus\overline{D}_0} n(x)$ and $n^* = \sup_{D\setminus\overline{D}_0} n(x)$.

If we consider the new variable $u := w - v$ in $D\backslash\overline{D}_0$, then u is in $H^1(D\backslash\overline{D}_0)$, $u = 0$ on Γ and v satisfies the mixed boundary problem depending on u in $D\backslash\overline{D}_0$

$$\begin{cases} \nabla \cdot (I - A)\nabla v + k^2(1-n)v = \nabla \cdot A\nabla u + k^2 nu & \text{in } D\backslash\overline{D}_0, \\ \nu \cdot (A - I)\nabla v = \nu \cdot A\nabla u & \text{on } \Gamma, \\ -v = u & \text{on } \Sigma. \end{cases} \tag{4.14}$$

We define

$$H^1_\Gamma(D\backslash\overline{D}_0) := \left\{u \in H^1(D\backslash\overline{D}_0) \text{ such that } u = 0 \text{ on } \Gamma\right\}$$

and

$$H^1_\Sigma(D\backslash\overline{D}_0) := \left\{u \in H^1(D\backslash\overline{D}_0) \text{ such that } u = 0 \text{ on } \Sigma\right\}.$$

The next step is to solve the mixed boundary value problem (4.14) for v as a function of u. To this end, for a fixed $u \in H^1_\Gamma(D\backslash\overline{D}_0)$, we define the lifting function $\theta \in H^1(D\backslash\overline{D}_0)$ such that $\theta = -u$ on Σ. Setting $v_0 := v - \theta$, the variational formulation of (4.14) as a problem for v_0 now becomes: find $v_0 \in H^1_\Sigma(D\backslash\overline{D}_0)$ such that

$$\begin{aligned} &\int_{D\backslash\overline{D}_0} \big((A - I)\nabla v_0 \cdot \nabla\overline{\varphi} - k^2(n-1)v_0\overline{\varphi}\big)\, dx \\ &= -\int_{D\backslash\overline{D}_0} \big(A\nabla u \cdot \nabla\overline{\varphi} - k^2 nu\overline{\varphi}\big)\, dx - \int_{D\backslash\overline{D}_0} \big((A - I)\nabla\theta \cdot \nabla\overline{\varphi} - k^2(n-1)\theta\overline{\varphi}\big)\, dx \end{aligned} \tag{4.15}$$

for all $\varphi \in H^1_\Sigma(D\backslash\overline{D}_0)$.

First, we want to show that problem (4.15) is well-posed using Lax-Milgram theorem. Since the right-hand side is obviously a continuous function of φ in $H^1_\Sigma(D\backslash\overline{D}_0)$, it only remains to show that the left-hand side is coercive. In the next theorem, we see that the latter is always true for $n < 1$ or for $n > 1$ small enough. Setting

$$\mu := \inf_{\varphi \in H^1_\Sigma(D\backslash\overline{D}_0)} \frac{||\nabla\varphi||^2_{D\backslash\overline{D}_0}}{||\varphi||^2_{D\backslash\overline{D}_0}},$$

we have that for all $\varphi \in H^1_\Sigma(D\backslash\overline{D}_0)$,

$$\frac{\mu}{\mu+1}||\varphi||^2_{H^1(D\backslash\overline{D}_0)} \leq ||\nabla\varphi||^2_{D\backslash\overline{D}_0}.$$

Note that $\mu > 0$ coincides with the first eigenvalue of $-\Delta$ in $D\backslash\overline{D}_0$ with mixed Neumann-Dirichlet boundary conditions.

Let B_r be a ball of radius r included in $D\backslash\overline{D}_0$ and let $\hat{k}' > 0$ be the first transmission eigenvalue of the interior transmission problem for B_r with $A = \frac{\gamma_*}{2}I$ and $n = 1$:

$$\begin{cases} \nabla \cdot \frac{\gamma_*}{2}\nabla w + k^2 w = 0 & \text{in } B_r \\ \Delta v + k^2 v = 0 & \text{in } B_r \\ w = v & \text{on } \partial B_r \\ \nu \cdot \frac{\gamma_*}{2}\nabla w = \nu \cdot \nabla v & \text{on } \partial B_r \end{cases} \tag{4.16}$$

The existence of such $\hat{k}' > 0$ is proven in [19], [26]. In the case when $n-1$ is positive, i.e $n_* > 1$, we further assume that

$$n^* - 1 \leq \frac{\gamma_* \mu}{2\hat{k}'^2}. \tag{4.17}$$

Lemma 4.2.6. *For every u in $H^1_\Gamma(D\backslash\overline{D}_0)$ and $k \geq 0$ satisfying $k \leq \hat{k}'$ if $n > 1$, there exists a unique solution $v_0 \in H^1_\Sigma(D\backslash\overline{D}_0)$ of* (4.15) *and consequently a unique $v_u := v_0 + \theta \in H^1(D\backslash\overline{D}_0)$ of* (4.14).

Proof. We denote

$$B_k(v,\varphi) := \int_{D\backslash\overline{D}_0} \left((A-I)\nabla v \cdot \nabla\overline{\varphi} - k^2(n-1)v\overline{\varphi}\right) dx$$

First assume that $1-n>0$. Then

$$\begin{aligned} B_k(v,v) &\geq (\gamma_* - 1)||\nabla v||^2_{D\backslash\overline{D}_0} \\ &\geq (\gamma_* - 1)\frac{\mu}{\mu+1}||v||^2_{H^1(D\backslash\overline{D}_0)}. \end{aligned}$$

Thus B_k is coercive for $k \geq 0$ if $n-1<0$. From Lax-Milgram theorem, we deduce that there exists a unique solution v_0 of (4.15) depending continuously on u.

Now assume that $n-1>0$ and more precisely that n satisfies (4.17)

$$\begin{aligned} B_k(v,v) &\geq (\gamma_* - 1)||\nabla v||^2_{D\backslash\overline{D}_0} - (k^2)(n^*-1)||v||^2_{D\backslash\overline{D}_0} \\ &\geq \left((\gamma_* - 1)^2 - \frac{\hat{k}'^2(n^*-1)}{\mu}\right)||\nabla v||^2_{D\backslash\overline{D}_0} \\ &\geq \left(\frac{\gamma_*}{2} - 1\right)\frac{\mu}{\mu+1}||v||^2_{H^1(D\backslash\overline{D}_0)}. \end{aligned}$$

In this case B_k is coercive for $0 \leq k \leq \hat{k}'$ if $n-1>0$ and the result again follows from the Lax-Milgram theorem. □

Hence we can now define a linear bounded operator A_k by

$$\begin{array}{rccc} A_k: & H^1_\Gamma(D\backslash\overline{D}_0) & \to & H^1(D\backslash\overline{D}_0) \\ & u & \mapsto & v_u := v_0 + \theta. \end{array}$$

for $k \geq 0$ if $n-1<0$ and $0 \leq k \leq \hat{k}'$ if $n-1>0$.

Assume now that k^2 is not a Dirichlet eigenvalue for $-\Delta$ is D_0, and let v be the unique solution in $H^1(D_0)$ to

$$\begin{cases} \Delta v + k^2 v = 0 & \text{in } D_0 \\ v = \varphi & \text{on } \Sigma \end{cases} \tag{4.18}$$

for some $\varphi \in H^{1/2}(\Sigma)$, In this case, we define the Dirichlet to Neumann operator T_k by

$$\begin{array}{rccc} T_k: & H^{1/2}(\Sigma) & \to & H^{-1/2}(\Sigma) \\ & \varphi & \mapsto & \dfrac{\partial v}{\partial \nu} \end{array}$$

where v is solution to (4.18).

Using the Riesz representation theorem, we can define the operator

$$L_k : H^1_\Gamma(D\backslash\overline{D}_0) \to H^1_\Gamma(D\backslash\overline{D}_0)$$

by

$$\langle L_k u, \varphi\rangle_{H^1(D\backslash\overline{D}_0)} = \int_{D\backslash\overline{D}_0} \left(-\nabla v_u \cdot \nabla\overline{\varphi} + k^2 v_u \overline{\varphi}\right) dx - \int_\Sigma T_k v_u \overline{\varphi} ds$$

for all $\varphi \in H^1_\Gamma(D\backslash\overline{D}_0)$, where last integral is understood in the sense of $H^{-1/2}(\Sigma)$, $H^{1/2}(\Sigma)$ duality.

It is obvious that the mapping $k \to L_k$ is continuous in the domain of definition, i.e. for $k \geq 0$ if $n-1<0$ and $0 \leq k \leq \hat{k}'$ if $n-1>0$ such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_0. The next theorem introduces an equivalent formulation to (TEPA).

Theorem 4.2.7. *Assume that $k \geq 0$ if $n-1<0$ and $0 \leq k \leq \hat{k}'$ if $n-1>0$, such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_0.*

(i) *Let (w, v) be a solution of* (TEPA) *for some $k > 0$. Then $u := w - v \in H^1_\Gamma(D\backslash\overline{D}_0)$ solves $L_k u = 0$.*

(ii) *Let $u \in H^1_\Gamma(D\backslash\overline{D}_0)$ such that $L_k u = 0$. If $v := A_k u$, the pair $w := (u+v, v)$ is solution to* (TEPA).

Proof. (i) If (w, v) is a solution of (TEPA), then, $v = A_k u$ where $u := w - v$ and solves the Helmholtz equation in D. In particular, v solves Helmholtz equation in $D\backslash\overline{D}_0$ and $\dfrac{\partial v}{\partial \nu} = T_k v$ on Σ. Then, for all $\varphi \in H^1_\Gamma(D\backslash\overline{D}_0)$,

$$\begin{aligned} 0 &= \int_{D\backslash\overline{D}_0} (\Delta v + k^2 v)\overline{\varphi} dx \\ &= \int_{D\backslash\overline{D}_0} \left(-\nabla v \cdot \nabla\overline{\varphi} + k^2 v\overline{\varphi}\right) dx - \int_\Sigma \frac{\partial v}{\partial \nu}\overline{\varphi} ds = \langle L_k u, \varphi\rangle_{H^1(D\backslash\overline{D}_0)} . \end{aligned}$$

Then $L_k u = 0$.

(ii) Let $u \in H^1_\Gamma(D\backslash\overline{D}_0)$ such that $L_k u = 0$. We define $v := A_k u$ in $D\backslash\overline{D}_0$ and in D_0, v is defined as the solution to

$$\begin{cases} \Delta v + k^2 v = 0 & \text{in } D_0 \\ v = A_k u & \text{on } \Sigma. \end{cases}$$

Then, v is in $H^1(D)$ and since $L_k u = 0$, v satisfies $\Delta v + k^2 v = 0$ in D. Besides, $v = A_k u$ in $D\backslash\overline{D}_0$ implies that the pair $w := (u+v, v)$ is solution to (TEPA). □

The following theorem states some properties of the operator L_k.

Theorem 4.2.8. *Assume that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_0, and $k \geq 0$ if $n-1<0$ and $0 \leq k < \hat{k}'$ if $n-1>0$.*

(i) *The operator* $L_k : H^1_\Gamma(D\backslash\overline{D}_0) \to H^1_\Gamma(D\backslash\overline{D}_0)$ *is self-adjoint.*

(ii) $L_k - L_0 : H^1_\Gamma(D\backslash\overline{D}_0) \to H^1_\Gamma(D\backslash\overline{D}_0)$ *is compact.*

(iii) *The operator* $L_0 : H^1_\Gamma(D\backslash\overline{D}_0) \to H^1_\Gamma(D\backslash\overline{D}_0)$ *is coercive.*

Proof. (i) Let u_1, $u_2 \in H^1_\Gamma(D\backslash\overline{D}_0)$ and $v_1 = A_k u_1$, $v_2 = A_k u_2$. Thus

$$\langle L_k u_1, u_2\rangle_{H^1(D\backslash\overline{D}_0)} = -\int_{D\backslash\overline{D}_0} \big((I-A)\nabla v_1\cdot\nabla\bar{u}_2 - k^2(1-n)v_1\bar{u}_2\big)\,dx \\ -\int_{D\backslash\overline{D}_0}\big(A\nabla v_1\cdot\nabla\bar{u}_2 - k^2 n v_1\bar{u}_2\big)\,dx - \int_\Sigma T_k(v_1)\bar{u}_2 ds. \quad (4.19)$$

From the equality (4.15), we have for $i = 1, 2$ and all $\varphi \in H^1_\Sigma(D\backslash\overline{D}_0)$

$$\int_{D\backslash\overline{D}_0}\big(A\nabla u_i\cdot\nabla\bar{\varphi} - k^2 n u_i\bar{\varphi}\big)\,dx = \int_{D\backslash\overline{D}_0}\big((I-A)\nabla v_i\cdot\nabla\bar{\varphi} - k^2(1-n)v_i\bar{\varphi}\big)\,dx.$$

Taking $i = 2$ with $\varphi = v_1$ and $i = 1$ with $\varphi = u_2$ in the above, the expression (4.19) for L_k becomes

$$\begin{aligned}\langle L_k u_1, u_2\rangle_{H^1(D\backslash\overline{D}_0)} &= \int_{D\backslash\overline{D}_0}\big((A-I)\nabla v_2\cdot\nabla\bar{v}_1 - k^2(n-1)v_2\bar{v}_1\big)\,dx \\ &-\int_{D\backslash\overline{D}_0}\big(A\nabla u_1\cdot\nabla\bar{u}_2 - k^2 n u_1\bar{u}_2\big)\,dx + \int_\Sigma T_k(v_1)\bar{v}_2 ds \\ &= \int_{D\backslash\overline{D}_0}\big((A-I)\nabla v_2\cdot\nabla\bar{v}_1 - k^2(n-1)v_2\bar{v}_1\big)\,dx \\ &-\int_{D\backslash\overline{D}_0}\big(A\nabla u_1\cdot\nabla\bar{u}_2 - k^2 n u_1\bar{u}_2\big)\,dx + \int_{D_0}\big(\nabla v_1\cdot\nabla v_2 - k^2 v_1\bar{v}_2\big)\,dx\end{aligned}$$

which is a symmetric expression for u_1 and u_2.

(ii) The compactness of $L_k - L_0$ is obtained from the compact embedding of $H^1(D\backslash\overline{D}_0)$ into $L^2(D\backslash\overline{D}_0)$. Indeed, let (u_j) be a sequence of $H^1_\Gamma(D\backslash\overline{D}_0)$ weakly converging to zero in $H^1_\Gamma(D\backslash\overline{D}_0)$. Since $H^1_\Gamma(D\backslash\overline{D}_0)$ is compactly embedded in $L^2(D\backslash\overline{D}_0)$, we deduce that the sequence (u_j) strongly converges to zero in $L^2(D\backslash\overline{D}_0)$. Let us denote $v_k^j := A_k u_j \in H^1(D\backslash\overline{D}_0)$ and $v_0^j := A_0 u_j \in H^1(D\backslash\overline{D}_0)$. Since the operators A_k and A_0 are continuous from $H^1_\Gamma(D\backslash\overline{D}_0)$ into $H^1(D\backslash\overline{D}_0)$, we deduce that v_k^j and v_0^j weakly converge to zero in $H^1(D\backslash\overline{D}_0)$ and consequently, strongly converge to zero in $L^2(D\backslash\overline{D}_0)$. Furthermore, from (4.15), v_k^j and v_0^j satisfy for all $\varphi \in H^1_\Sigma(D\backslash\overline{D}_0)$,

$$\int_{D\backslash\overline{D}_0}\big((A-I)\nabla v_k^j\cdot\nabla\overline{\varphi} - k^2(n-1)v_k^j\overline{\varphi}\big)\,dx = -\int_{D\backslash\overline{D}_0}\big(A\nabla u_j\cdot\nabla\overline{\varphi} - k^2 n u_j\overline{\varphi}\big)\,dx$$

and

$$\int_{D\backslash\overline{D}_0}(A-I)\nabla v_0^j\cdot\nabla\overline{\varphi}dx = -\int_{D\backslash\overline{D}_0}A\nabla u_j\cdot\nabla\overline{\varphi}dx.$$

Letting $\tilde{v}_j := v_0^j - v_k^j$, and taking the difference between the two previous equations yield

$$\int_{D\setminus\overline{D}_0} \left((A-I)\nabla\tilde{v}_j \cdot \nabla\overline{\varphi} + k^2(n-1)v_k^j\overline{\varphi}\right) dx = -k^2 \int_{D\setminus\overline{D}_0} nu_j\overline{\varphi}dx. \tag{4.20}$$

Now, for $\varphi = \tilde{v}_j$ in (4.20), applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\int_{D\setminus\overline{D}_0} (A-I)\nabla\tilde{v}_j \cdot \nabla\overline{\tilde{v}_j}dx\right| &= k^2 \left|\int_{D\setminus\overline{D}_0} \left((1-n)v_k^j + nu_j\right)\overline{\tilde{v}_j}dxdx\right| \\ &\leq k^2||(1-n)v_k^j + nu_j||_{D\setminus\overline{D}_0}||\tilde{v}_j||_{D\setminus\overline{D}_0}.\end{aligned}$$

Since $||(1-n)v_k^j + nu_j||_{D\setminus\overline{D}_0}$ is bounded and $||\tilde{v}_j||_{D\setminus\overline{D}_0}$ tends to zero, from the fact that $A-I$ is positive definite, we deduce that $\nabla\tilde{v}_j$ converges to zero in $L^2(D\setminus\overline{D}_0)$ and consequently $\tilde{v}_j$ converges to zero in $H^1(D\setminus\overline{D}_0)$.
Now, since for all $\varphi \in H^1_\Sigma(D\setminus\overline{D}_0)$,

$$\langle(L_k - L_0)u_j, \varphi\rangle_{H^1(D\setminus\overline{D}_0)} = \int_{D\setminus\overline{D}_0} \nabla\tilde{v}_j \cdot \nabla\overline{\varphi}dx + k^2 \int_{D\setminus\overline{D}_0} v_k^j\overline{\varphi}dx + \int_\Sigma (T_0v_0^j - T_kv_k^j)\overline{\varphi}ds,$$

we have that

$$\begin{aligned}||(L_k - L_0)u_j||_{H^1(D\setminus\overline{D}_0)} &= \sup_{||\varphi||_{H^1(D\setminus\overline{D}_0)}=1} \langle(L_k - L_0)u_j, \varphi\rangle_{H^1(D\setminus\overline{D}_0)} \\ &\leq ||\nabla\tilde{v}_j||_{D\setminus\overline{D}_0} + k^2||v_k^j||_{D\setminus\overline{D}_0} + ||\tilde{v}_j||_{H^{1/2}(\Sigma)}.\end{aligned}$$

The right-hand side tends to zero and consequently $(L_k - L_0)u_j$ strongly tends to zero in $H^1(D\setminus\overline{D}_0)$. Then, $L_k - L_0$ is compact.

(iii) Now we show that L_0 is coersive. To this end for $u \in H^1_\Gamma(D\setminus\overline{D}_0)$ we have that

$$\begin{aligned}\langle L_0u, u\rangle_{H^1(D\setminus\overline{D}_0)} &= -\int_{D\setminus\overline{D}_0} \nabla v_u \cdot \nabla\bar{u}dx - \int_\Sigma \frac{\partial v_u}{\partial\nu}\bar{u}ds \\ &= -\int_{D\setminus\overline{D}_0} \nabla v_u \cdot \nabla\bar{u}dx + \int_\Sigma \frac{\partial v_u}{\partial\nu}\bar{v_u}ds \\ &= -\int_{D\setminus\overline{D}_0} \nabla w_u \cdot \nabla\bar{u}dx + \int_{D\setminus\overline{D}_0} |\nabla u|^2dx + \int_{D_0} |\nabla v_u|^2dx.\end{aligned}$$

Replacing v_u by $w_u - u$ in (4.15) for $k = 0$ and $\varphi = w_u$, we obtain

$$\int_{D\setminus\overline{D}_0} \nabla u \cdot \nabla\bar{w}_udx = \int_{D\setminus\overline{D}_0} (I-A)\nabla w_u \cdot \nabla\bar{w}_udx$$

Therefore

$$\langle L_0u, u\rangle = \int_{D\setminus\overline{D}_0} (A-I)\nabla w_u \cdot \nabla\bar{w}_udx + \int_{D\setminus\overline{D}_0} |\nabla u|^2dx + \int_{D_0} |\nabla v_u|^2dx. \tag{4.21}$$

Since $(A-I)$ is positive definite, we deduce that L_0 is coercive, which ends the proof of the theorem

$\square$

Note that the mapping $k \to L_k$ is continuous in its domain of definition, i.e. for $k \geq 0$ if $n-1<0$ and $0 \leq k \leq \hat{k}'$ if $n-1>0$, such that k^2 is not a Dirichlet eigenvalue for $-\Delta$ in D_0. The proof of existence of transmission eigenvalues is based on the following theorem which is a modified version of Theorem 4.1.3 [19].

Theorem 4.2.9. *Let $L_k : H^1_\Gamma(D\backslash\overline{D}_0) \to H^1_\Gamma(D\backslash\overline{D}_0)$ be as defined above. If*

(a) *there exists k_0 such that L_{k_0} is positive on $H^1_\Gamma(D\backslash\overline{D}_0)$, and*

(b) *there exists k_1 such that L_{k_1} is non positive on some m-dimensional subspace of $H^1_\Gamma(D\backslash\overline{D}_0)$.*

Then there exists m transmission eigenvalues in $[k_0, k_1]$ counting with their multiplicity provided that the entire interval $[k_0, k_1]$ belongs to the domain of definition of the mapping $k \to L_k$.

Theorem 4.2.10. *Assume that $A - I > 0$ and that either $n_* < n < n^* < 1$ or $1 < n_* < n < n^* \leq 1 + \frac{\gamma_*\mu}{2\hat{k}'^2}$. Then there exists at least one transmission eigenvalue provided that the area of D_0 is small enough.*

Proof. We have shown in Theorem 4.2.8 that L_0 is coercive, thus the assumption (a) of Theorem 4.2.9 is satisfied for $k_0 = 0$.

First assume that $n < 1$. Let B_r be the largest ball included in $D\backslash\overline{D}_0$ of radius r and let us denote by $\hat{k}$ the first transmission eigenvalue of the interior transmission problem in B_r with $A = \gamma_* I$ and $n = n^*$, i.e.

$$\begin{cases} \nabla\cdot\gamma_*\nabla w + k^2 n^* w = 0 & \text{in } B_r \\ \Delta v + k^2 v = 0 & \text{in } B_r \\ w = v & \text{on } \partial B_r \\ \nu\cdot\gamma_*\nabla w = \nu\cdot\nabla v & \text{on } \partial B_r. \end{cases} \tag{4.22}$$

Assume now that the area of D_0 is small enough such that the first Dirichlet eigenvalue for $-\Delta$ in D_0 is greater than $\hat{k}$ (this is possible since due to the Faber-Krahn inequality the first Dirichlet eigenvalue for $-\Delta$ in D_0 is greater than $C/\text{area}D_0$) Thus the operator L_k is well defined for all $k \in [0,\, \hat{k}]$. denote by $\hat{w}$ and $\hat{v}$ the corresponding eigenvectors and we set $\hat{u} := \hat{w} - \hat{v} \in H^1_0(B_r)$. We shall show that we can find $u \in H^1_\Gamma(D\backslash\overline{D}_0)$ such that $\langle L_{\hat{k}}u, u\rangle \leq 0$ so that the assumption (b) of Theorem 4.2.9 is satisfied.

From the equation satisfied by $\hat{v}$ in B_r and using the fact that $\hat{u} = 0$ on ∂B_r and $\hat{v} = \hat{w} - \hat{u}$, we first have

$$0 = \int_{B_r} \left(\Delta\hat{v} + k^2\hat{v}\right)\overline{\hat{u}}dx = \int_{B_r}\left(\nabla\hat{v}\cdot\nabla\overline{\hat{u}} - \hat{k}^2\hat{v}\overline{\hat{u}}\right)dx \tag{4.23}$$

$$= \int_{B_r}\left(\nabla\hat{w}\cdot\nabla\overline{\hat{u}} - \hat{k}^2\hat{w}\overline{\hat{u}} - |\nabla\hat{u}|^2 + \hat{k}^2|\hat{u}|^2\right)dx. \tag{4.24}$$

On the other hand, replacing $\hat{v}$ by $\hat{w} - \hat{u}$ in the variational formulation satisfied by $\hat{v}$ and $\hat{w}$ we have

$$\int_{B_r}\left(\nabla\hat{u}\cdot\nabla\overline{\varphi} - \hat{k}^2\hat{u}\overline{\varphi}\right)dx = \int_{B_r}\left((1-\gamma_*)\nabla\hat{w}\cdot\nabla\overline{\varphi} - \hat{k}^2(1-n^*)\hat{w}\overline{\varphi}\right)dx$$

for all $\varphi \in H^1(B_r)$. In particular for $\varphi = \hat{w}$, we obtain

$$\int_{B_r} \left(\nabla \hat{w} \cdot \nabla \overline{\hat{u}} - \hat{k}^2 \hat{w} \overline{\hat{u}} \right) dx = \int_{B_r} \left((1-\gamma_*) |\nabla \hat{w}|^2 - \hat{k}^2 (1-n^*) |\hat{w}|^2 \right) dx. \tag{4.25}$$

From (4.24) and (4.25), we finally get the equality

$$\int_{B_r} \left((1-\gamma_*) |\nabla \hat{w}|^2 - \hat{k}^2 (1-n^*) |\hat{w}|^2 - |\nabla \hat{u}|^2 + \hat{k}^2 \hat{u}|^2 \right) dx = 0. \tag{4.26}$$

Now we denote by $\tilde{u}$ the extension of $\hat{u}$ by zero to all of $D \backslash \overline{D}_0$. Since $\tilde{u} \in H^1_\Gamma(D \backslash \overline{D}_0)$, we can define $\tilde{v} := v_{\tilde{u}}$ the corresponding solution to

$$\begin{cases} \nabla \cdot (I - A) \nabla v + \hat{k}^2 (1-n) v = \nabla \cdot A \nabla \tilde{u} + \hat{k} n \tilde{u} & \text{in } D \backslash \overline{D}_0 \\ \nu \cdot (I - A) \nabla v = \nu \cdot A \nabla \tilde{u} & \text{on } \Gamma \\ v = -\tilde{u} = 0 & \text{on } \Sigma \end{cases}$$

and we set $\tilde{w} := \tilde{u} + \tilde{v} \in H^1_\Sigma(D \backslash \overline{D}_0)$. We first remark that replacing $\tilde{v}$ by $\tilde{w} - \tilde{u}$ in (4.15) and for $\varphi = \tilde{u}$, yields

$$\int_{D \backslash \overline{D}_0} \left((A - I) \nabla \tilde{w} \cdot \nabla \overline{\tilde{u}} - \hat{k}^2 (n-1) \tilde{w} \overline{\tilde{u}} \right) dx = - \int_{D \backslash \overline{D}_0} \left(|\nabla \tilde{u}|^2 - \hat{k}^2 |\tilde{u}|^2 \right) dx.$$

Consequently, replacing $\tilde{v}$ by $\tilde{w} - \tilde{u}$ in the expression of $L_{\hat{k}}$ and using the definition of $\tilde{u}$, we obtain

$$\begin{aligned} \langle L_{\hat{k}} \tilde{u}, \tilde{u} \rangle_{H^1(D \backslash \overline{D}_0)} &= - \int_{D \backslash \overline{D}_0} \left(\nabla \tilde{v} \cdot \nabla \overline{\tilde{u}} - \hat{k}^2 \tilde{v} \tilde{u} \right) dx \\ &= - \int_{D \backslash \overline{D}_0} \left(\nabla \tilde{w} \cdot \nabla \overline{\tilde{u}} - \hat{k}^2 \tilde{w} \tilde{u} - |\nabla \tilde{u}|^2 + \hat{k}^2 |\tilde{u}|^2 \right) dx \\ &= \int_{D \backslash \overline{D}_0} \left((A - I) \nabla \tilde{w} \cdot \nabla \overline{\tilde{w}} - \hat{k}^2 (n-1) |\tilde{w}|^2 + |\nabla \tilde{u}|^2 - \hat{k}^2 |\tilde{u}|^2 \right) dx \\ &= \int_{D \backslash \overline{D}_0} \left((A - I) \nabla \tilde{w} \cdot \nabla \overline{\tilde{w}} - \hat{k}^2 (n-1) |\tilde{w}|^2 \right) dx + \int_{B_r} \left(|\nabla \hat{u}|^2 - \hat{k}^2 |\hat{u}|^2 \right) dx. \end{aligned}$$

Now, considering again (4.15) with $\tilde{v} = \tilde{w} - \tilde{u}$ and using the definition of $\tilde{u}$, for all $\varphi \in H^1_\Sigma(D \backslash \overline{D}_0)$, we have

$$\begin{aligned} &\int_{D \backslash \overline{D}_0} \left((A - I) \nabla \tilde{w} \cdot \nabla \overline{\varphi} - \hat{k}^2 (n-1) \tilde{w} \overline{\varphi} \right) dx = - \int_{D \backslash \overline{D}_0} \left(\nabla \tilde{u} \cdot \nabla \overline{\varphi} - \hat{k}^2 \tilde{u} \overline{\varphi} \right) dx \\ &= - \int_{B_r} \left(\nabla \hat{u} \cdot \nabla \overline{\varphi} - \hat{k}^2 \hat{u} \overline{\varphi} \right) dx = \int_{B_r} \left((\gamma_* - 1) \nabla \hat{w} \cdot \nabla \overline{\varphi} - \hat{k}^2 (n^* - 1) \hat{w} \overline{\varphi} \right) dx. \end{aligned}$$

In particular, for $\varphi = \tilde{w} \in H^1_\Sigma(D \backslash \overline{D}_0)$ we obtain

$$\begin{aligned} &\int_{D \backslash \overline{D}_0} \left((A - I) \nabla \tilde{w} \cdot \nabla \overline{\tilde{w}} - \hat{k}^2 (n-1) |\tilde{w}|^2 \right) dx \\ &\qquad = \int_{B_r} \left((\gamma_* - 1) \nabla \hat{w} \cdot \nabla \overline{\tilde{w}} - \hat{k}^2 (n^* - 1) \hat{w} \overline{\tilde{w}} \right) dx \end{aligned} \tag{4.27}$$

The Cauchy-Schwarz inequality applied to the right-hand side of (4.27) gives

$$\begin{aligned}
&\int_{D\setminus\overline{D}_0} \Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}^2(n-1)|\tilde{w}|^2\Big)\,dx = \int_{B_r} \Big((\gamma_*-1)\nabla\hat{w}\cdot\nabla\overline{\tilde{w}} + \hat{k}^2(1-n^*)\hat{w}\overline{\tilde{w}}\Big)\,dx \\
&\le \left(\int_{B_r} \Big((\gamma_*-1)|\nabla\hat{w}|^2 + \hat{k}^2(1-n^*)|\hat{w}|^2\Big)\,dx\right)^{1/2} \left(\int_{B_r} \Big((\gamma_*-1)|\nabla\tilde{w}|^2 + \hat{k}^2(1-n^*)|\tilde{w}|^2\Big)\,dx\right)^{1/2} \\
&\le \left(\int_{B_r} \Big((\gamma_*-1)|\nabla\hat{w}|^2 + \hat{k}^2(1-n^*)|\hat{w}|^2\Big)\,dx\right)^{1/2} \left(\int_{D\setminus\overline{D}_0} \Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}^2(n-1)|\tilde{w}|^2\Big)\,dx\right)^{1/2}
\end{aligned}$$

and finally

$$\begin{aligned}
&\int_{D\setminus\overline{D}_0} \Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}^2(n-1)|\tilde{w}|^2\Big)\,dx \\
&\qquad\qquad \le \int_{B_r} \Big((\gamma_*-1)|\nabla\hat{w}|^2 + \hat{k}^2(1-n^*)|\hat{w}|^2\Big)\,dx. \qquad (4.28)
\end{aligned}$$

Therefore, from (4.28) and (4.26), we obtain that

$$\begin{aligned}
\langle L_{\hat{k}}\tilde{u},\tilde{u}\rangle_{H^1(D\setminus\overline{D}_0)} &= \int_{D\setminus\overline{D}_0} \Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}^2(n-1)|\tilde{w}|^2\Big)\,dx \\
&\quad + \int_{B_r} \Big(|\nabla\hat{u}|^2 - \hat{k}^2|\hat{u}|^2\Big)\,dx \\
&\le \int_{B_r} \Big((\gamma_*-1)|\nabla\hat{w}|^2 + \hat{k}^2(1-n^*)|\hat{w}|^2 + |\nabla\hat{u}|^2 - \hat{k}^2|\hat{u}|^2\Big)\,dx = 0.
\end{aligned}$$

We can conclude that there exists a transmission eigenvalue in $(0,\hat{k}]$.

Now assume that $1 < n_* < n < n^* \le 1 + \frac{\gamma_*\mu}{2\hat{k}'^2}$. Again, we assume that the area of D_0 is small enough such that the first Dirichlet eigenvalue for $-\Delta$ in D_0 is greater than $\hat{k}'$. (We recall that $\hat{k}'$ is the first transmission eigenvalue of the interior transmission problem for B_r with $A = \frac{\gamma_*}{2}$ and $n = 1$ given in (4.16).) We denote by $\hat{w}$ and $\hat{v}$ the eigenvectors corresponding to $\hat{k}'$ and set $\hat{u} := \hat{w} - \hat{v} \in H_0^1(B_r)$. From the equation satisfied by $\hat{v}$ and using the fact that $\hat{u} = 0$ on ∂B_r and $\hat{v} = \hat{w} - \hat{u}$ we first have

$$0 = \int_{B_r} \Big(\nabla\hat{v}\cdot\nabla\overline{\hat{u}} - \hat{k}'^2\hat{v}\overline{\hat{u}}\Big)\,dx = \int_{B_r} \Big(\nabla\hat{w}\cdot\nabla\overline{\hat{u}} - \hat{k}'^2\hat{w}\overline{\hat{u}} - |\nabla\hat{u}|^2 + \hat{k}'^2|\hat{u}|^2\Big)\,dx.$$

On the other hand, replacing $\hat{v}$ by $\hat{w} - \hat{u}$ in the variational formulation satisfied by $\hat{v}$ and $\hat{w}$ we have

$$\int_{B_r} \Big(\nabla\hat{u}\cdot\nabla\overline{\varphi} - \hat{k}'^2\hat{u}\overline{\varphi}\Big)\,dx = \int_{B_r} \Big(1 - \frac{\gamma_*}{2}\Big)\nabla\hat{w}\cdot\nabla\overline{\varphi}dx$$

for all $\varphi \in H^1(B_r)$. In particular for $\varphi = \hat{w}$, we obtain

$$\int_{B_r} \Big(\nabla\hat{w}\cdot\nabla\overline{\hat{u}} - \hat{k}'^2\hat{w}\overline{\hat{u}}\Big)\,dx = \int_{B_r} \Big(1 - \frac{\gamma_*}{2}\Big)|\nabla\hat{w}|^2dx. \qquad (4.29)$$

Combining the above equations, we finally obtain

$$\int_{B_r} \left(\Big(1 - \frac{\gamma_*}{2}\Big)|\nabla\hat{w}|^2 - |\nabla\hat{u}|^2 + \hat{k}'^2\hat{u}|^2\right)\,dx = 0. \qquad (4.30)$$

Now we denote by $\tilde{u}$ the extension of $\hat{u}$ by zero to all of $D\backslash\overline{D}_0$. Since $\tilde{u} \in H^1_\Sigma(D\backslash\overline{D}_0)$, we can define $\tilde{v} := v_{\tilde{u}}$ the corresponding solution to

$$\begin{cases} \nabla\cdot(I-A)\nabla v + \hat{k}'^2(1-n)v = \nabla\cdot A\nabla\tilde{u} + \hat{k}' n\tilde{u} & \text{in } D\backslash\overline{D}_0 \\ \nu\cdot(I-A)\nabla v = \nu\cdot A\nabla\tilde{u} & \text{on } \Gamma \\ v = -\tilde{u} = 0 & \text{on } \Sigma \end{cases} \tag{4.31}$$

and we set $\tilde{w} := \tilde{u} + \tilde{v}$. We first remark that replacing $\tilde{v}$ by $\tilde{w} - \tilde{u}$ in (4.15), we have

$$\begin{aligned} \int_{D\backslash\overline{D}_0} \Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\varphi} - \hat{k}'^2(n-1)\tilde{w}\overline{\varphi}\Big)\,dx &= -\int_{D\backslash\overline{D}_0} \Big(\nabla\tilde{u}\cdot\nabla\overline{\varphi} - \hat{k}'^2\tilde{u}\overline{\varphi}\Big)\,dx \\ &= -\int_{B_r} \Big(\nabla\hat{u}\cdot\nabla\overline{\varphi} - \hat{k}'^2\hat{u}\overline{\varphi}\Big)\,dx \\ &= \int_{B_r} \Big(\frac{\gamma_*}{2} - 1\Big)\nabla\hat{w}\cdot\nabla\overline{\varphi}dx \end{aligned}$$

In particular, for $\varphi = \tilde{w} \in H^1_\Sigma(D\backslash\overline{D}_0)$, we obtain

$$\int_{D\backslash\overline{D}_0} \Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}'^2(n-1)|\tilde{w}|^2\Big)\,dx = \int_{B_r} \Big(\frac{\gamma_*}{2} - 1\Big)\nabla\hat{w}\cdot\nabla\overline{\tilde{w}}dx. \tag{4.32}$$

The Cauchy-Schwarz inequality applied to the right-hand side of (4.32) gives

$$\begin{aligned} &\int_{D\backslash\overline{D}_0} \Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}'^2(n-1)|\tilde{w}|^2\Big)\,dx = \int_{B_r} \Big(\frac{\gamma_*}{2} - 1\Big)\nabla\hat{w}\cdot\nabla\overline{\tilde{w}}dx \\ &\le \left(\int_{B_r} \Big(\frac{\gamma_*}{2} - 1\Big)|\nabla\hat{w}|^2dx\right)^{1/2} \left(\int_{B_r} \Big(\frac{\gamma_*}{2} - 1\Big)|\nabla\tilde{w}|^2dx\right)^{1/2} \\ &= \left(\int_{B_r} \Big(\frac{\gamma_*}{2} - 1\Big)|\nabla\hat{w}|^2dx\right)^{1/2} \left(\int_{B_r} \Big((\gamma_*-1)|\nabla\tilde{w}|^2 - \frac{\gamma_*}{2}|\nabla\tilde{w}|^2\Big)\,dx\right)^{1/2} \\ &\le \left(\int_{B_r} \Big(\frac{\gamma_*}{2} - 1\Big)|\nabla\hat{w}|^2dx\right)^{1/2} \left(\int_{D\backslash\overline{D}_0} \Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}'^2(n-1)|\tilde{w}|^2\Big)\,dx\right)^{1/2} \end{aligned}$$

and finally

$$\int_{D\backslash\overline{D}_0} \Big((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{k}'^2(n-1)|\tilde{w}|^2\Big)\,dx \le \int_{B_r} \Big(\frac{\gamma_*}{2} - 1\Big)|\nabla\hat{w}|^2dx. \tag{4.33}$$

Therefore, we obtain

$$
\begin{aligned}
\langle L_{\hat{k}'}\tilde{u},\tilde{u}\rangle_{H^1(D\backslash\overline{D}_0)} &= -\int_{D\backslash\overline{D}_0}\left(\nabla\tilde{v}\cdot\nabla\overline{\tilde{u}}-\hat{k}'^2\tilde{v}\tilde{u}\right)dx\\
&= -\int_{D\backslash\overline{D}_0}\left(\nabla\tilde{w}\cdot\nabla\overline{\tilde{u}}-\hat{k}'^2\tilde{w}\tilde{u}-|\nabla\tilde{u}|^2+\hat{k}'^2|\tilde{u}|^2\right)dx\\
&= \int_{D\backslash\overline{D}_0}\left((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}}-\hat{k}'^2(n-1)|\tilde{w}|^2+|\nabla\tilde{u}|^2-\hat{k}'^2|\tilde{u}|^2\right)dx\\
&= \int_{D\backslash\overline{D}_0}\left((A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}}-\hat{k}'^2(n-1)|\tilde{w}|^2\right)dx\\
&\quad+\int_{B_r}\left(|\nabla\hat{u}|^2-\hat{k}'^2|\hat{u}|^2\right)dx\\
&\le \int_{B_r}\left(\left(\frac{\gamma_*}{2}-1\right)|\nabla\hat{w}|^2+|\nabla\hat{u}|^2-\hat{k}'^2|\hat{u}|^2\right)dx=0.
\end{aligned}
$$

Thus we can conclude that if $1 < n_* < n < n^* \le 1+\frac{\gamma_*\mu}{2\hat{k}'^2}$ there exists a transmission eigenvalue in $(0,\hat{k}']$. □

Remark 4.2.3. *As the area of D_0 goes to 0, in the case when $0 < n_* < n^* < 1$ it is possible to prove the existence of more and more transmission eigenvalues. In this case since the first Dirichlet eigenvalue for $-\Delta$ in D_0 goes to infinity one can take r such that $M(r)$ disjoint balls of radius r are included in $D\backslash\overline{D}_0$ and no Dirichlet eigenvalues are in $[0,\ \hat{k}]$. This way the assumption (b) of Theorem 4.2.9 is satisfied in a $M(r)$-dimensional subspace of $H^1_\Gamma(D\backslash\overline{D}_0)$ and thus there exists $M(r)$ transmission eigenvalues in $[0,\ \hat{k}]$ (counting multiplicity). The smaller the area of D_0 is the smaller r can be chosen and the larger $M(r)$ becomes. The same remark holds true for the case when $1 < n_*$ provided that n^* is small enough, more specifically $n^* < 1+\frac{\gamma_*\mu}{2\hat{k}'^2}$.*

Remark 4.2.4. *The entire argument in the proof of Theorem 4.2.10 holds true if $\hat{k}$ or $\hat{k}'$ is the first transmission eigenvalue of* (4.22) *or* (4.16)*, respectively, where B_r is replaced with an arbitrary region $B \subset D\backslash\overline{D}_0$ (such transmission eigenvalues are known to exists [19]). Depending on the geometry of $D\backslash\overline{D}_0$ one can choose B such that the corresponding $\hat{k}$ or $\hat{k}'$ are smaller than the ones for the ball B_r (see the estimates on the first transmission eigenvalue in [16], [17] and [19]) which would enable to prove the existence of at least one transmission eigenvalue for larger D_0.*

Chapter 5

Surface integral formulation of the interior transmission problem - The case where the contrast changes sign

This chapter is dedicated to the study of the interior transmission problem using a surface integral equation formulation. The main original motivation behind this study was the design of a numerical method to solve ITP in the case of piece-wise constant index of refraction and compute transmission eigenvalues for general geometries. This numerical study is presented in Chapter 6. We adopted the integral equation approach since an efficient forward solver for electromagnetic scattering problems based on this technique is already developed at CERFACS, namely the CESC software [1].

Then, it turned out that the surface integral formulation of the problem also presents some theoretical interests. For instance, establishing the equivalence between this formulation and the original problem in the case of transverse magnetic polarizations requires the introduction of non standard results on potentials. This is due to the fact that the space of (variational) solutions (as already indicated in Chapter 1) is $L^2(D)$ with Laplacien in $L^2(D)$, where D is the domain of the inclusion. Hence the natural spaces for solutions to the integral equation would be $H^{-1/2}(\partial D) \times H^{-3/2}(\partial D)$, since the unknowns correspond with the traces and conormal traces of the (variational) solutions. Regularity, continuity and coercivity properties of the used potentials in those trace spaces are one the main novel ingredients of our study. We relied in particular on the theory of pseudo-differential operators to derive regularity properties. Then by using appropriate density arguments, classical traces formula are generalized to potentials with densities having weaker regularities. Coercivity properties of the potentials are analysed in the cases of purely imaginary wavenumbers. Let us already emphasize here that an alternative (theoretical) approach to treat this case would have been to consider potentials with kernels related to the fundamental solution of the biharmonic operator. However, this approach would have been less intuitive (in the case of ITP) and less appropriate for the numerical considerations of next chapter.

The second, and probably more important, interest of this integral equation formulation is related to the study of ITP for relaxed assumptions on the sign of the contrasts. More specifically we allow the difference between the index of refraction of the inclusion and of the background to change sign inside D. The variational method, employed in Chapter 3 to treat the case of inclusions with cavities, fails to establish the Fredholm nature of the ITP in those situations. Using the surface integral approach we are able to prove that the ITP is of Fredholm type if the contrast is constant and positive (or negative) only in the neighborhood of the boundary. We deduce in particular that the set of transmission eigenvalues is still discrete in some specific cases where uniqueness can be shown for a particular wavenumber. The main drawback of this method is that it can only treat the question of discreteness of the set of transmission eigenvalues and not existence. This type of results is similar to the one recently established by Sylvester [50] in the transverse magnetic case and the one by Chesnel-Bonnet-Ben Dhia-Haddar [5] in the case of anisotropic scalar case. The method in [50] is based on the notion of upper triangular compact operators, but the result can also be derived using classical analytical Fredholm theory and the use of appropriate inf-sup conditions, as shown in Kirsch [39]. The technique in [5] is based on the notion of T-coercivity, used in Section 4.2 of Chapter 4. Let us also indicate that in the case of anisotropic scalar case, results on the discreteness of the set of transmission eigenvalues have been obtained by [42] with weaker conditions. Roughly speaking, in that work, one only needs definite sign of the contrasts on a neighborhood of a point on the boundary, but the imaginary part of the refractive index cannot be identically zero.

In this chapter, we shall only consider the scalar problem. However the technique is extendable to the full Maxwell problem. The latter will be only presented in a formal setting in Chapter 6 for the sake of numerical experimentation and validation.

Consider a simply connected and bounded region $D \subset \mathbb{R}^d$, ($d = 2$ or $d = 3$) with smooth boundary $\Gamma := \partial D$. We recall that the general form of the scalar isotropic interior transmission problem can be written as

$$\begin{cases} \nabla \cdot \dfrac{1}{\mu(x)} \nabla w + k^2 n(x) w = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ w = v & \text{on } \Gamma, \\ \dfrac{1}{\mu} \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} & \text{on } \Gamma, \end{cases} \tag{5.1}$$

where $v, w \in H^1(D)$ if $\mu \neq 1$ and $v, w \in L^2(D)$ such that $u := w - v \in H^2(D)$ if $\mu = 1$.

In a first part, in order to introduce the surface integral equation method, we shall treat the simple case where n and μ are constant. We distinguish the case $\mu \neq 1$, for which the basic tools are the same as for classical transmission problems, from the case $\mu = 1$ and $n \neq 1$, where new ingredients have to be used. We then consider the more general cases where the latter assumptions hold only on a neighborhood of the boundary.

The outline of this chapter is the following. In section 5.1, we recall some classical results from potential theory associated with the Helmholtz operator. These results are used to treat the case $\mu \neq 1$ presented in Section 5.2 and lead to the discreteness of the set of transmission eigenvalues. After extended regularity results on the potentials for densities in $H^{-3/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, we treat in Section 5.3 the case $\mu = 1$ and n constant and also show the discreteness of the set of transmission eigenvalues. Finally, in Section 5.4, we show the same type a result for $\mu \neq 1$ and n piece-wise constant.

5.1 Some classical results from potential theory

We denote by $D^+ := \{\mathbb{R}^d \setminus \overline{D}\} \cap B_R$ a bounded exterior domain, where B_R denotes a ball containing D and denote by $D^- := D$ the interior domain. Let ν be the unit normal to Γ directed to the exterior of D.

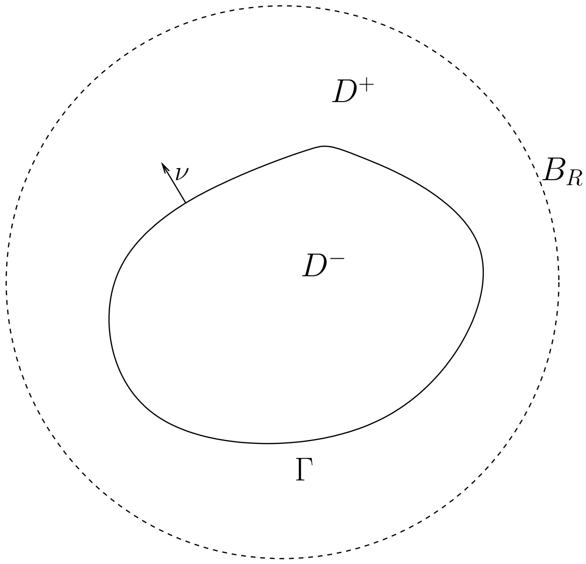


Figure 5.1: Domains and notation

If u is a regular function defined in $D^+ \cup D^-$, we denote by

$$u^\pm(x_\Gamma) := \lim_{h\downarrow 0^\pm} u(x_\Gamma + h\nu(x_\Gamma)), \qquad x_\Gamma \in \Gamma,$$
$$\frac{\partial u^\pm}{\partial \nu}(x_\Gamma) := \lim_{h\downarrow 0^\pm} \nabla u(x_\Gamma + h\nu(x_\Gamma)) \cdot \nu(x_\Gamma), \qquad x_\Gamma \in \Gamma.$$

Moreover, we define the jumps on Γ

$$[u]_\Gamma(x_\Gamma) := u^+(x_\Gamma) - u^-(x_\Gamma) \qquad x_\Gamma \in \Gamma,$$
$$\left[\frac{\partial u}{\partial \nu}\right]_\Gamma (x_\Gamma) := \frac{\partial u^+}{\partial \nu}(x_\Gamma) - \frac{\partial u^-}{\partial \nu}(x_\Gamma) \qquad x_\Gamma \in \Gamma.$$

We shall keep this notation for non regular functions if these trace operators can be continously extended (in an appropriate function space) to these functions.

Let Φ_k be the outgoing Green function associated with the Helmholtz operator with wavenumber $k \in \mathbb{C}$ with non negative real and imaginary parts. We recall that

$$\Phi_k(x,y) = \frac{e^{\mathrm{i}k|x-y|}}{4\pi|x-y|} \text{ for } d = 3 \text{ and } \Phi_k(x,y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|) \text{ for } d = 2,$$

where $H_0^{(1)}$ denotes the Hankel function of the first kind of order 0. We then define the single and double layer potentials for regular densities φ, respectively by

$$
\begin{aligned}
(\mathrm{SL}_k\varphi)(x) &:= \int_\Gamma \Phi_k(x,y)\varphi(y)ds(y), \qquad x \in \mathbb{R}^d \setminus \Gamma \\
(\mathrm{DL}_k\varphi)(x) &:= \int_\Gamma \frac{\partial \Phi_k}{\partial \nu(y)}(x,y)\varphi(y)ds(y) \qquad x \in \mathbb{R}^d \setminus \Gamma.
\end{aligned}
$$

The following theorem summarizes some classical results from potential theory that can be found for instance in [44], [46] or [35].

Theorem 5.1.1. *The single-layer potential* $\mathrm{SL}_k : H^{-1/2}(\Gamma) \to H^1(D^\pm)$ *and the double layer potential* $\mathrm{DL}_k : H^{1/2}(\Gamma) \to H^1(D^\pm)$ *are bounded and give rise to bounded linear operators*

$$
\begin{aligned}
&S_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), && K_k : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \\
&K'_k : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), && T_k : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma),
\end{aligned}
$$

such that for all $\varphi \in H^{-1/2}(\Gamma)$ *and* $\psi \in H^{1/2}(\Gamma)$,

$$
\begin{cases}
(\mathrm{SL}_k\varphi)^\pm = S_k\varphi & \text{and} \quad (\mathrm{DL}_k\psi)^\pm = K_k\psi \pm \dfrac{1}{2}\psi \qquad \text{in } H^{1/2}(\Gamma), \\
\dfrac{\partial(\mathrm{SL}_k\varphi)}{\partial\nu}^\pm = K'_k\varphi \mp \dfrac{1}{2}\varphi & \text{and} \quad \dfrac{\partial(\mathrm{DL}_k\psi)}{\partial\nu}^\pm = T_k\psi \qquad \text{in } H^{-1/2}(\Gamma).
\end{cases}
$$

We recall that for regular densities φ and ψ, the surface potentials S_k, K_k, K'_k and T_k can be expressed as

$$
\begin{aligned}
(S_k\varphi)(x) &= \int_\Gamma \Phi_k(x,y)\varphi(y)ds(y), \\
(K_k\psi)(x) &= \int_\Gamma \frac{\partial\Phi_k}{\partial\nu(y)}(x,y)\psi(y)ds(y), \\
(K'_k\varphi)(x) &= \int_\Gamma \frac{\partial\Phi_k}{\partial\nu(x)}(x,y)\varphi(y)ds(y), \\
(T_k\psi)(x) &= \lim_{\varepsilon\to 0}\int_{\Gamma, |y-x|>\varepsilon} \frac{\partial^2\Phi_k}{\partial\nu(y)\nu(x)}(x,y)\psi(y)ds(y)
\end{aligned}
$$

for all $x \in \Gamma$. We also recall that K'_k is the transpose of K_k in the sense that for regular densities φ and ψ on Γ,

$$
\int_\Gamma K_k\varphi\,\psi\,ds = \int_\Gamma \varphi\,K'_k\psi\,ds.
$$

It is also well known that since the principal singular term in the kernels of these surface potentials cancel in the difference between two potentials, then this difference defines compact operators. We need in the sequel precise information on the exact regularity of this difference. We shall use for that the theory of pseudo-differential operators (as presented in [35]). We provide in Appendix E some of the key results from this theory that will be used here.

First, one needs to extend the normal ν inside D. To this end, one consider the cutoff function $\chi \in \mathcal{C}^infty(D)$ such that $\chi = 1$ in $\mathcal{O}$ and $\chi = 0$ in $D \backslash \overline{\mathcal{O}}'$ where $\mathcal{O}$ and $\mathcal{O}'$ are two neighborhoods of the boundary Γ such that $\mathcal{O} \subset \mathcal{O}'$.

Let us now introduce the volumetric potentials

$$
\begin{aligned}
(\widehat{\mathrm{SL}}_k\varphi)(x) &:= \int_D \Phi_k(x,y)\varphi(y)dy, \qquad x \in \mathbb{R}^d,\\
(\widehat{\mathrm{DL}}_k\varphi)(x) &:= \int_D \frac{\partial \Phi_k}{\partial \nu(y)}(x,y)\varphi(y)dy \qquad x \in \mathbb{R}^d,
\end{aligned}
$$

which define pseudo-differential operators of order -2 and -1 respectively. We then define for two given wavenumbers k and k' (with non negative real parts)

$$
\begin{aligned}
\widehat{\mathcal{SL}}_{k,k'} &:= \widehat{\mathrm{SL}}_k - \widehat{\mathrm{SL}}_{k'}, \qquad \mathcal{SL}_{k,k'} := \mathrm{SL}_k - \mathrm{SL}_{k'},\\
\widehat{\mathcal{DL}}_{k,k'} &:= \widehat{\mathrm{DL}}_k - \widehat{\mathrm{DL}}_{k'}, \qquad \mathcal{DL}_{k,k'} := \mathrm{DL}_k - \mathrm{DL}_{k'}.
\end{aligned}
$$

Theorem 5.1.2. *The pseudo-differential operators $\widehat{\mathcal{SL}}_{k,k'}$ and $\widehat{\mathcal{DL}}_{k,k'}$ are respectively of order -4 and -3.*

Proof. First, we consider the case $d = 3$. We use the power series of the exponential

$$e^t = \sum_{p=0}^{\infty} \frac{t^n}{n!}.$$

Let $k \neq k'$ and denote $z = x - y$. Then, the kernel of $\widehat{\mathcal{SL}}_{k,k'}$ have the expansion

$$
\begin{aligned}
a(x,z) &:= \frac{e^{\mathrm{i}k|z|} - e^{\mathrm{i}k'|z|}}{4\pi|z|}\\
&= \frac{\mathrm{i}}{4\pi}(k-k') - \frac{1}{4\pi}\sum_{j=0}^{\infty} \frac{\mathrm{i}^j}{(j+2)!}(k^{j+2} - k'^{j+2})|z|^{j+1}\\
&= \frac{\mathrm{i}}{4\pi}(k-k') + \sum_{j=0}^{\infty} a_{j+1}(x,z)
\end{aligned}
$$

where

$$a_{j+1}(x,z) := \frac{-\mathrm{i}^j}{4\pi(j+2)!}(k^{j+2} - k'^{j+2})|z|^{j+1}, \quad \text{for all } j \geq 0,$$

which satisfies

$$a_p(x,tz) = t^p a(x,z).$$

From Theorem E.2.1, we deduce that

$$\widehat{\mathrm{SL}}_{k,k'}\varphi(x) = \int_D a(x, x-y)\varphi(y)dy$$

where a is a pseudo homogeneous kernel of degree 1 is a pseudo-differential operator of order -4.

Using the power series of the exponential, we have

$$\begin{aligned}\frac{\partial \Phi_k(x,y)}{\partial \nu(x)} &= \frac{\mathrm{i}k}{4\pi|x-y|^2}(x-y)\cdot\nu(x)\left(1-\frac{1}{\mathrm{i}k|x-y|}\right)e^{\mathrm{i}k|x-y|}\\ &= \frac{-1}{4\pi}\frac{z}{|z|^3}\cdot\nu(x) - \frac{k^2}{8\pi}\frac{z}{|z|}\cdot\nu(x) - \frac{\mathrm{i}k^3}{12\pi}z\cdot\nu(x)\\ &\quad + \frac{z\cdot\nu(x)}{4\pi}\sum_{p=1}^{\infty}\frac{p+2}{(p+3)!}(\mathrm{i}k)^{p+3}|z|^p.\end{aligned}$$

Consequently, the kernel of $\widehat{\mathcal{DL}}_{k,k'}$ have the expansion

$$\begin{aligned}b(x,z) &:= \frac{\partial \Phi_k(x,y)}{\partial \nu(x)} - \frac{\partial \Phi_{k'}(x,y)}{\partial \nu(x)}\\ &= \frac{k'^2-k^2}{8\pi}\frac{z}{|z|}\cdot\nu(x) + \frac{\mathrm{i}}{12\pi}(k'^3-k^3) + \frac{z\cdot\nu(x)}{4\pi}\sum_{p=1}^{\infty}\frac{(p+2)\mathrm{i}^{p+1}}{(p+3)!}(k^{p+3}-k'^{p+3})|z|^p\\ &= \frac{\mathrm{i}}{12\pi}(k'^3-k^3) + \sum_{j=0}^{\infty} b_j(x,z)\end{aligned}$$

where

$$b_j(x,z) := \begin{cases} \frac{k'^2-k^2}{8\pi}\frac{z\cdot\nu(x)}{|z|} & \text{if } j=0\\ 0 & \text{if } j=1\\ \frac{-\mathrm{i}^p z\cdot\nu(x)}{4\pi}\frac{j+1}{(j+2)!}(k'^{j+2}-k^{j+2})|z|^{j-1} & \text{if } j\geq 2\end{cases}$$

which satisfies

$$b_p(x,tz) = t^p b(x,z).$$

From Theorem E.2.1, we deduce that

$$\widehat{\mathcal{DL}}^*_{k,k'}\varphi(x) = \int_D b(x,x-y)\varphi(y)dy$$

where b is a pseudo-homogeneous kernel of degree 0, is a pseudo-differential operator of order -3. Consequently, from Theorem E.2.2, $\widehat{\mathcal{DL}}_{k,k'}$ is also a pseudo-differential operator of order -3.

Now, for $d=2$, the kernel of $\widehat{\mathcal{SL}}_{k,k'}$ is

$$a(x,z) := \frac{\mathrm{i}}{4}\left(H_0^{(1)}(k|z|) - H_0^{(1)}(k'|z|)\right).$$

From [2], for all $t\in\mathbb{C}$ we have

$$H_0^{(1)}(t) = \sum_{p=0}^{\infty}\frac{(-1)^p}{p!^2}\left(\frac{t}{2}\right)^{2p}\theta(p) + \frac{2\mathrm{i}}{\pi}\ln(t)\sum_{p=0}^{\infty}\frac{(-1)^p}{p!^2}\left(\frac{t}{2}\right)^{2p}$$

where

$$\theta(p) := 1 + \frac{2\mathrm{i}}{\pi}C - \frac{2\mathrm{i}}{\pi}\sum_{m=1}^{p}\frac{1}{m}.$$

Then,

$$\begin{aligned}
a(x,z) &= \frac{\mathrm{i}}{4}\sum_{p=0}^{\infty}\frac{(-1)^{p+1}}{(p+1)!^2}\left(k^{2p+2}-k'^{2p+2}\right)\left(\frac{|z|}{2}\right)^{2p+2}\theta(p)) \\
&+ \frac{1}{2\pi}\sum_{p=0}^{\infty}\frac{(-1)^{p}}{(p+1)!^2}\left(k^{2p+2}\ln(k)-k'^{2p+2}\ln(k')\right)\left(\frac{|z|}{2}\right)^{2p+2} \\
&+ \frac{1}{2\pi}(\ln k - \ln k') \\
&+ \frac{1}{2\pi}\ln|z|\sum_{p=0}^{\infty}\frac{(-1)^{p}}{(p+1)!^2}\left(k^{2p+2}-k'^{2p+2}\right)\left(\frac{|z|}{2}\right)^{2p+2} \\
&= f(x,z) + \sum_{j=0}^{\infty} p_{j+2}(x,z)\ln|z|
\end{aligned}$$

where $f \in C^\infty(D \times \mathbb{R}^d)$ and

$$p_{j+2}(x,z) = \begin{cases} 0 & \text{if } j \text{ is odd,} \\ \frac{1}{2\pi}\frac{(-1)^{p+1}}{(p+1)!^2}\left(k^{j+2}-k'^{j+2}\right)\left(\frac{|z|}{2}\right)^{j+2} & \text{if } j = 2p. \end{cases}$$

The function p_q satisfies $p_q(x,tz) = t^q p_q(x,z)$ and consequently the kernel of $\widehat{\mathcal{SL}}_{k,k'}$ is a pseudo-homogeneous kernel of degree 2. From Theorem E.2.1, we deduce that $\widehat{\mathcal{SL}}_{k,k'}$ is a pseudo-differential operator of order -4.

Now remark that

$$\frac{\partial}{\partial\nu(x)}\left(H_0^{(1)}(k|x-y|)\right) = k\frac{z\cdot\nu(x)}{|z|}H_0^{(1)'}(k|x-y|)$$

and from [2], for all $t \in \mathbb{C}$ we have

$$H_0^{(1)'}(t) = \sum_{p=1}^{\infty}\frac{(-1)^p}{p!^2}p\left(\frac{t}{2}\right)^{2p-1}\tilde{\theta}(p) + \frac{2\mathrm{i}}{\pi}\ln(t)\sum_{p=1}^{\infty}\frac{(-1)^p}{p!^2}p\left(\frac{t}{2}\right)^{2p-1}.$$

where $\tilde{\theta}(p) = \theta(p) + \frac{\mathrm{i}}{\pi}$. Then, the kernel of $\widehat{\mathcal{DL}}_{k,k'}$ is

$$\begin{aligned}
b(x,z) &:= \frac{\partial}{\partial\nu(x)}\left(H_0^{(1)}(k|z|) - H_0^{(1)}(k'|z|)\right) \\
&= z\cdot\nu(x)\left(\frac{\mathrm{i}}{4}\sum_{p=0}^{\infty}\frac{(-1)^{p+1}}{p!(p+1)!}\left(k^{2p+2}-k'^{2p+2}\right)\frac{|z|^{2p}}{2^{2p+1}}\tilde{\theta}(p)\right. \\
&\quad + \frac{1}{2\pi}\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!(p+1)!}\left(k^{2p+2}\ln k-k'^{2p+2}\ln k'\right)\frac{|z|^{2p}}{2^{2p+1}} \\
&\quad \left.+\frac{1}{2\pi}\ln|z|\sum_{p=0}^{\infty}\frac{(-1)^{p}}{p!(p+1)!}\left(k^{2p+2}-k'^{2p+2}\right)\frac{|z|^{2p}}{2^{2p+1}}\right) \\
&= f(x,z) + \sum_{j=0}^{\infty} p_{j+1}(x,z)\ln|z|
\end{aligned}$$

where

$$f(x,z) = \frac{\mathrm{i}}{4}\sum_{p=0}^{\infty}\frac{(-1)^{p+1}}{p!(p+1)!}\left(k^{2p+2}-k'^{2p+2}\right)\frac{|z|^{2p}}{2^{2p+1}}z\cdot\nu(x)\tilde{\theta}(p+1)$$
$$+\frac{1}{2\pi}\sum_{p=0}^{\infty}\frac{(-1)^{p+1}}{p!(p+1)!}\left(k^{2p+2}\ln(k)-k'^{2p+2}\ln(k')\right)\frac{|z|^{2p}}{2^{2p+1}}z\cdot\nu(x)$$

is a function in $C^\infty(D\times D)$, and

$$p_{j+1}(x,z) = \begin{cases} 0 & \text{if } j \text{ is odd,}\\ \dfrac{1}{2\pi}\dfrac{(-1)^p}{p!(p+1)!}\left(k^{j+2}-k'^{j+2}\right)\dfrac{|z|^j}{2^{j+1}}z\cdot\nu(x) & \text{if } j=2p. \end{cases}$$

The function p_q satisfies $p_q(x,tz) = t^q p_q(x,z)$ and consequently the kernel of $\widehat{\mathcal{DL}}^*_{k,k'}$ is a pseudo-homogeneous kernel of degree 1. From Theorem E.2.1, we deduce that $\widehat{\mathcal{DL}}_{k,k'}$ is a pseudo-differential operator of order -3.

$\square$

We then deduce from the application of [35, Theorem 8.5.8] (see also Theorem E.2.3 in Appendix E) the following mapping properties .

Corollary 5.1.3. *The operators $\mathcal{SL}_{k,k'}: H^{-1/2}(\Gamma)\to H^3(D)$ and $\mathcal{DL}_{k,k'}: H^{1/2}(\Gamma)\to H^3(D)$ are continuous.*

Combining this result with classical trace theorems and definitions contained in Theorem 5.1.1 we deduce the following regularity properties for the differences of surface potentials.

Corollary 5.1.4. *Let k and k' be two complex numbers with non negative real parts. Then the mappings*

$$\begin{array}{lcl} S_k - S_{k'} & : & H^{-1/2}(\Gamma)\to H^{5/2}(\Gamma),\\ K_k - K_{k'} & : & H^{1/2}(\Gamma)\to H^{5/2}(\Gamma),\\ K'_k - K'_{k'} & : & H^{-1/2}(\Gamma)\to H^{3/2}(\Gamma),\\ T_k - T_{k'} & : & H^{1/2}(\Gamma)\to H^{3/2}(\Gamma), \end{array}$$

are continuous.

5.2 Surface integral equation formulation of ITP in the case $\mu \neq 1$

We consider in this section the simpler case where μ and n are constant in D and $\mu\neq 1$. We first write an equivalent formulation of the problem in terms of surface integral equations then prove that the operator associated with this formulation if Fredholm of index 0. Let k be the wavenumber appearing in system (5.1), we shall denote

$$k_0 := k \quad \text{and} \quad k_1 := \sqrt{\mu n}k.$$

5.2.1 Derivation of the surface integral equation

Consider $(v, w) \in H^1(D) \times H^1(D)$ a solution to (5.1) and set

$$\alpha := \frac{\partial v}{\partial \nu}|_\Gamma = \frac{1}{\mu}\frac{\partial w}{\partial \nu}|_\Gamma \in H^{-1/2}(\Gamma)$$

and

$$\beta := v|_\Gamma = w|_\Gamma \in H^{1/2}(\Gamma).$$

Since v and w satisfy

$$\Delta v + k_0^2 v = 0 \text{ and } \Delta w + k_1^2 w = 0 \text{ in } D,$$

then it is well known [26] that these solutions can be expressed using the following integral representation

$$\begin{array}{ll} v = \mathrm{SL}_{k_0}\alpha - \mathrm{DL}_{k_0}\beta & \text{in } D, \\ w = \mu\mathrm{SL}_{k_1}\alpha - \mathrm{DL}_{k_1}\beta & \text{in } D. \end{array} \tag{5.2}$$

From the boundary conditions of (5.1) $w = v$ and $\frac{\partial v}{\partial \nu}|_\Gamma = \frac{1}{\mu}\frac{\partial w}{\partial \nu}|_\Gamma$ and the jump properties of the potentials recalled in Theorem 5.1.1, one easily verifies that α and β satisfy

$$Z(k)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \tag{5.3}$$

where

$$Z(k) := \begin{pmatrix} \mu S_{k_1} - S_{k_0} & -K_{k_1} + K_{k_0} \\ -K'_{k_1} + K'_{k_0} & 1/\mu T_{k_1} - T_{k_0} \end{pmatrix}.$$

Equation (5.3) forms the surface integral equation formulation of (5.1). The equivalence between the two formulations is ensured after guaranteing that non trivial solutions of (5.3) define, through (5.2), non trivial solutions to (5.1). Using Green's formula it is easily seen that solutions to (5.1) correspond with non radiating solutions. More precisely, if we define the far field operators $P_i^\infty : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to L^2(\Omega)$ for $i = 0, 1$ by

$$\begin{aligned} P_0^\infty(\alpha, \beta)(\hat{x}) &= \frac{1}{4\pi}\int_\Gamma \left(\beta(y)\frac{\partial e^{-\mathrm{i}k\hat{x}\cdot y}}{\partial \nu(y)} - \alpha(y)e^{-\mathrm{i}k\hat{x}\cdot y}\right) ds(y), \\ P_1^\infty(\alpha, \beta)(\hat{x}) &= \frac{1}{4\pi}\int_\Gamma \left(\beta(y)\frac{\partial e^{-\mathrm{i}k_1\hat{x}\cdot y}}{\partial \nu(y)} - \frac{1}{\mu}\alpha(y)e^{-\mathrm{i}k_1\hat{x}\cdot y}\right) ds(y), \end{aligned}$$

then we have

$$P_0^\infty(\alpha, \beta) = 0 \text{ and } P_1^\infty(\alpha, \beta) = 0.$$

The following theorem indicates that one of the latter conditions is sufficient to ensure the equivalence between the two formulations of ITP (see also Remark 5.2.2).

Theorem 5.2.1. *Assume that the wavenumber k is real and positive. The three following assertions are equivalent:*

(i) *There exists $(v, w) \in H^1(D) \times H^1(D)$ a non trivial solution to* (5.1)

(ii) *There exists* $(\alpha,\beta)\neq(0,0)$ *in* $H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$ *such that*

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=0 \text{ and } P_0^\infty(\alpha,\beta)=0.$$

(iii) *There exists* $(\alpha,\beta)\neq(0,0)$ *in* $H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$ *such that*

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=0 \text{ and } P_1^\infty(\alpha,\beta)=0.$$

Proof. It only remains to show that (ii) implies (i) and that (iii) implies (i). Assume that there exist $\alpha\in H^{-1/2}(\Gamma)$ and $\beta\in H^{1/2}(\Gamma)$ satisfying

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=0.$$

We define

$$v:=\mathrm{SL}_{k_0}\alpha-\mathrm{DL}_{k_0}\beta \text{ and } w:=\mu\mathrm{SL}_{k_1}\alpha-\mathrm{DL}_{k_1}\beta \text{ in } \mathbb{R}^d\setminus D.$$

The regularity of the single and double layer potentials shows that v and w are in $H^1(D)$ and they satisfy $\Delta v+k^2v=0$ and $\nabla\cdot\dfrac{1}{\mu}\nabla w+k^2nw=0$ in D.

First assume that $P_0^\infty(\alpha,\beta)=0$. We shall show that $v\neq 0$. From Rellich's lemma, we deduce that $v=0$ in $\mathbb{R}^d\backslash D$. Assume that $v=0$ also in D. We have in particular that

$$[v]_\Gamma=\left[\frac{\partial v}{\partial\nu}\right]_\Gamma=0$$

and from the jump properties of the single and double layer potentials we also have that

$$[v]_\Gamma=-\beta \text{ and } \left[\frac{\partial v}{\partial\nu}\right]_\Gamma=-\alpha.$$

This contradicts the fact that $(\alpha,\beta)\neq(0,0)$. Then $v\neq 0$ in D.

If we assume now that $P_1^\infty(\alpha,\beta)=0$ we can similarly show that $w\neq 0$. Indeed, from Rellich's lemma, we deduce that $w=0$ in $\mathbb{R}^d\backslash\bar{D}$. Now if we assume that $w=0$ also in D, we have in particular that

$$[w]_\Gamma=\left[\frac{\partial w}{\partial\nu}\right]_\Gamma=0.$$

From the expression of w and the jump properties of the single and double layer potentials, we also have

$$[w]_\Gamma=-\beta \quad\text{and}\quad \left[\frac{\partial w}{\partial\nu}\right]_\Gamma=-\mu\alpha$$

which contradicts the fact that $(\alpha,\beta)\neq(0,0)$. Then $w\neq 0$ in D. □

Remark 5.2.1. *Since* v *and* w *have the same Cauchy data on* Γ*, if either* v *or* w *is different from zero, then the other one is necessarily different from zero too.*

Remark 5.2.2. *Another possibility to ensure equivalence between the surface integral and volumetric formulations of ITP would have been to use the so called Calderòn projectors. More precisely, its is well-known (see for instance [44]) that the pairs* $(\alpha, \beta) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *that coincide with normal traces and traces of solutions* $u \in H^1(D)$ *to the Helmholtz equation* $\Delta u + k^2 u = 0$ *in* D *can be characterized as elements of the kernel of the Calderòn projector (for the exterior problem)*

$$P(k) = \begin{pmatrix} S_k & -K_k - I/2 \\ K_k' - I/2 & -T_k \end{pmatrix}.$$

We preferred to rather use the farfield operator since it is easier to handle in numerical applications and is also more convenient to use in the case of $\mu = 1$ *(studied in Section 5.3).*

5.2.2 Fredholm property of the operator $Z(k)$

In order to show the discreteness of the set of transmission eigenvalues, we want to use the analytic Fredholm theory. To this end, we need to decompose $Z(k)$ as the sum of a coercive operator and a compact operator. The following lemmas show that for any purely imaginary $k := \mathrm{i}\kappa$, with $\kappa \in \mathbb{R}$, the trace of the single layer potential and the normal derivative of the double layer potential are coercive on their corresponding spaces. The result of these lemmas are classical (see for instance [46, Section 33] for the case $\kappa = 0$), and their proof is given here for the reader convenience. A similar procedure will be used later for the case $\mu = 1$ and we found it useful to present the two proofs in order to make a parallel between both cases. We shall assume in the sequel that $\kappa \neq 0$.

Remark 5.2.3. *In the following, an operator* $A : H \to H'$ *is said to be coercive if*

$$|\langle Ax, x\rangle_{H,H'}| \geq C||x||_H^2$$

for all $x \in H$ *where* $\langle \cdot, \cdot \rangle_{H,H'}$ *denotes the duality pairing between* H *and its dual* H'.

Lemma 5.2.2. *The operator* $S_{\mathrm{i}\kappa} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ *is coercive.*

Proof. Let $\alpha \in H^{-1/2}(\Gamma)$. Let us consider the following problem find $u \in H^1(\mathbb{R}^d \setminus \Gamma)$ such that

$$\begin{cases} \Delta u - \kappa^2 u = 0 & \text{in } \mathbb{R}^d \backslash \Gamma, \\ [u]_\Gamma = 0 & \text{on } \Gamma, \\ \left[\dfrac{\partial u}{\partial \nu}\right]_\Gamma = -\alpha & \text{on } \Gamma. \end{cases}$$

The equivalent variational formulation is: find $u \in H^1(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} (\nabla u \cdot \nabla \overline{\varphi} + \kappa^2 u \overline{\varphi})\, dx = \int_\Gamma \alpha \overline{\varphi}\, ds(x) \tag{5.4}$$

for all $\varphi \in H^1(\mathbb{R}^d)$. The sesquilinear form of the left-hand side is clearly coercive and continuous while the antilinear form of the left-hand side is continuous by trace theorems.

The Lax-Milgram theorem ensures the existence of a unique solution $u \in H^1(\mathbb{R}^d)$ and the representation theorem B.1.1 tells us that we can write $u = \mathrm{SL}_{\mathrm{i}\kappa}\alpha$ and in particular $u_{|\Gamma} = S_{\mathrm{i}\kappa}\alpha$.

Now, let $\beta \in H^{1/2}(\Gamma)$ such that $||\beta||_{H^{1/2}(\Gamma)} = 1$. We can find a function $\varphi \in H^1(\mathbb{R}^d)$ such that $\varphi_{|\Gamma} = \beta$. Then,

$$\begin{aligned}|\langle \alpha, \beta\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}| &= \left|\int_{\mathbb{R}^d} (\nabla u \cdot \nabla\overline{\varphi} + \kappa^2 u\overline{\varphi})dx\right| \\ &\leq C||u||_{H^1(\mathbb{R}^d)}||\varphi||_{H^1(\mathbb{R}^d)} \\ &\leq C||u||_{H^1(\mathbb{R}^d)}\end{aligned}$$

since $||\varphi||_{H^1(\mathbb{R}^d)} \leq ||\beta||_{H^{1/2}(\Gamma)} = 1$. We deduce that

$$||\alpha||_{H^{-1/2}(\Gamma)} \leq ||u||_{H^1(\mathbb{R}^d)}.$$

Finally, we can conclude on the coercivity of $S_{\mathrm{i}\kappa}$

$$\begin{aligned}\left|\langle S_{\mathrm{i}\kappa}\alpha, \alpha\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)}\right| &= \left|\langle u_{|\Gamma}, \alpha\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)}\right| \\ &= \left|\int_{\mathbb{R}^d} (|\nabla u|^2 + \kappa^2|u|^2)dx\right| \\ &\geq C||u||^2_{H^1(\mathbb{R}^d)} \\ &\geq C||\alpha||^2_{H^{-1/2}(\Gamma)}.\end{aligned}$$

□

Lemma 5.2.3. *The operator* $T_{\mathrm{i}\kappa} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ *is coercive.*

Proof. Let $\beta \in H^{1/2}(\Gamma)$. The proof of the coercivity is similar to the previous proof by considering here $u \in H^1(\mathbb{R}^d \setminus \Gamma)$

$$\begin{cases} \Delta u - \kappa^2 u = 0 & \text{in } \mathbb{R}^d \backslash \Gamma \\ [u]_\Gamma = \beta & \text{on } \Gamma \\ \left[\dfrac{\partial u}{\partial \nu}\right]_\Gamma = 0 & \text{on } \Gamma. \end{cases}$$

Let $u_0 \in H^1(\mathbb{R}^d \setminus \Gamma)$ such that $[u_0]_\Gamma = \beta$ on Γ and $\|u_0\|_{H^1(\mathbb{R}^d \backslash \Gamma)} \leq C\|\beta\|_{H^{1/2}(\Gamma)}$. The equivalent variational formulation of the latter problem is to find $u \in H^1(\mathbb{R}^d \setminus \Gamma)$ such that $u - u_0 \in H^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} (\nabla u \cdot \nabla\overline{\varphi} + \kappa^2 u\overline{\varphi})dx = 0 \tag{5.5}$$

for all $\varphi \in H^1(\mathbb{R}^d)$. From Lax-Milgram theorem one easily deduce the existence and uniqueness of such solutions $u \in H^1(\mathbb{R}^d \setminus \Gamma)$. Moreover, from the representation theorem B.1.1, this solution can be written as $u = \mathrm{DL}_{\mathrm{i}\kappa}\beta$. In particular, $\dfrac{\partial u}{\partial \nu}|_\Gamma = T_{\mathrm{i}\kappa}\beta$.

Now, let $\alpha \in H^{-1/2}(\Gamma)$ such that $||\alpha||_{H^{-1/2}(\Gamma)} = 1$ and consider $\varphi \in H^1(\mathbb{R}^d \setminus \Gamma)$ such that

$$\begin{cases} \Delta\varphi - \kappa^2\varphi = 0 & \text{in } \mathbb{R}^d\backslash\Gamma \\ \left(\dfrac{\partial\varphi}{\partial\nu}\right)^{\pm} = \alpha & \text{on } \Gamma. \end{cases}$$

The existence of this solution is guaranteed by the Lax-Milgram theorem as in the previous cases. Then, with C denoting a constant independant from α and β but with a value that can change from one line to another,

$$\begin{aligned} |\langle\alpha, \beta\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}| &= \left|\int_{\mathbb{R}^d}(\nabla\varphi\cdot\nabla\overline{u} + \kappa^2\varphi\overline{u})dx\right| \\ &\leq C||u||_{H^1(\mathbb{R}^d\backslash\Gamma)}||\varphi||_{H^1(\mathbb{R}^d\backslash\Gamma)} \\ &\leq C||u||_{H^1(\mathbb{R}^d\backslash\Gamma)} \end{aligned}$$

since $||\varphi||_{H^1(\mathbb{R}^d\backslash\Gamma)} \leq C||\alpha||_{H^{-1/2}(\Gamma)} = C$. We deduce that

$$||\beta||_{H^{1/2}(\Gamma)} \leq C||u||_{H^1(\mathbb{R}^d\backslash\Gamma}.$$

Finally we can conclude on the coercivity of $T_{\mathrm{i}\kappa}$

$$\begin{aligned} \left|\langle T_{\mathrm{i}\kappa}\beta, \beta\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\right| &= \left|\langle\frac{\partial u}{\partial\nu}, \beta\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\right| \\ &= \left|\int_{\mathbb{R}^d}(|\nabla u|^2 + \kappa^2|u|^2)dx\right| \\ &\geq C||u||^2_{H^1(\mathbb{R}^d\backslash\Gamma)} \\ &\geq C||\beta||^2_{H^{1/2}(\Gamma)}. \end{aligned}$$

□

Using an appropriate decomposition of the operator $Z(k)$, we can show that it is Fredholm.

Lemma 5.2.4. *The operator $Z(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is Fredholm of index zero, and is analytic on $k \in \mathbb{C}\backslash\mathbb{R}^-$.*

Proof. We can write

$$\begin{aligned} Z(k) = &\begin{pmatrix} (\mu-1)S_{\mathrm{i}|k_0|} & 0 \\ 0 & (1/\mu-1)T_{\mathrm{i}|k_0|} \end{pmatrix} + \begin{pmatrix} \mu\left(S_{k_1} - S_{\mathrm{i}|k_0|}\right) & 0 \\ 0 & 1/\mu\left(T_{k_1} + T_{\mathrm{i}|k_0|}\right) \end{pmatrix} \\ &+ \begin{pmatrix} S_{\mathrm{i}|k_0|} - S_{k_0} & 0 \\ 0 & T_{\mathrm{i}|k_0|} - T_{k_0} \end{pmatrix} + \begin{pmatrix} 0 & K_{k_1} - K_{k_0} \\ K'_{k_1} - K'_{k_0} & 0 \end{pmatrix} \end{aligned} \tag{5.6}$$

From the two previous lemma, the first operator of the right-hand side is invertible from $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ into $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. From Theorem 5.1.2 and 5.1.4, we deduce that for $k \neq k'$, the mappings

$$\begin{array}{lcl} S_k - S_{k'} & : & H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \\ K_k - K_{k'} & : & H^{1/2}(\Gamma) \to H^{1/2}(\Gamma), \\ K'_k - K'_{k'} & : & H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma), \\ T_k - T_{k'} & : & H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) \end{array}$$

are compact. Therefore, the three last operators in the decomposition of $Z(k)$ are compact from $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ into $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. Consequently, the operator $Z(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is Fredholm of index zero.

The analyticity of the operator $Z(k)$ is a consequence of the analyticity of the kernels of the potentials and the fact that the derivative with respect to k does not increase the singularity of the surface potentials. $\square$

To apply the analytic Fredholm theorem and conclude on the discreteness of the set of transmission eigenvalues, we need one more result that ensures the injectivity of $Z(k)$ for at least one k.

Lemma 5.2.5. *Assume that $\mu - 1$ and $1 - n$ are either positive or negative and let k be a positive real. Then the operator $Z(\mathrm{i}k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is injective.*

Proof. Assume that $Z(\mathrm{i}k)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$. Let us define

$$v := \mathrm{SL}_{\mathrm{i}k_0}\alpha - \mathrm{DL}_{\mathrm{i}k_0}\beta \quad \text{in } \mathbb{R}^d \setminus \Gamma$$

and

$$w := \mu\mathrm{SL}_{\mathrm{i}k_1}\alpha - \mathrm{DL}_{\mathrm{i}k_1}\beta \quad \text{in } \mathbb{R}^d \setminus \Gamma.$$

The relation $Z(\mathrm{i}k)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$ implies, that on Γ we have

$$w^\pm = v^\pm \text{ and } \frac{1}{\mu}\frac{\partial w^\pm}{\partial \nu} = \frac{\partial v^\pm}{\partial \nu}.$$

Consequently the pair $(w, v) \in H^1(\mathbb{R}^d \setminus \Gamma)^2$ is solution to

$$\begin{cases} \Delta v - k^2 v = 0 & \text{in } \mathbb{R}^d \backslash \Gamma \\ \nabla \cdot \dfrac{1}{\mu}\nabla w - k^2 n w = 0 & \text{in } \mathbb{R}^d \backslash \Gamma \\ w^\pm = v^\pm & \text{on } \Gamma \\ \dfrac{1}{\mu}\dfrac{\partial w^\pm}{\partial \nu} = \dfrac{\partial v^\pm}{\partial \nu} & \text{on } \Gamma. \end{cases}$$

Let us define the Hilbert space

$$\mathbb{H} := \left\{ (w, v) \in H^1(\mathbb{R}^d \setminus \Gamma)^2, \;\; w^\pm = v^\pm \text{ on } \Gamma \right\}.$$

We then observe that (w, v) in $\mathbb{H}$ and

$$a((w, v), (\varphi, \psi)) = 0 \quad \forall\, (\varphi, \psi) \in \mathbb{H},$$

where

$$a((w, v), (\varphi, \psi)) := \int_{\mathbb{R}^d} \frac{1}{\mu}\nabla w \cdot \nabla\overline{\varphi} + k^2 n w \overline{\varphi} - \int_{\mathbb{R}^d} \nabla v \cdot \nabla\overline{\psi} + k^2 v \overline{\psi}.$$

1. Case where $\mu < 1$ and $n > 1$.

Let Ω be a neighbourhood of Γ. Let us define the cutoff function χ with compact support in Ω such that $\chi = 1$ on Γ and

$$T: \begin{array}{ccc} \mathbb{H}\times\mathbb{H} & \to & \mathbb{H}\times\mathbb{H} \\ (w,v) & \mapsto & (w, -v+2\chi w) \end{array}.$$

$$\begin{aligned} a((w,v),T(w,v)) = & \int_{\mathbb{R}^d\setminus\Gamma} \left(\frac{1}{\mu}|\nabla w|^2 + k^2 n|w|^2 + |\nabla v|^2 + k^2|v|^2\right) dx \\ & - 2\int_\Omega \nabla v\cdot\nabla(\chi\overline{w}) - 2k^2\int_\Omega \chi v\overline{w} \\ \geq & \frac{1}{\mu}||\nabla w||^2_{L^2(\mathbb{R}^d)} + k^2 n||w||^2_{L^2(\mathbb{R}^d)} + ||\nabla v||^2_{L^2(\mathbb{R}^d)} + k^2||v||^2_{L^2(\mathbb{R}^d)} \\ & - \alpha||\nabla v||^2_{L^2(\mathbb{R}^d)} - \frac{1}{\alpha}||\nabla w||^2_{L^2(\mathbb{R}^d)} - C\eta||\nabla v||^2_{L^2(\mathbb{R}^d)} - \frac{C}{\eta}||w||^2_{L^2(\mathbb{R}^d)} \\ - & k^2\beta||v||^2_{L^2(\mathbb{R}^d)} - \frac{k^2}{\beta}||w||^2_{L^2(\mathbb{R}^d)} \\ \geq & \left(\frac{1}{\mu} - \frac{1}{\alpha}\right)||\nabla w||^2_{L^2(\mathbb{R}^d)} + \left(k^2\left(n - \frac{1}{\beta}\right) - \frac{C}{\eta}\right)||w||^2_{L^2(\mathbb{R}^d)} \\ & + (1-\alpha-C\eta)\,||\nabla v||^2_{L^2(\mathbb{R}^d)} + k^2(1-\beta)||v||^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Let $\mu < \alpha < 1$, $1/n < \beta < 1$ and η such that $1-\alpha-C\eta > 0$ fixed. Then if k is large enough to have $k^2\left(n - \frac{1}{\beta}\right) - \frac{C}{\eta} > 0$, we deduce that a is coercive. As a consequence, $w = 0$ and $v = 0$ are the only solutions. From the equality $[v]_\Gamma = -\beta$ and $\left[\frac{\partial v}{\partial \nu}\right]_\Gamma = -\alpha$ we get that $\alpha = \beta = 0$ and finally $Z(\mathrm{i}k) : H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ is injective.

2. Case where $\mu > 1$ and $n < 1$.

Let Ω be a neighbourhood of Γ. Let us define the cutoff function χ with compact support in Ω such that $\chi = 1$ on Γ and

$$T: \begin{array}{ccc} \mathbb{H}\times\mathbb{H} & \to & \mathbb{H}\times\mathbb{H} \\ (w,v) & \mapsto & (-w+2\chi v, v) \end{array}.$$

$$
\begin{aligned}
a((w,v),T(w,v)) = &\int_{\mathbb{R}^d\setminus\Gamma}\left(\frac{1}{\mu}|\nabla w|^2 + k^2 n|w|^2 + |\nabla v|^2 + k^2|v|^2\right)dx \\
&- 2\int_\Omega \nabla v\cdot\nabla(\chi\overline{w}) - 2k^2\int_\Omega \chi v\overline{w} \\
\geq\ &\frac{1}{\mu}||\nabla w||^2_{L^2(\mathbb{R}^d)} + k^2 n||w||^2_{L^2(\mathbb{R}^d)} + ||\nabla v||^2_{L^2(\mathbb{R}^d)} + k^2||v||^2_{L^2(\mathbb{R}^d)} \\
&- \alpha||\nabla v||^2_{L^2(\mathbb{R}^d)} - \frac{1}{\alpha}||\nabla w||^2_{L^2(\mathbb{R}^d)} - C\eta||\nabla v||^2_{L^2(\mathbb{R}^d)} - \frac{C}{\eta}||w||^2_{L^2(\mathbb{R}^d)} \\
&- k^2\beta||v||^2_{L^2(\mathbb{R}^d)} - \frac{k^2}{\beta}||w||^2_{L^2(\mathbb{R}^d)} \\
\geq\ &\left(\frac{1}{\mu} - \frac{1}{\alpha}\right)||\nabla w||^2_{L^2(\mathbb{R}^d)} + \left(k^2\left(n - \frac{1}{\beta}\right) - \frac{C}{\eta}\right)||w||^2_{L^2(\mathbb{R}^d)} \\
&+ (1-\alpha-C\eta)\,||\nabla v||^2_{L^2(\mathbb{R}^d)} + k^2(1-\beta)||v||^2_{L^2(\mathbb{R}^d)}.
\end{aligned}
$$

Let $\mu < \alpha < 1$, $1/n < \beta < 1$ and η such that $1-\alpha-C\eta > 0$ fixed. Then if k is large enough to have $k^2\left(n - \frac{1}{\beta}\right) - \frac{C}{\eta} > 0$, we deduce that a is coercive. As a consequence, $w = 0$ and $v = 0$ are the only solutions. From the equality $[v]_\Gamma = -\beta$ and $\left[\frac{\partial v}{\partial\nu}\right]_\Gamma = -\alpha$ we get that $\alpha = \beta = 0$ and finally $Z(\mathrm{i}k) : H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ is injective.

□

We now can state a concluding theorem for this section, which is a classical result on ITP [18] related to the discreteness of transmission eigenvalues for contrasts that does not change sign.

Theorem 5.2.6. *Assume that $\mu - 1$ and $1 - n$ are either positive or negative. The set of transmission eigenvalues is discrete.*

Proof. From Lemma 5.2.4, the operator $Z(k) : H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ is Fredholm and analytic on $k \in \mathbb{C}\setminus\mathbb{R}^-$. Moreover, Lemma 5.2.5 ensures the existence of a k such that $Z(k)$ is injective. Applying the analytic Fredholm theorem, $Z(k)$ is injective for all k except for a discrete set. The discreteness of the set of transmission eigenvalues follows. □

5.3 The case $\mu = 1$

In this section, we assume that $\mu = 1$. The interior transmission problem we consider is then

$$
\begin{cases}
\Delta w + k^2 n w = 0 & \text{in } D,\\
\Delta v + k^2 v = 0 & \text{in } D,\\
w = v & \text{on } \Gamma,\\
\dfrac{\partial w}{\partial\nu} = \dfrac{\partial v}{\partial\nu} & \text{on } \Gamma,
\end{cases}
\tag{5.7}
$$

where $v, w \in L^2(D)$ such that $u := w - v \in H^2(D)$. As in the previous section we treat first the case where n is constant and shall assume that either $n \neq 1$.

We first observe that the analysis done in the previous section cannot be carried to the current case since the operator $Z(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is compact for $\mu = 1$ (see decomposition (5.6)). This is somehow predictable since we already know (See Chapter 1) that the natural spaces for the solutions is $v \in L^2(D)$ and $\Delta v \in L^2(D)$, therefore, the boundary values $\alpha := \dfrac{\partial v}{\partial \nu}|_\Gamma$ and $\beta := v|_\Gamma$ now live respectively in $H^{-3/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ and not in the classical spaces $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Consequently one needs to analyse the operator $Z(k)$ as acting on $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

The first step would then to analyse/generalize the properties of the single and the double layer potentials in these spaces.

5.3.1 Single and double layer potentials and trace properties for densities in $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$

We have already seen in Section 5.1 that the volumetric potentials $\widehat{\mathrm{SL}}_k$ and $\widehat{\mathrm{DL}}_k$ are pseudo-differential operators of order -2 and -1 respectively. This implies in particular that (See [35, Theorem 8.5.8])

$$\mathrm{SL}_k : H^{-3/2}(\Gamma) \to L^2(D^\pm)$$
$$\mathrm{DL}_k : H^{-1/2}(\Gamma) \to L^2(D^\pm)$$

are continuous. Moreover, by obvious density arguments, for any densities $\varphi \in H^{-3/2}(\Gamma)$ and $\psi \in H^{-1/2}(\Gamma)$, $\mathrm{SL}_k\varphi$ and $\mathrm{DL}_k\psi$ satisfy the Helmholtz equation in the distributional sense in $\mathbb{R}^d \setminus \Gamma$. Therefore, if one defines

$$L^2_\Delta(D^\pm) := \left\{u \in L^2(D^\pm),\ \Delta u \in L^2(D^\pm)\right\}$$

equipped with the graph norm, then one easily deduces that

$$\mathrm{SL}_k : H^{-3/2}(\Gamma) \to L^2_\Delta(D^\pm)$$
$$\mathrm{DL}_k : H^{-1/2}(\Gamma) \to L^2_\Delta(D^\pm)$$

are continuous. More importantly, one can generalize the results of Theorem 5.1.1 in the following sense.

Theorem 5.3.1. *The single-layer potential* $\mathrm{SL}_k : H^{-3/2}(\Gamma) \to L^2_\Delta(D^\pm)$ *and the double layer potential* $\mathrm{DL}_k : H^{-1/2}(\Gamma) \to L^2_\Delta(D^\pm)$ *are bounded and give rise to bounded linear operators*

$$S_k : H^{-3/2}(\Gamma) \to H^{-1/2}(\Gamma), \qquad K_k : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma),$$
$$K'_k : H^{-3/2}(\Gamma) \to H^{-3/2}(\Gamma), \qquad T_k : H^{-1/2}(\Gamma) \to H^{-3/2}(\Gamma),$$

such that for all $\varphi \in H^{-3/2}(\Gamma)$ *and* $\psi \in H^{-1/2}(\Gamma)$,

$$\begin{cases} (\mathrm{SL}_k\varphi)^\pm = S_k\varphi & \text{and} \quad (\mathrm{DL}_k\psi)^\pm = K_k\psi \pm \dfrac{1}{2}\psi \quad \text{in } H^{-1/2}(\Gamma), \\ \dfrac{\partial(\mathrm{SL}_k\varphi)}{\partial\nu}^\pm = K'_k\varphi \mp \dfrac{1}{2}\varphi & \text{and} \quad \dfrac{\partial(\mathrm{DL}_k\psi)}{\partial\nu}^\pm = T_k\psi \quad \text{in } H^{-3/2}(\Gamma). \end{cases}$$

Proof. The first part of the theorem is already proven. The jump and trace properties will be deduced from Theorem 5.1.1 using a density argument. More specifically, let $\varphi \in H^{-3/2}(\Gamma)$ and $\varphi_n \in H^{-1/2}(\Gamma)$ such that $\varphi_n \xrightarrow[n\to\infty]{} \varphi$ in $H^{-3/2}(\Gamma)$. Using the continuity of SL_k from $H^{-3/2}(\Gamma)$ into $L^2(D^\pm)$, we have that

$$\mathrm{SL}_k\varphi_n \xrightarrow[n\to\infty]{} \mathrm{SL}_k\varphi \text{ in } L^2(D^\pm)$$

and

$$\Delta\mathrm{SL}_k\varphi_n \xrightarrow[n\to\infty]{} \Delta\mathrm{SL}_k\varphi \text{ in } L^2(D^\pm).$$

For all $v \in L^2_\Delta(D^\pm)$, we define the trace $v^\pm$ of v on Γ by

$$\langle v^\pm, \varphi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} = \mp\int_{D^\pm} v\Delta w \pm \int_{D^\pm} w\Delta v$$

where $w \in H^2(D^\pm)$ such that $w = 0$ and $\dfrac{\partial w}{\partial \nu} = \varphi$ on Γ. Furthermore,

$$\begin{aligned} ||v^\pm||_{H^{-1/2}(\Gamma)} &:= \sup_{||\varphi||_{H^{1/2}(\Gamma)}=1} \langle v^\pm, \varphi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \\ &\le C(||v||_{L^2(D^\pm)} + ||\Delta v||_{L^2(D^\pm)}) \end{aligned}$$

We deduce that

$$\begin{aligned} ||(\mathrm{SL}_k\varphi_n)^\pm - (\mathrm{SL}_k\varphi)^\pm||_{H^{-1/2}(\Gamma)} \le C\big(&||\mathrm{SL}_k\varphi_n - \mathrm{SL}_k\varphi||_{L^2(D^\pm)} \\ &+ ||\Delta\mathrm{SL}_k\varphi_n - \Delta\mathrm{SL}_k\varphi||_{L^2(D^\pm)}\big) \end{aligned}$$

and consequently $(\mathrm{SL}_k\varphi_n)^\pm \xrightarrow[n\to\infty]{} (\mathrm{SL}_k\varphi)^\pm$ in $H^{-1/2}(\Gamma)$. We get that

$$0 = [\mathrm{SL}_k\varphi_n]_\Gamma \xrightarrow[n\to\infty]{} [\mathrm{SL}_k\varphi]_\Gamma \text{ in } H^{-1/2}(\Gamma)$$

and finally we have the jump property

$$[\mathrm{SL}_k\varphi]_\Gamma = 0.$$

For all $v \in L^2(D^\pm)$ we define the normal derivative $\dfrac{\partial v}{\partial \nu}^\pm$ of v on Γ by

$$\langle \frac{\partial v}{\partial \nu}^\pm, \varphi\rangle_{H^{-3/2}(\Gamma),H^{3/2}(\Gamma)} = \pm\int_{D^\pm} v\Delta w \mp \int_{D^\pm} w\Delta v$$

where $w \in H^2(D^\pm)$ such that $w = \varphi$ and $\dfrac{\partial w}{\partial \nu} = 0$ on Γ. Furthermore,

$$\begin{aligned} \left|\left|\frac{\partial v}{\partial \nu}^\pm\right|\right|_{H^{-3/2}(\Gamma)} &:= \sup_{||\varphi||_{H^{3/2}}=1} \langle \frac{\partial v}{\partial \nu}^\pm, \varphi\rangle_{H^{-3/2}(\Gamma),H^{3/2}(\Gamma)} \\ &\le C\left(||v||_{L^2(D^\pm)} + ||\Delta v||_{L^2(D^\pm)}\right) \end{aligned}$$

We deduce that

$$\left|\left|\frac{\partial(\mathrm{SL}_k\varphi_n)}{\partial\nu}^{\pm} - \frac{\partial(\mathrm{SL}_k\varphi)}{\partial\nu}^{\pm}\right|\right|_{H^{-3/2}(\Gamma)} \leq C\left(||\mathrm{SL}_k\varphi_n - \mathrm{SL}_k\varphi||_{L^2(D^\pm)} \right.$$
$$\left. + \, ||\Delta(\mathrm{SL}_k\varphi_n) - \Delta(\mathrm{SL}_k\varphi)||_{L^2(D^\pm)}\right)$$

and consequently $\dfrac{\partial(\mathrm{SL}_k\varphi_n)}{\partial\nu}^{\pm} \xrightarrow[n\to\infty]{} \dfrac{\partial(\mathrm{SL}_k\varphi)}{\partial\nu}^{\pm}$ in $H^{-1/2}(\Gamma)$. We get that

$$\left[\frac{\partial(\mathrm{SL}_k\varphi_n)}{\partial\nu}\right]_\Gamma \xrightarrow[n\to\infty]{} \left[\frac{\partial(\mathrm{SL}_k\varphi)}{\partial\nu}\right]_\Gamma \text{ in } H^{-3/2}(\Gamma).$$

However $\left[\dfrac{\partial(\mathrm{SL}_k\varphi_n)}{\partial\nu}\right]_\Gamma = -\varphi_n \xrightarrow[n\to\infty]{} -\varphi$ in $H^{-3/2}(\Gamma)$. Finally we get the jump property

$$\left[\frac{\partial(\mathrm{SL}_k\varphi)}{\partial\nu}\right]_\Gamma = -\varphi.$$

Finally, let us prove the jump properties of the double-layer potential. Let $\psi \in H^{-1/2}(\Gamma)$ and $\psi_n \in H^{1/2}(\Gamma)$ such that $\psi_n \xrightarrow[n\to\infty]{} \psi$ in $H^{-1/2}(\Gamma)$. Using the continuity of DL_k from $H^{-1/2}(\Gamma)$ into $L^2(D^\pm)$, we have that

$$\mathrm{DL}_k\psi_n \xrightarrow[n\to\infty]{} \mathrm{DL}_k\psi \text{ in } L^2(D^\pm)$$

and

$$\Delta\mathrm{DL}_k\psi_n \xrightarrow[n\to\infty]{} \Delta\mathrm{DL}_k\psi \text{ in } L^2(D^\pm).$$

We deduce that

$$||(\mathrm{DL}_k\psi_n)^\pm - (\mathrm{DL}_k\psi)^\pm||_{H^{-1/2}(\Gamma)} \leq C\left(||\mathrm{DL}_k\psi_n - \mathrm{DL}_k\psi||_{L^2(D^\pm)}\right.$$
$$\left. + \, ||\Delta\mathrm{DL}_k\psi_n - \Delta\mathrm{DL}_k\psi||_{L^2(D^\pm)}\right)$$

and consequently $(\mathrm{DL}_k\psi_n)^\pm \xrightarrow[n\to\infty]{} (\mathrm{DL}_k\psi)^\pm$ in $H^{-1/2}(\Gamma)$. We get that

$$\psi_n = [\mathrm{DL}_k\psi_n]_\Gamma \xrightarrow[n\to\infty]{} [\mathrm{DL}_k\psi]_\Gamma \text{ in } H^{-1/2}(\Gamma).$$

Furthermore, $\psi_n \xrightarrow[n\to\infty]{} \psi$ in $H^{-1/2}(\Gamma)$ and finally we have the jump property

$$[\mathrm{SL}_k\psi]_\Gamma = \psi.$$

We also have that

$$\left|\left|\frac{\partial(\mathrm{DL}_k\psi_n)}{\partial\nu}^{\pm} - \frac{\partial(\mathrm{DL}_k\psi)}{\partial\nu}^{\pm}\right|\right|_{H^{-3/2}(\Gamma)} \leq C\left(||\mathrm{DL}_k\psi_n - \mathrm{DL}_k\psi||_{L^2(D^\pm)}\right.$$
$$\left. + \, ||\Delta(\mathrm{DL}_k\psi_n) - \Delta(\mathrm{DL}_k\psi)||_{L^2(D^\pm)}\right)$$

and consequently $\dfrac{\partial(\mathrm{DL}_k\psi_n)}{\partial\nu}^{\pm} \underset{n\to\infty}{\longrightarrow} \dfrac{\partial(\mathrm{DL}_k\psi)}{\partial\nu}^{\pm}$ in $H^{-3/2}(\Gamma)$. We get that

$$\left[\frac{\partial(\mathrm{DL}_k\psi_n)}{\partial\nu}\right]_\Gamma \underset{n\to\infty}{\longrightarrow} \left[\frac{\partial(\mathrm{DL}_k\psi)}{\partial\nu}\right]_\Gamma \text{ in } H^{-3/2}(\Gamma).$$

Finally we get the jump property

$$\left[\frac{\partial(\mathrm{DL}_k\psi)}{\partial\nu}\right]_\Gamma = 0.$$

□

Now we have generalized the properties of the potentials in the weaker spaces $H^{-3/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, we can treat the interior transmission problem and study the discreteness of transmission eigenvalues.

Regularity of the single and double-layer potentials

Using Theorem 5.1.2 and Theorem E.2.3, we can generalize the regularity results on $\mathcal{SL}_{k,k'}$ and $\mathcal{DL}_{k,k'}$ for densities in $H^{-3/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ respectively.

Corollary 5.3.2.

$$\mathcal{SL}_{k,k'} : H^{-3/2}(\Gamma) \to H^2(D)$$

and

$$\mathcal{DL}_{k,k'} : H^{-1/2}(\Gamma) \to H^2(D)$$

are continuous for all $k \neq k'$.

In the case where $\mu = 1$, we need to find more regular operators for the compact part in the Fredholm decomposition of the operator corresponding to the interior transmission problem. To this end, we eliminate the principal part of the asymptotic developments of the kernels of the potentials and consider the operators $\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ and $\widehat{\mathcal{DL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{DL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ where

$$\gamma(k,k') := \frac{k^2 - k'^2}{|k|^2 - |k'|^2}.$$

Theorem 5.3.3. *$\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ and $\widehat{\mathcal{DL}}_{k,k'} + \tilde{\gamma}(k,k')\widehat{\mathcal{DL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ are pseudo-differential operators of order -5 for all $k \neq k'$, $k,k' \in \mathbb{C}\backslash\mathbb{R}^-$.*

Proof. The proof follows the same idea as in the proof of Theorem 5.1.2. First, let us consider the case $d = 3$.

We consider the kernel of $\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$. It can be written of the form

$$\begin{aligned}
\tilde{a}(x,z) &:= \frac{e^{\mathrm{i}k|z|} - e^{\mathrm{i}k'|z|}}{4\pi|z|} + \frac{e^{-|kz|} - e^{-|k'z|}}{4\pi|z|} \\
&= \frac{1}{4\pi}\left[\mathrm{i}(k-k') - \frac{k^2 - k'^2}{|k| + |k'|}\right] \\
&\quad - \frac{1}{4\pi}\sum_{j=0}^{\infty}\frac{1}{(j+3)!}\left[\mathrm{i}^{j+1}(k^{j+3} - k'^{j+3}) + (-1)^j\left(|k|^{j+3} - |k'|^{j+3}\right)\gamma(k,k')\right]|z|^{j+2} \\
&= \frac{1}{4\pi}\left[\mathrm{i}(k-k') - \frac{k^2 - k'^2}{|k| + |k'|}\right] + \sum_{j=0}^{\infty}\tilde{a}_{j+2}(x,z)
\end{aligned}$$

where

$$\tilde{a}_{j+2}(x,z) := -\frac{1}{4\pi(j+3)!}\left[\mathrm{i}^{j+1}(k^{j+3} - k'^{j+3}) + (-1)^j\left(|k|^{j+3} - |k'|^{j+3}\right)\gamma(k,k')\right]|z|^{j+2},$$

for all $j \geq 0$, which satisfies

$$\tilde{a}_p(x,tz) = t^p\tilde{a}(x,z).$$

From Theorem E.2.1, we deduce that

$$\left(\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}\right)\varphi(x) = \int_D \tilde{a}(x,x-y)\varphi(y)dy$$

where $\tilde{a}$ is a pseudo-homogeneous kernel of degree 2 is a pseudo-differential operator of order -5.

Now, the kernel of $\widehat{\mathcal{DL}}^*_{k,k'} + \gamma(k,k')\widehat{\mathcal{DL}}^*_{\mathrm{i}|k|,\mathrm{i}|k'|}$ is of the form

$$\begin{aligned}
\tilde{b}(x,z) &:= \frac{\partial\Phi_k(x,y)}{\partial\nu(x)} - \frac{\partial\Phi_{k'}(x,y)}{\partial\nu(x)} + \frac{\partial\Phi_{\mathrm{i}|k|}(x,y)}{\partial\nu(x)} - \frac{\partial\Phi_{\mathrm{i}|k'|}(x,y)}{\partial\nu(x)} \\
&= \frac{-1}{12\pi}\left[\mathrm{i}(k^3 - k'^3) + (|k|^3 - |k'|^3)\gamma(k,k')\right] \\
&\quad + \frac{z\cdot\nu(x)}{4\pi}\sum_{p=1}^{\infty}\frac{(p+2)}{(p+3)!}\left[\mathrm{i}^{p+1}(k^{p+3} - k'^{p+3}) + (-1)^p\left(|k|^{p+3} - |k'|^{p+3}\right)\gamma(k,k')\right]|z|^p \\
&= \frac{-1}{12\pi}\left[\mathrm{i}(k^3 - k'^3) + (|k|^3 - |k'|^3)\gamma(k,k')\right] + \sum_{j=0}^{\infty}\tilde{b}_{j+2}(x,z)
\end{aligned}$$

where

$$\tilde{b}_{j+2}(x,z) := \frac{z\cdot\nu(x)}{4\pi}\frac{(j+3)}{(j+4)!}\left[-\mathrm{i}^j(k^{p+4} - k'^{p+4}) + (-1)^{j+1}\left(|k|^{j+4} - |k'|^{j+4}\right)\gamma(k,k')\right]|z|^{j+1}$$

which satisfies

$$\tilde{b}_p(x,tz) = t^p\tilde{b}(x,z).$$

From Theorem E.2.1, we deduce that

$$\left(\widehat{\mathcal{DL}}^*_{k,k'} + \gamma(k,k')\widehat{\mathcal{DL}}^*_{\mathrm{i}|k|,\mathrm{i}|k'|}\right)\varphi(x) = \int_D \tilde{b}(x, x-y)\varphi(y)dy$$

where $\tilde{b}$ is a pseudo-homogeneous kernel of degree 2, is a pseudo-differential operator of order -5. Consequently, from Theorem E.2.2, $\widehat{\mathcal{DL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{DL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ is also a pseudo-differential operator of order -5.

Let us now consider the case $d = 2$. The kernel of $\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ is

$$\begin{aligned}
\tilde{a}(x,z) &= \frac{\mathrm{i}}{4}\left(H_0^{(1)}(k|z|) - H_0^{(1)}(k'|z|) + H_0^{(1)}(\mathrm{i}|kz|) - H_0^{(1)}(\mathrm{i}|k'z|)\right) \\
&= \frac{\mathrm{i}}{4}\sum_{p=1}^{\infty}\frac{1}{(p+1)!^2}\left[(-1)^{p+1}\left(k^{2p+2} - k'^{2p+2}\right) + \left(|k|^{2p+2} - |k'|^{2p+2}\right)\gamma(k,k')\right]\left(\frac{|z|}{2}\right)^{2p+2}\theta(p)) \\
&\quad + \frac{1}{2\pi}\sum_{p=0}^{\infty}\frac{1}{(p+1)!^2}\left[(-1)^p\left(k^{2p+2}\ln(k) - k'^{2p+2}\ln(k')\right)\right. \\
&\qquad \left. + \left(|k|^{2p+2}\ln(\mathrm{i}|k|) - |k'|^{2p+2}\ln(\mathrm{i}|k'|)\right)\gamma(k,k')\right]\left(\frac{|z|}{2}\right)^{2p+2} \\
&\quad + \frac{1}{2\pi}(\ln k - \ln k' + \ln|k| - \ln|k'|) \\
&\quad + \frac{1}{2\pi}\ln|z|\sum_{p=0}^{\infty}\frac{1}{(p+2)!^2}\left[(-1)^{p+1}\left(k^{2p+4} - k'^{2p+4}\right) - \left(|k|^{2p+4} - |k'|^{2p+4}\right)\gamma(k,k')\right]\left(\frac{|z|}{2}\right)^{2p+4} \\
&= \tilde{f}(x,z) + \sum_{j=0}^{\infty}\tilde{p}_{j+4}(x,z)\ln|z|
\end{aligned}$$

where $\tilde{f} \in C^\infty(D \times \mathbb{R}^d)$ and $\tilde{p}_{j+4}(x,z) = 0$ if j is odd and

$$\begin{aligned}
\tilde{p}_{j+4}(x,z) = \frac{1}{2\pi(p+2)!^2}&\left[(-1)^{p+1}\left(k^{j+4} - k'^{j+4}\right)\right. \\
&\left. - \left(|k|^{j+4} - |k'|^{j+4}\right)\gamma(k,k')\right]\left(\frac{|z|}{2}\right)^{j+4}\ln|z| \quad \text{if } j = 2p.
\end{aligned}$$

The function $\tilde{p}_q$ satisfies $\tilde{p}_q(x,tz) = t^q\tilde{p}_q(x,z)$ and consequently the kernel of $\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ is a pseudo-homogeneous kernel of degree 4. From Theorem E.2.1, we deduce that $\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ is a pseudo-differential operator of order -6 (then also of order -5).

Now, we consider the kernel of $\widehat{\mathcal{DL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{DL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ which is given by

$$
\begin{aligned}
\tilde{b}(x,z) &:= \frac{\mathrm{i}}{4}\frac{\partial}{\partial\nu(x)}\left(H_0^{(1)}(k|z|) - H_0^{(1)}(k'|z|) + H_0^{(1)}(\mathrm{i}|k||z|) - H_0^{(1)}(\mathrm{i}|k'||z|)\right) \\
&= z\cdot\nu(x)\Bigg(\frac{\mathrm{i}}{4}\sum_{p=0}^{\infty}\frac{1}{(p+1)!(p+2)!}\Big[(-1)^p\left(k^{2p+4} - {k'}^{2p+4}\right) \\
&\qquad\qquad + \left(|k|^{2p+4} - |k'|^{2p+4}\right)\gamma(k,k')\Big]\frac{|z|^{2p+1}}{2^{2p+3}}\tilde{\theta}(p+1) \\
&\quad + \frac{1}{2\pi}\sum_{p=0}^{\infty}\frac{1}{p!(p+1)!}\Big[(-1)^p\left(k^{2p+2}\ln k - {k'}^{2p+2}\ln k'\right) \\
&\qquad\qquad - \left(|k|^{2p+2}\ln(\mathrm{i}|k|) - |k'|^{2p+2}\ln(\mathrm{i}|k'|)\right)\gamma(k,k')\Big]\frac{|z|^{2p}}{2^{2p+1}} \\
&\quad + \frac{1}{2\pi}\ln|z|\sum_{p=0}^{\infty}\frac{1}{(p+1)!(p+2)!}\Big[(-1)^{p+1}\left(k^{2p+4} - {k'}^{2p+4}\right) \\
&\qquad\qquad - \left(|k|^{2p+4} - |k'|^{2p+4}\right)\gamma(k,k')\Big]\frac{|z|^{2p+2}}{2^{2p+3}}\Bigg) \\
&= \tilde{f}(x,z) + \sum_{j=0}^{\infty}\tilde{p}_{j+3}(x,z)\ln|z|
\end{aligned}
$$

where $\tilde{f}$ is a function in $C^\infty(D\times\mathbb{R}^d)$, and $\tilde{p}_{j+3}(x,z) = 0$ if j is odd and

$$
\begin{aligned}
\tilde{p}_{j+3}(x,z) = \frac{1}{(p+1)!(p+2)!}&\Big[(-1)^{p+1}\left(k^{2p+4} - {k'}^{2p+4}\right) \\
&- \left(|k|^{2p+4} - |k'|^{2p+4}\right)\gamma(k,k')\Big]\frac{|z|^{2p+2}}{2^{2p+3}}z\cdot\nu(x) \quad \text{if } j = 2p.
\end{aligned}
$$

The function $\tilde{p}_q$ satisfies $\tilde{p}_q(x,tz) = t^q\tilde{p}_q(x,z)$ and consequently the kernel of $\widehat{\mathcal{DL}}^*_{k,k'} + \gamma(k,k')\widehat{\mathcal{DL}}^*_{\mathrm{i}|k|,\mathrm{i}|k'|}$ is a pseudo-homogeneous kernel of degree 3. From Theorem E.2.1, we deduce that $\widehat{\mathcal{DL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{DL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ is a pseudo-differential operator of order -5. □

We can immediately deduce the following corollary from Theorem E.2.3 of $\widehat{\mathcal{SL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{SL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$ and $\widehat{\mathcal{DL}}_{k,k'} + \gamma(k,k')\widehat{\mathcal{DL}}_{\mathrm{i}|k|,\mathrm{i}|k'|}$.

Corollary 5.3.4.

$$\mathcal{SL}_{k,k'} + \gamma(k,k')\mathcal{SL}_{\mathrm{i}|k|,\mathrm{i}|k'|} : H^{-3/2}(\Gamma) \to H^3(D)$$

and

$$\mathcal{DL}_{k,k'} + \gamma(k,k')\mathcal{DL}_{\mathrm{i}|k|,\mathrm{i}|k'|} : H^{-1/2}(\Gamma) \to H^4(D)$$

are continuous for all $k \neq k'$, $k,k' \in \mathbb{C}\backslash\mathbb{R}^-$.

5.3.2 Surface integral formulation

The procedure to derive a surface integral formulation of problem (5.7) is similar to the case $\mu \neq 0$. Consider $(v, w) \in L^2(D) \times L^2(D)$ a solution to (5.7) and set

$$\alpha := \frac{\partial v}{\partial \nu}|_\Gamma = \frac{\partial w}{\partial \nu}|_\Gamma \in H^{-3/2}(\Gamma)$$

and

$$\beta := v|_\Gamma = w|_\Gamma \in H^{-1/2}(\Gamma).$$

Since v and w satisfy

$$\Delta v + k_0^2 v = 0 \text{ and } \Delta w + k_1^2 w = 0 \text{ in } D,$$

similarly to the case $\mu \neq 1$, these solutions can be written

$$\begin{array}{ll} v = \mathrm{SL}_{k_0}\alpha - \mathrm{DL}_{k_0}\beta & \text{in } D, \\ w = \mathrm{SL}_{k_1}\alpha - \mathrm{DL}_{k_1}\beta & \text{in } D. \end{array} \tag{5.8}$$

Then $u := w - v$ can be written in the form

$$u = \mathcal{SL}_{k_1,k_0}\alpha - \mathcal{DL}_{k_1,k_0}\beta.$$

From the boundary conditions of (5.7), $u|_\Gamma = 0$ and $\dfrac{\partial u}{\partial \nu}|_\Gamma = 0$, α and β satisfy

$$Z(k) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \tag{5.9}$$

where

$$Z(k) := \begin{pmatrix} S_{k_1} - S_{k_0} & -K_{k_1} + K_{k_0} \\ -K'_{k_1} + K'_{k_0} & T_{k_1} - T_{k_0} \end{pmatrix}.$$

Again, the far fields generated by v and w are equal to zero and if we define the far field operators

$$P_0^\infty(\alpha,\beta)(\hat{x}) = \frac{1}{4\pi}\int_\Gamma \left(\beta(y)\frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} - \alpha(y) e^{-ik\hat{x}\cdot y} \right) ds(y),$$
$$P_1^\infty(\alpha,\beta)(\hat{x}) = \frac{1}{4\pi}\int_\Gamma \left(\beta(y)\frac{\partial e^{-\mathrm{i}k_1\hat{x}\cdot y}}{\partial \nu(y)} - \frac{1}{\mu}\alpha(y) e^{-\mathrm{i}k_1\hat{x}\cdot y} \right) ds(y),$$

we have

$$P_0^\infty(\alpha,\beta) = 0 \text{ and } P_1^\infty(\alpha,\beta) = 0.$$

Theorem 5.3.5. *The three following assertions are equivalent.*

(i) *There exist $v, w \in L^2(D)$ such that $w - v \in H^2(D)$ a non trivial solution to (5.7).*

(ii) *There exist $\alpha \neq 0$ in $H^{-3/2}(\Gamma)$ and $\beta \neq 0$ in $H^{-1/2}(\Gamma)$ such that*

$$Z(k) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \textit{ and } P_0^\infty(\alpha,\beta) = 0.$$

(iii) *There exist* $\alpha \neq 0$ *in* $H^{-3/2}(\Gamma)$ *and* $\beta \neq 0$ *in* $H^{-1/2}(\Gamma)$ *such that*

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0 \text{ and } P_1^\infty(\alpha,\beta) = 0.$$

Proof. It only remains to show that (ii) implies (i) and (iii) implies (i). Assume that there exist $\alpha \in H^{-3/2}(\Gamma)$ and $\beta \in H^{-1/2}(\Gamma)$ satisfying

$$Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0.$$

We define

$$u := \mathcal{SL}_{k_1,k_0}\alpha - \mathcal{DL}_{k_1,k_0}\beta.$$

We have that $(\Delta + k^2)(\Delta + k^2 n)u = 0$ and from Theorem 5.1.2 and Corollary E.2.3, we obtain that $u \in H^2(D)$. Moreover, the relation $Z(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$ implies that $u_{|\Gamma} = 0$ and $\dfrac{\partial u}{\partial \nu}_{|\Gamma} = 0$ on Γ. Now, we must show that $u \neq 0$ or equivalently that $v \neq 0$ or $w \neq 0$ where

$$v := \mathrm{SL}_{k_0}\alpha - \mathrm{DL}_{k_0}\beta \text{ and } w := \mathrm{SL}_{k_1}\alpha - \mathrm{DL}_{k_1}\beta.$$

First, assume that $P_0^\infty(\alpha,\beta) = 0$. From Rellich's lemma, we deduce that $v = 0$ in $\mathbb{R}^d\backslash D$. Assume that $v = 0$ also in D. Then we have in particular that

$$[v]_\Gamma = \left[\frac{\partial v}{\partial \nu}\right]_\Gamma = 0$$

and from the jump properties of the single and double layer potential we also have that

$$[v]_\Gamma = -\beta \text{ and } \left[\frac{\partial v}{\partial \nu}\right]_\Gamma = -\alpha.$$

This contradicts the fact that $(\alpha,\beta) \neq (0,0)$. Then $v \neq 0$ in D.

Similarly, it can be shown that $P_1^\infty(\alpha,\beta) = 0$ implies that $w \neq 0$. □

5.3.3 Discreteness of the set of transmission eigenvalues

Similarly to the case where $\mu \neq 1$, we want to show that $Z(k)$ is of Fredholm type. Again, we can show here that the diagonal part of $Z(\mathrm{i}k)$ is coercive for k real. Let κ and κ' be two different real numbers.

Lemma 5.3.6. $S_{\mathrm{i}\kappa} - S_{\mathrm{i}\kappa'} : H^{-3/2}(\Gamma) \to H^{3/2}(\Gamma)$ *is coercive.*

Proof. Let α be in $H^{-3/2}(\Gamma)$. Let us consider the following problem:

$$\begin{cases}(\Delta - \kappa^2)(\Delta - \kappa'^2)u = 0 & \text{in } \mathbb{R}^d\backslash\Gamma,\\ [\Delta u]_\Gamma = 0 & \text{on } \Gamma,\\ \left[\dfrac{\partial(\Delta u)}{\partial \nu}\right]_\Gamma = \alpha(\kappa'^2 - \kappa^2) & \text{on } \Gamma.\end{cases} \tag{5.10}$$

The variational formulation is as follows: find $u \in H^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d \setminus \Gamma} (\Delta u - \kappa^2 u)(\Delta \overline{\varphi} - \kappa'^2 \overline{\varphi}) dx = - \int_\Gamma (\kappa'^2 - \kappa^2) \alpha \overline{\varphi} ds(x). \tag{5.11}$$

The left-hand side of (5.11) is continuous and coercive.

$$\begin{aligned} \int_{\mathbb{R}^d \setminus \Gamma} (\Delta u - \kappa^2 u)(\Delta \overline{u} - \kappa'^2 \overline{u}) dx &= ||\Delta u||^2_{L^2(\mathbb{R}^d)} + (\kappa^2 + \kappa'^2) ||\nabla u||^2_{L^2(\mathbb{R}^d)} + \kappa^2 \kappa'^2 ||u||^2_{L^2(\mathbb{R}^d)} \\ &\geq C ||u||^2_{H^2(\mathbb{R}^d)}. \end{aligned}$$

From Lax-Milgram theorem, we deduce that there exists a unique $u \in H^2(\mathbb{R}^d)$ solution to (5.11) which is $u = \mathcal{SL}_{\mathrm{i}\kappa', \mathrm{i}\kappa} \alpha$. In particular, $u|_\Gamma = (S_{\mathrm{i}\kappa'} - S_{\mathrm{i}\kappa})\alpha$. For $\varphi = u$ we obtain:

$$\int_{\mathbb{R}^d \setminus \Gamma} (\Delta - \kappa^2) u (\Delta - \kappa'^2) \overline{u} dx + \int_\Gamma (\kappa'^2 - \kappa^2) \alpha \overline{u} ds(x) = 0 \tag{5.12}$$

From the inequality

$$\int_{\mathbb{R}^d \setminus \Gamma} (\Delta u - \kappa^2 u)(\Delta \overline{u} - \kappa'^2 \overline{u}) dx \geq C ||u||^2_{H^2(\mathbb{R}^d)}$$

and (5.12) we obtain

$$|\langle \alpha, u \rangle_{H^{-3/2}(\Gamma), H^{3/2}(\Gamma)}| \geq C' ||u||^2_{H^2(\mathbb{R}^d)} \tag{5.13}$$

Now show that there exists $C_1 > 0$ such that $||\alpha||_{H^{-3/2}(\Gamma)} \leq C_1 ||u||_{H^2(\mathbb{R}^d)}$. First remark that

$$||\alpha||_{H^{-3/2}(\Gamma)} = \sup \left\{ |\langle \alpha, \varphi \rangle_{H^{-3/2}(\Gamma), H^{3/2}(\Gamma)}| / \varphi \in H^{3/2}(\Gamma) \text{ and } ||\varphi||_{H^{3/2}(\Gamma)} = 1 \right\}.$$

Let $\varphi \in H^{3/2}(\Gamma)$. Then there exists $\tilde{\varphi} \in H^2(\mathbb{R}^d)$ such that $\tilde{\varphi}|_\Gamma = \varphi$. From (5.11) we have that

$$\begin{aligned} |\langle \alpha, \varphi \rangle_{H^{-3/2}(\Gamma), H^{3/2}(\Gamma)}| &= \frac{1}{|\kappa'^2 - \kappa^2|} \left| \int_{\mathbb{R}^d \setminus \Gamma} (\Delta u - \kappa^2 u)(\Delta \overline{\tilde{\varphi}} - \kappa'^2 \overline{\tilde{\varphi}}) dx \right| \\ &\leq C ||u||_{H^2(\mathbb{R}^d)} ||\tilde{\varphi}||_{H^2(\mathbb{R}^d)} \\ &\leq C_1 ||u||_{H^2(\mathbb{R}^d)} \end{aligned}$$

because $||\tilde{\varphi}|||_{H^2(\mathbb{R}^d)} \leq ||\varphi||_{H^{3/2}(\Gamma)} = 1$. Then

$$||\alpha||_{H^{-3/2}(\Gamma)} \leq C_1 ||u||_{H^2(\mathbb{R}^d)}.$$

We deduce now the coercivity of $S_{\mathrm{i}\kappa'} - S_{\mathrm{i}\kappa}$

$$\begin{aligned} \left| \langle (S_{\mathrm{i}\kappa'} - S_{\mathrm{i}\kappa}) \alpha, \alpha \rangle_{H^{-3/2}(\Gamma), H^{3/2}(\Gamma)} \right| &= \left| \langle \alpha, u \rangle_{H^{-3/2}(\Gamma), H^{3/2}(\Gamma)} \right| \\ &\geq C' ||u||^2_{H^2(\mathbb{R}^d)} \\ &\geq \frac{C'}{C_1} ||\alpha||^2_{H^{-3/2}(\Gamma)}. \end{aligned}$$

Then $S_{\mathrm{i}\kappa'} - S_{\mathrm{i}\kappa} : H^{-3/2}(\Gamma) \to H^{3/2}(\Gamma)$ is coercive. $\square$

Lemma 5.3.7. *$T_{\mathrm{i}\kappa'} - T_{\mathrm{i}\kappa} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is coercive.*

Proof. Let $\beta \in H^{-1/2}(\Gamma)$. Let us define

$$\begin{cases} (\Delta - \kappa^2)(\Delta - \kappa'^2 n)u = 0 & \text{in } \mathbb{R}^d \backslash \Gamma \\ [\Delta u]_\Gamma = \beta(\kappa'^2 - \kappa^2) & \text{on } \Gamma \\ \left[\dfrac{\partial(\Delta u)}{\partial \nu}\right]_\Gamma = 0 & \text{on } \Gamma \end{cases} \tag{5.14}$$

The variational formulation is as follows : find $u \in H^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d \backslash \Gamma} (\Delta u - \kappa^2 u)(\Delta \overline{\varphi} - \kappa'^2 \overline{\varphi}) dx = \int_\Gamma (\kappa'^2 - \kappa^2) \beta \frac{\partial \overline{\varphi}}{\partial \nu} ds(x). \tag{5.15}$$

It can easily be shown that there exists a unique $u \in H^2(\mathbb{R}^d)$ solution to (5.15) which is $u = \mathcal{DL}_{\mathrm{i}\kappa', \mathrm{i}\kappa} \beta$. In particular, $\dfrac{\partial u}{\partial \nu}|_\Gamma = (T_{\mathrm{i}\kappa'} - T_{\mathrm{i}\kappa})\beta$. For $\varphi = u$, we obtain

$$\int_{\mathbb{R}^d \backslash \Gamma} (\Delta u - \kappa^2 u)(\Delta \overline{u} - \kappa'^2 \overline{u}) dx = (\kappa'^2 - \kappa^2) \langle \beta, \frac{\partial u}{\partial \nu} \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}. \tag{5.16}$$

From the inequality

$$\int_{\mathbb{R}^d \backslash \Gamma} (\Delta u - \kappa^2 u)(\Delta \overline{u} - \kappa'^2 \overline{u}) dx \geq C ||u||^2_{H^2(\mathbb{R}^d)}$$

and (5.16) we get

$$|\langle \beta, \frac{\partial u}{\partial \nu} \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}| \geq C' ||u||^2_{H^2(\mathbb{R}^d)} \tag{5.17}$$

Now show that there exists $C_1 > 0$ such that $||\beta||_{H^{-1/2}(\Gamma)} \leq C_1 ||u||_{H^2(\mathbb{R}^d)}$. First, remark that

$$||\beta||_{H^{-1/2}(\Gamma)} = \sup \left\{ |\langle \beta, \varphi \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}| / \varphi \in H^{1/2}(\Gamma) \text{ and } ||\varphi||_{H^{1/2}(\Gamma)} = 1 \right\}.$$

Let $\varphi \in H^{1/2}(\Gamma)$. Then there exists $\tilde{\varphi} \in H^2(\mathbb{R}^d)$ such that $\dfrac{\partial \tilde{\varphi}}{\partial \nu}|_\Gamma = \varphi$. From (5.15) we have that

$$\begin{aligned} |\langle \beta, \varphi \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}| &= \frac{1}{|\kappa'^2 - \kappa^2|} \left| \int_{\mathbb{R}^d \backslash \Gamma} (\Delta u - \kappa^2 u)(\Delta \overline{\tilde{\varphi}} - \kappa'^2 \overline{\tilde{\varphi}}) dx \right| \\ &\leq C ||u||_{H^2(\mathbb{R}^d)} ||\tilde{\varphi}||_{H^2(\mathbb{R}^d)} \\ &\leq C_1 ||u||_{H^2(\mathbb{R}^d)} \end{aligned}$$

because $||\tilde{\varphi}|||_{H^2(\mathbb{R}^d)} \leq ||\varphi||_{H^{1/2}(\Gamma)} = 1$. Then

$$||\beta||_{H^{-1/2}(\Gamma)} \leq C_1 ||u||_{H^2(\mathbb{R}^d)}.$$

We now deduce the coercivity of $T_{\mathrm{i}\kappa'} - T_{\mathrm{i}\kappa}$

$$\begin{aligned}\left|\langle (T_{\mathrm{i}\kappa'} - T_{\mathrm{i}\kappa})\beta, \beta\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\right| &= \left|\langle \beta, \frac{\partial u}{\partial \nu}\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\right| \\ &\geq C'||u||^2_{H^2(\mathbb{R}^d)} \\ &\geq \frac{C'}{C_1}||\beta||^2_{H^{-1/2}(\Gamma)}.\end{aligned}$$

Then $T_{\mathrm{i}\kappa'} - T_{\mathrm{i}\kappa} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is coercive.

□

Similarly to the case where $\mu \neq 1$, the first idea is to write

$$Z(k) = -\gamma(k_1,k_0)\begin{pmatrix} (S_{\mathrm{i}|k_1|} - S_{\mathrm{i}|k_0|}) & 0 \\ 0 & (T_{\mathrm{i}|k_1|} - T_{\mathrm{i}|k_0|}) \end{pmatrix} + \begin{pmatrix} 0 & -K_{k_1} + K_{k_0} \\ -K'_{k_1} + K'_{k_0} & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} (S_{k_1} - S_{k_0}) + \gamma(k_1,k_0)(S_{\mathrm{i}|k_1|} - S_{\mathrm{i}|k_0|})) & 0 \\ 0 & (T_{k_1} - T_{k_0}) + \gamma(k_1,k_0)(T_{\mathrm{i}|k_1|} - T_{\mathrm{i}|k_0|})) \end{pmatrix}.$$

Nevertheless, on the contrary to the case where $\mu \neq 1$, according to Corollary 5.3.2

$$K_k - K_{k'} : H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma)$$

and

$$K'_k - K'_{k'} : H^{-3/2}(\Gamma) \to H^{1/2}(\Gamma)$$

for $k \neq k'$, are only continuous and not compact, then we instead shall make the following decomposition

$$Z(k) = -\gamma(k_1,k_0)Z(\mathrm{i}|k|) + (Z(k) + \gamma(k_1,k_0)Z(\mathrm{i}|k|)).$$

Indeed, from Corollaries 5.3.4, the operator $Z(k)+\gamma(k_1,k_0)Z(\mathrm{i}|k|) : H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is compact. Thus, it only remains to show that $Z(\mathrm{i}|k|) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is coercive.

Lemma 5.3.8. *$Z(\mathrm{i}|k|) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is coercive.*

Proof. The proof combines the two auxiliary system used in the two previous lemma. For a sake of presentation, we denote by $\kappa_0 := |k_0|$ and $\kappa_1 := |k_1|$.

Let α be in $H^{-3/2}(\Gamma)$ and $\beta \in H^{-1/2}(\Gamma)$. Let us consider the following problem:

$$\begin{cases} (\Delta - \kappa_0^2)(\Delta - \kappa_1^2)u = 0 & \text{in } \mathbb{R}^d\backslash\Gamma \\ [\Delta u]_\Gamma = \beta(\kappa_1^2 - \kappa_0^2) & \text{on } \Gamma \\ \left[\dfrac{\partial(\Delta u)}{\partial \nu}\right]_\Gamma = \alpha(\kappa_1^2 - \kappa_0^2) & \text{on } \Gamma. \end{cases} \tag{5.18}$$

The variational formulation is as follows : find $u \in H^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d\backslash\Gamma} (\Delta u - \kappa_0^2 u)(\Delta\overline{\varphi} - \kappa_1^2\overline{\varphi})dx = -\int_\Gamma (\kappa_1^2 - \kappa_0^2)\left(\alpha\overline{\varphi} - \beta\frac{\partial\overline{\varphi}}{\partial\nu}\right) ds(x). \tag{5.19}$$

Using Lax-Milgram theorem, it can easily be shown that there exists a unique $u \in H^2(\mathbb{R}^d)$ solution to (5.19) which can be written as $u = \mathcal{SL}_{\mathrm{i}\kappa_1,\mathrm{i}\kappa_0}\alpha - \mathcal{DL}_{\mathrm{i}\kappa_1,\mathrm{i}\kappa_0}\beta$. In particular,

$$u|_\Gamma = (S_{\mathrm{i}\kappa_1} - S_{\mathrm{i}\kappa_0})\alpha - (K_{\mathrm{i}\kappa_1} - K_{\mathrm{i}\kappa_0})\beta$$

and

$$\frac{\partial u}{\partial \nu}|_\Gamma = (K'_{\mathrm{i}\kappa_1} - K'_{\mathrm{i}\kappa_0})\alpha - (T_{\mathrm{i}\kappa_1} - T_{\mathrm{i}\kappa_0})\beta$$

For $\varphi = u$ in (5.19), we get

$$\int_{\mathbb{R}^d\setminus\Gamma} (\Delta u - \kappa_0^2 u)(\Delta\overline{u} - \kappa_1^2\overline{u})dx = -\int_\Gamma (\kappa_1^2 - \kappa_0^2)\left(\alpha\overline{u} - \beta\frac{\partial\overline{u}}{\partial\nu}\right) ds(x). \tag{5.20}$$

From the inequality

$$\int_{\mathbb{R}^d\setminus\Gamma} (\Delta u - \kappa_0^2 u)(\Delta\overline{u} - \kappa_1^2\overline{u})dx \geq C||u||^2_{H^2(\mathbb{R}^d)}$$

and (5.20) we obtain

$$\left|\langle\alpha, u\rangle_{H^{-3/2}(\Gamma),H^{3/2}(\Gamma)} - \langle\beta, \frac{\partial u}{\partial\nu}\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\right| \geq C'||u||^2_{H^2(\mathbb{R}^d)}. \tag{5.21}$$

Now let us show that there exists $C_1 > 0$ such that $||\alpha||_{H^{-3/2}(\Gamma)} \leq C_1||u||_{H^2(\mathbb{R}^d)}$. Let $\varphi \in H^{3/2}(\Gamma)$. Then, there exists $\tilde{\varphi} \in H^2(\mathbb{R}^d)$ such that $\tilde{\varphi}|_\Gamma = \varphi$ and $\dfrac{\partial\tilde{\varphi}}{\partial\nu}|_\Gamma = 0$. From (5.19), we have that

$$\begin{aligned}
|\langle\alpha, \varphi\rangle_{H^{-3/2}(\Gamma),H^{3/2}(\Gamma)}| &= \frac{1}{|\kappa_1^2 - \kappa_0^2|}\left|\int_{\mathbb{R}^d\setminus\Gamma} (\Delta u - \kappa_0^2 u)(\Delta\overline{\tilde{\varphi}} - \kappa_1^2\overline{\tilde{\varphi}})dx\right| \\
&\leq C||u||_{H^2(\mathbb{R}^d)}||\tilde{\varphi}||_{H^2(\mathbb{R}^d)} \\
&\leq C_1||u||_{H^2(\mathbb{R}^d)}
\end{aligned}$$

because $||\tilde{\varphi}|||_{H^2(\mathbb{R}^d)} \leq ||\varphi||_{H^{3/2}(\Gamma)} = 1$. Then

$$||\alpha||_{H^{-3/2}(\Gamma)} \leq C_1||u||_{H^2(\mathbb{R}^d)}.$$

Furthermore, we can show that $||\beta||_{H^{-1/2}(\Gamma)} \leq C_2||u||_{H^2(\mathbb{R}^d)}$. Indeed, let $\psi \in H^{1/2}(\Gamma)$. There exists $\tilde{\psi} \in H^2(\mathbb{R}^d)$ such that $\tilde{\psi}|_\Gamma = 0$ and $\frac{\partial\tilde{\psi}}{\partial\nu}|_\Gamma = \psi$. Then

$$\begin{aligned}
|\langle\beta, \psi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}| &= \frac{1}{|\kappa_1^2 - \kappa_0^2|}\left|\int_{\mathbb{R}^d\setminus\Gamma} (\Delta u - \kappa_0^2 u)(\Delta\overline{\tilde{\psi}} - \kappa_1^2\overline{\tilde{\psi}})dx\right| \\
&\leq C||u||_{H^2(\mathbb{R}^d)}||\tilde{\psi}||_{H^2(\mathbb{R}^d)} \\
&\leq C_2||u||_{H^2(\mathbb{R}^d)}
\end{aligned}$$

since $||\tilde{\psi}|||_{H^2(\mathbb{R}^d)} \leq ||\psi||_{H^{1/2}(\Gamma)} = 1$. Then

$$||\beta||_{H^{-1/2}(\Gamma)} \leq C_2||u||_{H^2(\mathbb{R}^d)}.$$

We deduce now the coercivity of $Z(\mathrm{i}|k|)$.

$$\begin{aligned}
\left|\langle Z(\mathrm{i}|k|)\begin{pmatrix}\alpha\\ \beta\end{pmatrix},\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\rangle\right| &= \big|\langle (S_{\mathrm{i}\kappa_1}-S_{\mathrm{i}\kappa_0})\alpha-(K_{\mathrm{i}\kappa_1}-K_{\mathrm{i}\kappa_0})\beta,\alpha\rangle_{H^{-3/2}(\Gamma),H^{3/2}(\Gamma)}\\
&\quad+\langle -(K'_{\mathrm{i}\kappa_1}-K'_{\mathrm{i}\kappa_0})\alpha+(T_{\mathrm{i}\kappa_1}-T_{\mathrm{i}\kappa_0})\beta,,\beta\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)}\big|\\
&\geq \left|\langle u|_\Gamma,\alpha\rangle_{H^{3/2}(\Gamma),H^{-3/2}(\Gamma)}+\langle -\frac{\partial u}{\partial\nu}|_\Gamma,\beta\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)}\right|\\
&\geq C'||u||^2_{H^2(\mathbb{R}^d)}\\
&\geq \frac{C'}{C_1}||\alpha||^2_{H^{-3/2}(\Gamma)}+\frac{C'}{C_2}||\beta||^2_{H^{-1/2}(\Gamma)}.
\end{aligned}$$

Then $Z(\mathrm{i}|k|):H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma)\to H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$ is coercive. □

Lemma 5.3.9. *The operator $Z(k):H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma)\to H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$ is Fredholm of index zero and analytic on $k\in\mathbb{C}\backslash\mathbb{R}^-$.*

Proof. The analyticity is a direct consequence of the analyticity of the kernels of the integral operators. We can rewrite

$$\begin{aligned}
&Z(k)=-\gamma(k_0,k_1)Z(\mathrm{i}|k|)\\
&+\begin{pmatrix}(S_{k_1}-S_{k_0})+\gamma(k_0,k_1)(S_{\mathrm{i}|k_1|}-S_{\mathrm{i}|k_0|}) & -(K_{k_1}-K_{k_0})-\gamma(k_0,k_1)(K_{\mathrm{i}|k_1|}-K_{\mathrm{i}|k_0|})\\ -(K'_{k_1}-K'_{k_0})-\gamma(k_0,k_1)(K'_{\mathrm{i}|k_1|}-K'_{\mathrm{i}|k_0|}) & (T_{k_1}-T_{k_0})+\gamma(k_0,k_1)(T_{\mathrm{i}|k_1|}-T_{\mathrm{i}|k_0|})\end{pmatrix}
\end{aligned}$$

From Corollary 5.3.4 we deduce that

$$(S_{k_1}-S_{k_0})+\gamma(k_0,k_1)(S_{\mathrm{i}|k_1|}-S_{\mathrm{i}|k_0|}):H^{-3/2}(\Gamma)\to H^{7/2}(\Gamma)$$
$$(K_{k_1}-K_{k_0})+\gamma(k_0,k_1)(K_{\mathrm{i}|k_1|}-K_{\mathrm{i}|k_0|}):H^{-1/2}(\Gamma)\to H^{7/2}(\Gamma)$$
$$(K'_{k_1}-K'_{k_0})+\gamma(k_0,k_1)(K'_{\mathrm{i}|k_1|}-K'_{\mathrm{i}|k_0|}):H^{-3/2}(\Gamma)\to H^{5/2}(\Gamma)$$
$$(T_{k_1}-T_{k_0})+\gamma(k_0,k_1)(T_{\mathrm{i}|k_1|}-T_{\mathrm{i}|k_0|}):H^{-1/2}(\Gamma)\to H^{5/2}(\Gamma)$$

are continuous and then

$$(S_{k_1}-S_{k_0})+\gamma(k_0,k_1)(S_{\mathrm{i}|k_1|}-S_{\mathrm{i}|k_0|}):H^{-3/2}(\Gamma)\to H^{3/2}(\Gamma)$$
$$(K_{k_1}-K_{k_0})+\gamma(k_0,k_1)(K_{\mathrm{i}|k_1|}-K_{\mathrm{i}|k_0|}):H^{-1/2}(\Gamma)\to H^{3/2}(\Gamma)$$
$$(K'_{k_1}-K'_{k_0})+\gamma(k_0,k_1)(K'_{\mathrm{i}|k_1|}-K'_{\mathrm{i}|k_0|}):H^{-3/2}(\Gamma)\to H^{1/2}(\Gamma)$$
$$(T_{k_1}-T_{k_0})+\gamma(k_0,k_1)(T_{\mathrm{i}|k_1|}-T_{\mathrm{i}|k_0|}):H^{-1/2}(\Gamma)\to H^{1/2}(\Gamma)$$

are compact. Then $Z(k):H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma)\to H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$ is Fredholm. □

In order to use the analytic Fredholm theorem and conclude on the discreteness of transmission eigenvalues, we need to show that $Z(k)$ is injective for at least one k.

Lemma 5.3.10. *The operator $Z(\mathrm{i}|k|):H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma)\to H^{3/2}(\Gamma)\times H^{1/2}(\Gamma)$ is injective.*

Proof. Let (α, β) such that $Z(\mathrm{i}|k|)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$. We want to show that $(\alpha,\beta) = (0,0)$. Let u be defined in $\mathbb{R}^d \backslash \Gamma$ by $u := \mathcal{SL}_{\mathrm{i}|k|}\alpha - \mathcal{DL}_{\mathrm{i}|k|}\beta$. Then u satisfies $(\Delta - k_0^2)(\Delta - |k_1|^2)u = 0$ in $\mathbb{R}^d \backslash \Gamma$. Furthermore $Z(\mathrm{i}|k|)\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = 0$ implies that $u^+|_\Gamma = u^-|_\Gamma = \dfrac{\partial u^+}{\partial \nu}|_\Gamma = \dfrac{\partial u^-}{\partial \nu}|_\Gamma = 0$. We deduce that $u = 0$ in $\mathbb{R}^d$. Now we can write $u = w - v$ where

$$v := \mathrm{SL}_{\mathrm{i}|k_0|}\alpha - \mathrm{DL}_{\mathrm{i}|k_0|}\beta$$

and

$$w := \mathrm{SL}_{\mathrm{i}|k_1|}\alpha - \mathrm{DL}_{\mathrm{i}|k_1|}\beta$$

which satisfy $\Delta v - |k_0|^2 v = 0$ and $\Delta w - |k_1|^2 w = 0$ in $\mathbb{R}^d$. In particular

$$[v]_\Gamma = -\beta$$

and

$$\left[\frac{\partial v}{\partial \nu}\right]_\Gamma = -\alpha.$$

Then, from the equality $\Delta u - |k_1|^2 u = (|k_1|^2 - |k_0|^2)v$ we deduce that

$$0 = [\Delta u]_\Gamma = (|k_1|^2 - |k_0|^2)[v]_\Gamma = -(|k_1|^2 - |k_0|^2)\beta$$

and

$$0 = \left[\frac{\partial}{\partial \nu}(\Delta u)\right]_\Gamma = (|k_1|^2 - |k_0|^2)\left[\frac{\partial v}{\partial \nu}\right]_\Gamma = -(|k_1|^2 - |k_0|^2)\alpha.$$

□

Theorem 5.3.11. *Assume that $n \neq 1$. The set of transmission eigenvalues is discrete.*

Proof. This is a direct consequence of Lemma 5.3.9 and Lemma 5.3.10 using the analytic Fredholm theory. □

5.4 Cases where the contrasts change sign

5.4.1 The case of piecewise constant coefficients and $\mu \neq 1$

In this section, we show the discreteness of transmission eigenvalues when the contrast $n - 1$ changes sign for $\mu \neq 1$. We therefore consider the interior transmission problem

$$\begin{cases} \nabla \cdot \dfrac{1}{\mu}\nabla w + k^2 n w = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \Gamma \\ \dfrac{1}{\mu}\dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} & \text{on } \Gamma \end{cases} \tag{5.22}$$

where $v, w \in H^1(D)$. Before considering the case where n can be an L^∞ function, we shall concentrate in this section on the case of piecewise constant coefficients. More specifically we consider the case where n can have two different constant values $n_1 < 1$ and $n_2 > 1$. Let $D_1 \subset D$ such that $n := n_1 < 1$ in D_1 and $n := n_2 > 1$ in $D_2 := D \backslash D_1$. We shall assume that $\overline{D_1} \cap \partial D = \emptyset$. We set $\Gamma := \partial D$ and $\Sigma := \partial D_1$ and assume that these surfaces are regular. We denote by $k_i := k\sqrt{\mu n_i}$ for $i = 1, 2$ and $k_0 := k$.

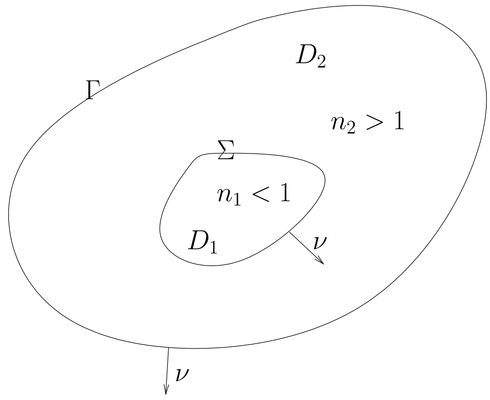


Figure 5.2: Geometry and notations

The single and double layer potentials defined on Γ and Σ are denoted by

$$\mathrm{SL}_k^\Gamma(\varphi)(x) = \int_\Gamma \Phi_k(x,y)\varphi(y)ds(y), \qquad \mathrm{SL}_k^\Sigma(\varphi)(x) = \int_\Sigma \Phi_k(x,y)\varphi(y)ds(y),$$

$$\mathrm{DL}_k^\Gamma(\varphi)(x) = \int_\Gamma \frac{\partial \Phi_k}{\partial \nu(y)}(x,y)\varphi(y)ds(y), \qquad \mathrm{DL}_k^\Sigma(\varphi)(x) = \int_\Sigma \frac{\partial \Phi_k}{\partial \nu(y)}(x,y)\varphi(y)ds(y)$$

for all $x \in D$.

Surface integral formulation of ITP

If we define

$$\alpha := \frac{\partial v}{\partial \nu}|_\Gamma = \frac{1}{\mu}\frac{\partial w}{\partial \nu}|_\Gamma \in H^{-1/2}(\Gamma), \qquad \beta := v|_\Gamma = w|_\Gamma \in H^{1/2}(\Gamma),$$

$$\alpha' := \frac{1}{\mu}\frac{\partial w}{\partial \nu}|_\Sigma \in H^{-1/2}(\Sigma), \qquad \beta' := w|_\Sigma \in H^{1/2}(\Sigma).$$

then the solutions to (5.22) can be written in the form

$$v = \mathrm{SL}_{k_0}^\Gamma \alpha - \mathrm{DL}_{k_0}^\Gamma \beta,$$

$$w = \begin{cases} \mu \mathrm{SL}_{k_2}^\Gamma \alpha - \mathrm{DL}_{k_2}^\Gamma \beta - \mu \mathrm{SL}_{k_2}^\Sigma \alpha' + \mathrm{DL}_{k_2}^\Sigma \beta' & \text{in } D_2, \\ \mu \mathrm{SL}_{k_1}^\Sigma \alpha' - \mathrm{DL}_{k_1}^\Sigma \beta' & \text{in } D_1. \end{cases}$$

We now need to make the difference between the surface potentials defined on Σ or on Γ. To this end, we define new notations

$$\begin{aligned}
S^{\Gamma}_{k_i}(\varphi)(x) &= \int_{\Gamma} \Phi_{k_i}(x,y)\varphi(y)ds(y), & S^{\Sigma\to\Gamma}_{k_i}(\varphi)(x) &= \int_{\Sigma} \Phi_{k_i}(x,y)\varphi(y)ds(y),\\
K^{\Gamma}_{k_i}(\varphi)(x) &= \int_{\Gamma} \frac{\partial \Phi_{k_i}}{\partial \nu(y)}(x,y)\varphi(y)ds(y), & K^{\Sigma\to\Gamma}_{k_i}(\varphi)(x) &= \int_{\Sigma} \frac{\partial \Phi_{k_i}}{\partial \nu(y)}(x,y)\varphi(y)ds(y),\\
K'^{\Gamma}_{k_i}(\varphi)(x) &= \int_{\Gamma} \frac{\partial \Phi_{k_i}}{\partial \nu(x)}(x,y)\varphi(y)ds(y), & K'^{\Sigma\to\Gamma}_{k_i}(\varphi)(x) &= \int_{\Sigma} \frac{\partial \Phi_{k_i}}{\partial \nu(x)}(x,y)\varphi(y)ds(y),\\
T^{\Gamma}_{k_i}(\varphi)(x) &= \int_{\Gamma} \frac{\partial^2 \Phi_{k_i}}{\partial \nu(y)\nu(x)}(x,y)\varphi(y)ds(y), & T^{\Sigma\to\Gamma}_{k_i}(\varphi)(x) &= \int_{\Sigma} \frac{\partial^2 \Phi_{k_i}}{\partial \nu(y)\nu(x)}(x,y)\varphi(y)ds(y),
\end{aligned}$$

for all $x \in \Gamma$.

From the boundary conditions of (5.22) and the continuity of w through Σ we get

$$\underbrace{\left[\begin{pmatrix} S^{\Gamma}_{k_0} & -K^{\Gamma}_{k_0} \\ -K'^{\Gamma}_{k_0} & T^{\Gamma}_{k_0}\end{pmatrix} - \begin{pmatrix} \mu S^{\Gamma}_{k_2} & -K^{\Gamma}_{k_2} \\ -K^{\Gamma}_{k_2} & 1/\mu T^{\Gamma}_{k_2}\end{pmatrix}\right]}_{Z^{\Gamma}_{02}(k)} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \underbrace{\begin{pmatrix} \mu S^{\Sigma\to\Gamma}_{k_2} & -K^{\Sigma\to\Gamma}_{k_2} \\ -K'^{\Sigma\to\Gamma}_{k_2} & 1/\mu T^{\Sigma\to\Gamma}_{k_2}\end{pmatrix}}_{Z^{\Sigma\to\Gamma}(k)} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = 0$$

$$\underbrace{\left[\begin{pmatrix} \mu S^{\Sigma}_{k_1} & -K^{\Sigma}_{k_1} \\ -K'^{\Sigma}_{k_1} & 1/\mu T^{\Sigma}_{k_1}\end{pmatrix} + \begin{pmatrix} \mu S^{\Sigma}_{k_2} & -K^{\Sigma}_{k_2} \\ -K'^{\Sigma}_{k_2} & 1/\mu T^{\Sigma}_{k_2}\end{pmatrix}\right]}_{Z'^{\Sigma}_{12}(k)} \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} - \underbrace{\begin{pmatrix} \mu S^{\Gamma\to\Sigma}_{k_2} & -K^{\Gamma\to\Sigma}_{k_2} \\ -K'^{\Gamma\to\Sigma}_{k_2} & 1/\mu T^{\Gamma\to\Sigma}_{k_2}\end{pmatrix}}_{Z^{\Gamma\to\Sigma}(k)} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$$

One can remark that the matrix $Z'^{\Sigma}_{12}(k)$ corresponds to the transmission problem

$$\begin{cases} \Delta w + k_1^2 w = 0 & \text{in } D_1 \\ \Delta v + k_2^2 v = 0 & \text{in } \mathbb{R}^d \setminus \overline{D_1} \\ w - v = h \in H^{1/2}(\Sigma) & \text{on } \Sigma \\ \dfrac{\partial w}{\partial \nu} - \dfrac{1}{\mu}\dfrac{\partial v}{\partial \nu} = g \in H^{-1/2}(\Sigma) & \text{on } \Sigma \\ \lim\limits_{r\to\infty} r^{\frac{d-1}{2}} \left(\dfrac{\partial v}{\partial r} - \mathrm{i}kv\right) = 0 & \end{cases} \tag{5.23}$$

with $w \in H^1(D_1)$ and $v \in H^1_{loc}(\mathbb{R}^d \setminus \overline{D_1})$. This is a classical scattering problem that has a unique solution $(w,v) \in H^1(D_1) \times H^1_{loc}(\mathbb{R}^d \setminus \overline{D_1})$.

Now, since w satisfies the Helmholtz equation in D_1 with wave number k_1 and v is a radiating solution to the Helmholtz equation with wave number k_2, they have an integral representation of the form

$$w(x) = \mathrm{SL}^{\Sigma}_{k_1} \frac{\partial w}{\partial \nu}|_{\Sigma}(x) - \mathrm{DL}^{\Sigma}_{k_1} w|_{\Sigma}(x)$$

$$v(x) = -\mathrm{SL}^{\Sigma}_{k_2} \frac{\partial v}{\partial \nu}|_{\Sigma}(x) - \mathrm{DL}^{\Sigma}_{k_2} v|_{\Sigma}(x).$$

Using the boundary conditions satisfied by w and v and the jump properties of the potentials, we obtain equivalence between solving (5.23) and solving the following integral equation

$$Z'^{\Sigma}_{12}(k)\left(\begin{array}{c}\dfrac{1}{\mu}\dfrac{\partial v}{\partial \nu}|_\Sigma\\ v|_\Sigma\end{array}\right)=\left(\begin{array}{cc}-\mu S_{k_1} & K_{k_1}+\dfrac{1}{2}I\\ (K'_{k_1}-\dfrac{1}{2}I) & -\dfrac{1}{\mu}T_{k_1}\end{array}\right)\left(\begin{array}{c}\dfrac{1}{\mu}g\\ h\end{array}\right)$$

where I denotes the identity operator on $H^{1/2}(\Sigma)$.

Consequently, the operator $Z'^{\Sigma}_{12}(k) : H^{-1/2}(\Sigma)\times H^{-1/2}(\Sigma)\to H^{1/2}(\Sigma)\times H^{-1/2}(\Sigma)$ is invertible and we can rewrite the problem as

$$\mathcal{Z}(k)\left(\begin{array}{c}\alpha\\ \beta\end{array}\right)=0 \tag{5.24}$$

where

$$\mathcal{Z}(k) := Z^{\Gamma}_{02}(k)+Z^{\Sigma\to\Gamma}(k)Z'^{\Sigma}_{12}(k)^{-1}Z^{\Gamma\to\Sigma}(k).$$

The matrix $Z^{\Gamma}_{02}(k)$ corresponds to the interior transmission problem for n constant equal to n_2 inside D. We will show that $\mathcal{Z}(k)$ is a compact perturbation of $Z^{\Gamma}_{02}(k)$.

The following theorem shows that equivalence between the interior transmission problem and the formulation with $\mathcal{Z}(k)$. Again, to get the equivalence, we need to add the condition that the far field generated by (α,β) vanishes.

Theorem 5.4.1. *The two following assertions are equivalent.*

(i) *There exists $(w,v)\in H^1(D)\times H^1(D)$ a non trivial solution to* (5.22)

(ii) *There exist $\alpha\neq 0$ in $H^{-1/2}(\Gamma)$ and $\beta\neq 0$ in $H^{1/2}(\Gamma)$ such that* (5.24) *is satisfied and*

$$P^{\infty}_0\left(\begin{array}{c}\alpha\\ \beta\end{array}\right)=0.$$

Proof. Let $\alpha\neq 0$ in $H^{-1/2}(\Gamma)$ and $\beta\neq 0$ in $H^{1/2}(\Gamma)$ satisfying (5.24). We set

$$\left(\begin{array}{c}\alpha'\\ \beta'\end{array}\right) := Z'^{\Sigma}_{12}(k)^{-1}Z^{\Gamma\to\Sigma}(k)\left(\begin{array}{c}\alpha\\ \beta\end{array}\right).$$

Let us define

$$v=\mathrm{SL}^{\Gamma}_{k_0}\alpha-\mathrm{DL}^{\Gamma}_{k_0}\beta$$

and

$$w=\begin{cases}\mathrm{SL}^{\Gamma}_{k_2}\alpha-\mathrm{DL}^{\Gamma}_{k_2}\beta-\mathrm{SL}^{\Sigma}_{k_2}\alpha'+\mathrm{DL}^{\Sigma}_{k_2}\beta' & \text{in } D_2,\\ \mathrm{SL}^{\Sigma}_{k_1}\alpha'-\mathrm{DL}^{\Sigma}_{k_1}\beta' & \text{in } D_1.\end{cases}$$

The regularity of the single and the double layer potentials shows that $v\in H^1(D)$ and $w\in H^1(D_1)\cap H^1(D_2)$. Furthermore

$$w|^{+}_{\Sigma}=S^{\Gamma\to\Sigma}_{k_2}\alpha-K^{\Gamma\to\Sigma}_{k_2}\beta-S^{\Sigma}_{k_2}\alpha'+K^{\Sigma}_{k_2}\beta'+\frac{1}{2}\beta',\qquad w|^{-}_{\Sigma}=S^{\Sigma}_{k_1}\alpha'-K^{\Sigma}_{k_1}\beta'+\frac{1}{2}\beta',$$

$$\frac{\partial w}{\partial \nu}|^{+}_{\Sigma}=K'^{\Gamma\to\Sigma}_{k_2}\alpha-T^{\Gamma\to\Sigma}_{k_2}\beta-K'^{\Sigma}_{k_2}\alpha'+\frac{1}{2}\alpha'+T^{\Sigma}_{k_2}\beta',\qquad \frac{\partial w}{\partial \nu}|^{-}_{\Sigma}=K'^{\Sigma}_{k_1}\alpha'+\frac{1}{2}\alpha'-T^{\Sigma}_{k_1}\beta',$$

and then

$$\left(\begin{array}{c} w|_\Sigma^+ - w|_\Sigma^- \\ \dfrac{\partial w}{\partial \nu}|_\Sigma^+ - \dfrac{\partial w}{\partial \nu}|_\Sigma^- \end{array}\right) = Z^{\Gamma\to\Sigma}(k)\left(\begin{array}{c} \beta \\ \alpha \end{array}\right) - Z'^{\Sigma}_{12}(k)\left(\begin{array}{c} \beta' \\ \alpha' \end{array}\right) = 0$$

by definition of α' and β'. We deduce that $w \in H^1(D)$.

Now we must show that $v \neq 0$ or $w \neq 0$. Assume that $P_0^\infty \left(\begin{array}{c} \alpha \\ \beta \end{array}\right) = 0$. From Rellich's lemma, we deduce that $v = 0$ in $\mathbb{R}^d \backslash D$. Assume that $v = 0$ also in D. We have in particular that

$$[v]_\Gamma = \left[\frac{\partial v}{\partial \nu}\right]_\Gamma = 0$$

and from the jump properties of the single and double layer potentials we also have that

$$[v]_\Gamma = -\beta \text{ and } \left[\frac{\partial v}{\partial \nu}\right]_\Gamma = -\alpha.$$

This contradicts the fact that $(\alpha, \beta) \neq (0,0)$. Then $v \neq 0$ in D and as a consequence we also have that $w \neq 0$ in D. □

Lemma 5.4.2. *The operator $\mathcal{Z}(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is Fredholm of index zero and analytic on $k \in \mathbb{C}\backslash\mathbb{R}^-$.*

Proof. From lemma 5.2.4, the operator $Z_{02}^\Gamma(k)$ is Fredholm from $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ into $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$. Finally, the operators

$$Z^{\Sigma\to\Gamma}(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma)$$

and

$$Z^{\Gamma\to\Sigma}(k) : H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$$

are compact due to the regularity of the kernels and

$$Z'^{\Sigma}_{12}(k)^{-1} : H^{1/2}(\Sigma) \times H^{-1/2}(\Sigma) \to H^{-1/2}(\Sigma) \times H^{1/2}(\Sigma)$$

is continuous. This shows that

$$Z^{\Sigma\to\Gamma}(k) Z'^{\Sigma}_{12}(k)^{-1} Z^{\Gamma\to\Sigma}(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$$

is compact. Consequently, the operator $\mathcal{Z}(k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is Fredholm. □

Discreteness of the set of transmission eigenvalues

Lemma 5.4.3. *Assume that $\mu - 1$ and $1 - n_2$ are either positive or negative. Then the operator $\mathcal{Z}(\mathrm{i}k) : H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is injective for $k \in \mathbb{R}^{*+}$.*

Proof. Assume that $\mathcal{Z}(\mathrm{i}k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=0$. Let us define

$$v := \mathrm{SL}^{\Gamma}_{\mathrm{i}k_0}\alpha - \mathrm{DL}^{\Gamma}_{\mathrm{i}k_0}\beta \text{ in } \mathbb{R}^d\backslash\Gamma$$

and

$$w := \begin{cases} \mu\mathrm{SL}^{\Gamma}_{\mathrm{i}k_2}\alpha - \mathrm{DL}^{\Gamma}_{\mathrm{i}k_2}\beta - \mu\mathrm{SL}^{\Sigma}_{\mathrm{i}k_2}\alpha' + \mathrm{DL}^{\Sigma}_{\mathrm{i}k_2}\beta' & \text{in } \mathbb{R}^d\backslash\left(\overline{D}_1\cup\Gamma\right) \\ \mu\mathrm{SL}^{\Sigma}_{\mathrm{i}k_1}\alpha' - \mathrm{DL}^{\Sigma}_{\mathrm{i}k_1}\beta' & \text{in } D_1 \end{cases}$$

where

$$\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix} := Z'^{\Sigma}_{12}(k)^{-1}Z^{\Gamma\to\Sigma}(k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

The relation $\mathcal{Z}(\mathrm{i}k)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}=0$ implies that on Γ we have $w^{\pm}=v^{\pm}$ and $\frac{1}{\mu}\frac{\partial w^{\pm}}{\partial\nu}=\frac{\partial v^{\pm}}{\partial\nu}$. Consequently, the pair $(w,v)\in\left(H^1(\mathbb{R}^d\backslash\overline{D})\cup H^1(D)\right)\times\left(H^1(\mathbb{R}^d\backslash\overline{D})\cup H^1(D)\right)$ is solution to

$$\begin{cases} \Delta v - k^2 v = 0 & \text{in } \mathbb{R}^d\backslash\Gamma \\ \nabla\cdot\dfrac{1}{\mu}\nabla w - k^2 n_2 w = 0 & \text{in } \mathbb{R}^d\backslash(\overline{D_1}\cup\Gamma) \\ \nabla\cdot\dfrac{1}{\mu}\nabla w - k^2 n_1 w = 0 & \text{in } D_1 \\ w^{\pm} = v^{\pm} & \text{on } \Gamma \\ \dfrac{1}{\mu}\dfrac{\partial w^{\pm}}{\partial\nu} = \dfrac{\partial v^{\pm}}{\partial\nu} & \text{on } \Gamma \end{cases}$$

Let us define the Hilbert space

$$\mathbb{H} := \left\{(w,v)\in\left(H^1(\mathbb{R}^d\backslash\overline{D})\cup H^1(D)\right)\times\left(H^1(\mathbb{R}^d\backslash\overline{D})\cup H^1(D)\right)/w^{\pm}=v^{\pm}\text{ on }\Gamma\right\}.$$

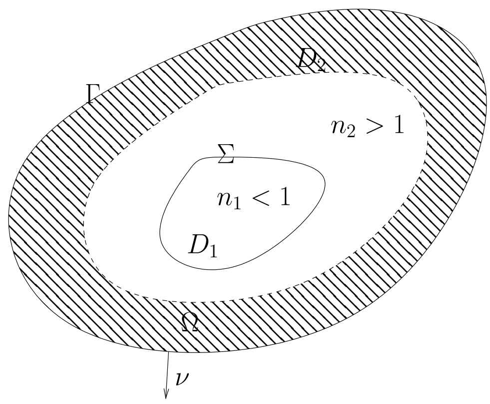
The corresponding variational formulation is: find (w,v) in $\mathbb{H}$ such that

$$a_k((w,v),(\varphi,\psi))=0$$

for all (φ,ψ) in $\mathbb{H}$ where

$$a_k((w,v),(\varphi,\psi)) := \int_{\mathbb{R}^d\backslash\Gamma}\left(\frac{1}{\mu}\nabla w\cdot\nabla\overline{\varphi}+k^2 n w\overline{\varphi}\right)dx - \int_{\mathbb{R}^d\backslash\Gamma}\left(\nabla v\cdot\nabla\overline{\psi}+k^2 v\overline{\psi}\right)dx.$$

Let Ω be a neighborhood of Γ such that $\Omega\cap D_1=\emptyset$.



Let us define the cutoff function χ with compact support in Ω such that $\chi = 1$ on Γ.

1. Case where $\mu < 1$ and $n_2 > 1$. Let us define the isomorphism

$$\begin{array}{rccc} T: & \mathbb{H}\times\mathbb{H} & \rightarrow & \mathbb{H}\times\mathbb{H} \\ & (w,v) & \mapsto & (w, -v+2\chi w). \end{array}$$

$$\begin{aligned}
a_k((w,v),T(w,v)) = & \int_{\mathbb{R}^d\setminus\Gamma}\left(\frac{1}{\mu}|\nabla w|^2 + k^2 n|w|^2 + |\nabla v|^2 + k^2|v|^2\right)dx \\
& - 2\int_\Omega \nabla v\cdot\nabla(\chi\overline{w}) - 2k^2\int_\Omega \chi v\overline{w} \\
\geq & \frac{1}{\mu}||\nabla w||^2_{L^2(\mathbb{R}^d)} + k^2 n_1||w||^2_{L^2(D_1)} + k^2 n_2||w||^2_{L^2(\mathbb{R}^d\setminus\overline{D}_1)} + ||\nabla v||^2_{L^2(\mathbb{R}^d)} \\
& + k^2||v||^2_{L^2(\mathbb{R}^d)} - \alpha||\nabla v||^2_{L^2(\mathbb{R}^d)} - \frac{1}{\alpha}||\nabla w||^2_{L^2(\mathbb{R}^d)} - C\eta||\nabla v||^2_{L^2(\mathbb{R}^d)} \\
& - \frac{C}{\eta}||w||^2_{L^2(\mathbb{R}^d\setminus\overline{D}_1)} - k^2\beta||v||^2_{L^2(\mathbb{R}^d)} - \frac{k^2}{\beta}||w||^2_{L^2(\mathbb{R}^d\setminus\overline{D}_1)} \\
\geq & (1-\alpha-C\eta)\,||\nabla v||^2_{L^2(\mathbb{R}^d)} + k^2(1-\beta)||v||^2_{L^2(\mathbb{R}^d)} + \left(\frac{1}{\mu}-\frac{1}{\alpha}\right)||\nabla w||^2_{L^2(\mathbb{R}^d)} \\
& + k^2 n_1||w||^2_{L^2(D_1)} + \left(k^2\left(n_2-\frac{1}{\beta}\right)-\frac{C}{\eta}\right)||w||^2_{L^2(\mathbb{R}^d\setminus\overline{D}_1)}.
\end{aligned}$$

Let $\mu < \alpha < 1$, $1/n_2 < \beta < 1$ and η such that $1-\alpha-C\eta > 0$ fixed. Then if k is large enough to have $k^2\left(n_2 - \frac{1}{\beta}\right) - \frac{C}{\eta} > 0$, we deduce that a_k is T-coercive. As a consequence, $w = 0$ and $v = 0$ are the only solutions. From the equality $[v]_\Gamma = -\beta$ and $\left[\frac{\partial v}{\partial\nu}\right]_\Gamma = -\alpha$ we get that $\alpha = \beta = 0$ and finally $\mathcal{Z}(\mathrm{i}k) : H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma) \rightarrow H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ is injective.

2. Case where $\mu > 1$ and $n_2 < 1$. Let us define the isomorphism

$$\begin{array}{rccc} T: & \mathbb{H}\times\mathbb{H} & \rightarrow & \mathbb{H}\times\mathbb{H} \\ & (w,v) & \mapsto & (-w+2\chi v, v). \end{array}$$

$$
\begin{aligned}
a_k((w,v),T(w,v)) &= \int_{\mathbb{R}^d} \left(|\nabla v|^2 + k^2|v|^2 + \frac{1}{\mu}|\nabla w|^2 + k^2 n|w|^2 \right) dx \\
&\quad - 2\int_\Omega \frac{1}{\mu}\nabla w\cdot\nabla(\chi\overline{v}) - 2k^2\int_\Omega n\chi w\overline{v} \\
&\geq ||\nabla v||^2_{L^2(\mathbb{R}^d)} + k^2||v||^2_{L^2(\mathbb{R}^d)} + \mu\left|\left|\frac{1}{\mu}\nabla w\right|\right|^2_{L^2(\mathbb{R}^d)} + k^2 n_1||w||^2_{L^2(D_1)} \\
&\quad - \frac{1}{\alpha}||\nabla v||^2_{L^2(\mathbb{R}^d)} - \alpha\left|\left|\frac{1}{\mu}\nabla w\right|\right|^2_{L^2(\mathbb{R}^d)} - \frac{C}{\eta}||v||^2_{L^2(\mathbb{R}^d)} - \beta C\left|\left|\frac{1}{\mu}\nabla w\right|\right|^2_{L^2(\mathbb{R}^d\setminus\overline{D}_1)} \\
&\quad - \frac{k^2}{\eta}||v||^2_{L^2(\mathbb{R}^d)} - k^2\eta||n_2 w||^2_{L^2(\mathbb{R}^d\setminus\overline{D}_1)} + \frac{k^2}{n_2}||n_2 w||^2_{L^2(\mathbb{R}^d\setminus\overline{D}_1)} \\
&\geq \left(1-\frac{1}{\alpha}\right)||\nabla v||^2_{L^2(\mathbb{R}^d)} + \left(k^2\left(1-\frac{1}{\eta}\right)-\frac{C}{\beta}\right)||v||^2_{L^2(\mathbb{R}^d)} + k^2 n_1||w||^2_{L^2(D_1)} \\
&\quad + (\mu-\alpha-\beta C)\,||\frac{1}{\mu}\nabla w||^2_{L^2(\mathbb{R}^d)} + k^2\left(\frac{1}{n_2}-\eta\right)||n_2 w||^2_{L^2(\mathbb{R}^d\setminus\overline{D}_1)}.
\end{aligned}
$$

Let $1 < \alpha < \mu$, $1 < \eta < 1/n_2$ and β such that $\mu - \alpha - \beta C > 0$ fixed. Then for k large enough to have $k^2\left(1-\frac{1}{\eta}\right) - \frac{C}{\beta} > 0$, a_k is T-coercive. Similarly we deduce that $\mathcal{Z}(\mathrm{i}k) : H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)$ is injective.

□

Theorem 5.4.4. *Assume that $\mu - 1$ and $1 - n_2$ are either positive or negative. Then, the set of transmission eigenvalues is discrete.*

Proof. This is a direct consequence of 5.4.2 and 5.4.3 using the analytic Fredholm theory.

□

5.4.2 The inhomogeneous case

The results derived in the previous section for the case of piecewise constant coefficients can be easily generalized to the case where μ and n are constant in a neighborhood $O \subset D$ of the boundary Γ. In this case one can consider Σ to be a regular surface lying in O so that the region between Σ and Γ is connected (for instance, in the case of regular boundary Γ, one can choose $\Sigma = \Gamma - \delta\nu$ where δ is a sufficiently small parameter). The previous analysis then holds true if one replaces $\Phi_{k_1}(\cdot,y)$ with the fundamental solution $\mathbb{G}(\cdot,y) \in H^1_{loc}(\mathbb{R}^d)\setminus\{y\}$ of

$$\nabla\cdot\frac{1}{\mu}\nabla\mathbb{G}(\cdot,y) + k^2 n\mathbb{G}(\cdot,y) = -\delta_y \text{ in } \mathbb{R}^d,$$

in the distributional sense and satisfying the Sommerfeld radiation condition (we extend μ and n outside D by their constant values in O). Since, for all $y \in \mathbb{R}^d$,

$$x \mapsto \mathbb{G}(x,y) - \Phi_{k_2}(x,y)$$

satisfies the Helmholtz equation (with constant coefficients) in O, then this function is a C^∞ function in O. By symmetry, the same holds for $y \mapsto \mathbb{G}(x,y) - \Phi_{k_2}(x,y)$. Therefore the (previously introduced) potentials defined on Σ with $\mathbb{G}$ replacing Φ_{k_1} keep exactly the same mapping properties. The invertibility of $Z'^{\Sigma}_{12}(k)$ is ensured as long as the forward scattering problem associated with μ and n is well posed. The latter is for instance true if one assumes n and μ to be bounded functions with non negative imaginary parts and positive definte real parts. Finally, the proof of injectivity of $\mathcal{Z}(ik)$ for positive k can be reproduced with minor obvious modifications in the current setting.

We therefore could state the following theorem

Theorem 5.4.5. *Assume that n and μ are bounded functions with non negative imaginary parts and positive definte real parts and further assume that $\mu - 1$ and $1 - n$ are either positive or negative constants in a neighborhood of Γ. Then, the set of transmission eigenvalues is discrete.*

5.4.3 The case of $\mu = 1$

Indeed our analysis for the case $\mu \neq 1$ extends to the case $\mu = 1$ when one uses the appropriate function spaces. For instance, following exactly the same procedure as in the previous section and applying the analysis done for the case $\mu = 1$ and $n = cte \neq 1$ in the neighborhood of Γ, then with

$$\mathcal{Z}(k) := Z^{\Gamma}_{02}(k) + Z^{\Sigma\to\Gamma}(k) Z'^{\Sigma}_{12}(k)^{-1} Z^{\Gamma\to\Sigma}(k)$$

we have:

Lemma 5.4.6. *Assume that $\mu = 1$ and n is a bounded function with non negative imaginary part and further assume that $1 - n$ is either a positive or negative constant in a neighborhood of Γ. Then, the operator $\mathcal{Z}(k) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is Fredholm and analytic on $k \in \mathbb{C}\backslash\mathbb{R}^-$.*

The only missing point here is to prove injectivity when $n - 1$ changes sign. We refer to [50] where injectivity is proved for purely imaginary wavenumbers with large enough modulus.

Chapter 6

A numerical method to compute transmission eigenvalues based on surface integral equations

This chapter is devoted to different methods to compute transmission eigenvalues. The first approach is introduced in Section 6.1 and uses solutions to the ITP. We shall implement the method introduced in Chapter 5 based on reformulating the ITP as a surface integral equation. Computing transmission eigenvalues is equivalent to solving a system of the form

$$Z(k)X = 0$$

where $Z(k)$ is an appropriate surface operator. Numerically, the idea is to compute the eigenvalues of $Z(k)$ and look for values of k for which the smallest eigenvalue is close to zero. However, in the case where the magnetic permeability contrast is zero, due to the compactness of the operator $Z(k)$, its eigenvalues accumulate to zero. Consequently the eigenvalue zero would be "lost" in the set of smallest eigenvalues, due to numerical errors. In order to get around his difficulty, we use a preconditioner $B(k)$ and solve a generalized eigenvalue problem of the form

$$Z(k)X = \lambda B(k)X.$$

Choosing $B(k)$ to be injective implies that the eigenvalues $\lambda = 0$ correponds with k being a transmission eigenvalue. Considering operators $B(k)$ with principal part that coincides with the principal part of $Z(k)$ would shift the accumulation point for λ out of zero.

Finally, the last method is inspired from Theorem 2.5.1 which characterizes the transmission eigenvalues from far field data. On the contrary to the LSM which takes a sample of points z and compute the norm of the regularized solution to the far field equation $||g_{z,k}||$ for only one k, here we fix one or several points z inside the obstacle and compute $||g_{z,k}||$ for a sample of wave numbers k. As suggested in Theorem 2.5.1, transmission eigenvalues are located from the peaks of $||g_{z,k}||$ against k.

In Appendix F, the scatterer is assumed to be a sphere. Since solutions to the Helmholtz equation and Maxwell's equations have analytical expansions using the spherical harmonics, one can characterize transmission eigenvalues as the zeros of determinants with analytical expressions. As a consequence, accurate values of transmission eigenvalues can be computed, which give reference values to validate the other methods developed in this chapter that treat more general geometries.

6.1 Computation of transmission eigenvalues in the scalar case

6.1.1 Integral equations representation

We shall adopt here the notations of Chapter 5 and provide details only for the case μ and n constant. The case of piecewise constant coefficients can be deduced as in Chapter 5 section 5.4.1.

We recall that $k_0 := k$ and $k_1 := k\sqrt{n\mu}$ and assume that k is a positive real. The corresponding interior transmission problem is

$$\begin{cases} \nabla \cdot \dfrac{1}{\mu} \nabla w + k^2 n w = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \Gamma \\ \dfrac{1}{\mu} \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} & \text{on } \Gamma. \end{cases} \tag{6.1}$$

As seen in Theorem 5.3.5 in Chapter 5, there exists a non trivial solution to this interior transmission problem if and only if there exists $(\alpha, \beta) \neq (0,0)$ in $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ when $\mu \neq 1$ and in $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ when $\mu = 1$ such that

$$Z(k) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \text{ and } P_0^\infty \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \tag{6.2}$$

where $Z(k)$ is given by

$$Z(k) := \begin{pmatrix} \mu S_{k_1} & -K_{k_1} \\ K'_{k_1} & -\dfrac{1}{\mu} T_{k_1} \end{pmatrix} - \begin{pmatrix} S_{k_0} & -K_{k_0} \\ K'_{k_0} & -T_{k_0} \end{pmatrix}$$

and P_0^∞ is the far field operator defined by

$$P_0^\infty \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (\hat{x}) := \int_\Gamma \alpha(y) e^{-\mathrm{i}k\hat{x}\cdot y} - \beta(y) \frac{\partial e^{-\mathrm{i}k\hat{x}\cdot y}}{\partial \nu(y)} ds(y).$$

A simple idea to compute transmission eigenvalue is to compute the eigenvalues of the operator $Z(k)$ for each k and the values of k for which $Z(k)$ has the eigenvalue zero contain the transmission eigenvalues.

Difficulties

Two difficulties arise here. The first one is that this method is not numerically efficient in the scalar case for $\mu = 1$ since we have seen in Chapter 5 that the operator $Z(k)$: $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is compact and its eigenvalues accumulate to zero and we are not able to distinguish the eigenvalue zero if it exists (see figure 6.1).

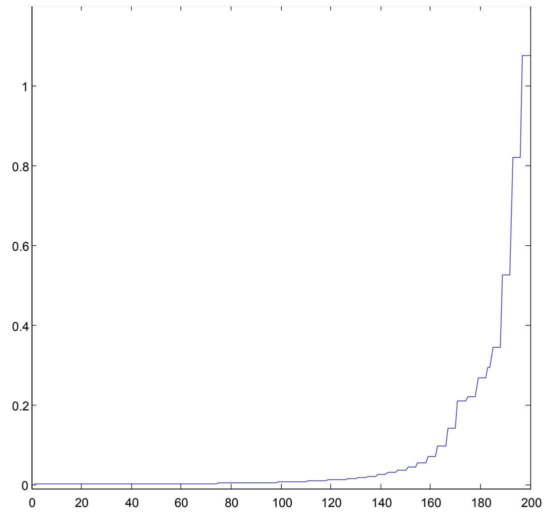


Figure 6.1: Eigenvalues of the operator $Z(k)$.

To get around this difficulty, we use a preconditioner in order to shift the accumulation away from 0. We consider the generalized eigenvalue problem of the form

$$Z(k)X = \lambda B(k)X.$$

We shall discuss in the next section an appropriate choice for the preconditioner $B(k)$.

The other difficulty is that only solving $Z(k)X = 0$ is not sufficient to get the transmission eigenvalues, we need to make sure that the far field pattern generated by X vanishes.

Let us first observe that (6.2) is equivalent to

$$\mathrm{Re}(Z(k)) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0 \text{ and } P_0^\infty \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$$

Indeed, if we only solve

$$\Re(Z(k))X = 0$$

we observe that we obtain other peaks not located at transmission eigenvalues which correspond to trivial solutions. However, it is shown in the Appendix C that

$$kP_j^{\infty *} P_j^\infty = \Im \begin{pmatrix} S_{k_j} & -K_{k_j} \\ -K'_{k_j} & T_{k_j} \end{pmatrix}.$$

Consequently, one possible method to solve is to consider the system

$$\Re(Z(k))X + \mathrm{i}\Im \begin{pmatrix} S_{k_0} & -K_{k_0} \\ -K'_{k_0} & T_{k_0} \end{pmatrix} X = 0.$$

Thus, we get the transmission eigenvalues by solving $\Re(Z(k))X = 0$ but the second part also ensures that the far field pattern generated by X vanishes for k_0. So the peaks found are only corresponding to transmission eigenvalues.

However, we observed numerically that solving with the whole matrix $Z(k)$ was sufficient but we therefore have no justification. Solving with the whole matrix $Z(k)$ implies that the far fields generated by X for k_0 and k_1 are equal but this does not necessarily implies that the solutions are not trivial in D.

For the electromagnetic case, we can observe numerically that the eigenvalues of $Z(k)$ also accumulate to zero but we have not proven yet that the operator $Z(k)$ is compact. This is one work that needs to be completed after the thesis. However, we use the same numerical procedure than for the scalar case that is to say that we compute the generalized eigenvalues of the problem of the form

$$Z(k)X = \lambda B(k)X.$$

6.1.2 Choice of the preconditioner in the case $\mu = 1$

In order to shift the accumulation of the eigenvalues of $Z(k)$, the idea is to change the problem into a generalized eigenvalue problem of the form

$$Z(k)X = \lambda B(k)X$$

where the preconditioner $B(k)$ is invertible so that the eigenvalue $\lambda = 0$ still corresponds to a transmission eigenvalues k and such that the eigenvalues now accumulate away from zero. To this end, we need for example a preconditioner with the same leading singularity so that $B(k)^{-1}Z(k)$ is Fredholm. In this way, we consider the interior transmission problem defined for pure imaginary wave numbers given by $w, v \in L^2(D)$, $w - v \in H^2(D)$,

$$\begin{cases} \Delta w - k^2 n w = 0 & \text{in } D \\ \Delta v - k^2 v = 0 & \text{in } D \\ w - v = g & \text{on } \Gamma \\ \dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = h & \text{on } \Gamma \end{cases} \tag{6.3}$$

where $g \in H^{-1/2}(\Gamma)$ and $h \in H^{-3/2}(\Gamma)$. The operator corresponding to the problem is $B(k) := Z(\mathrm{i}k)$. We have seen in Lemma 5.3.10 that this operator $Z(\mathrm{i}k)$ is injective from $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ into $H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ due to the fact that it is coercive.

The next step is to prove that $Z(\mathrm{i}k)^{-1}Z(k)$ is Fredholm.

Theorem 6.1.1. *Let $k > 0$. Then $B(k)^{-1}Z(k) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$ is Fredholm of index 0.*

Proof. We recall that $B(k) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is a bijection and that $Z(k) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is continuous. We can rewrite the operator $B(k)^{-1}Z(k)$ in the form

$$B(k)^{-1}Z(k) = -I + B(k)^{-1}(Z(k) + B(k))$$

where I denotes the identity operator on $H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$. According to Lemma 5.3.9, $B(k) + Z(k) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$ is compact. (If k is real, $\gamma(k_0, k_1) = 1$ in Lemma 5.3.9.) This implies the desired result. □

Theorem 6.1.2. *The only possible accumulation point of the eigenvalues $\lambda \in \mathbb{C}$ such that there exists $X \in H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$, $X \neq 0$ and*

$$Z(k)X = \lambda B(k)X$$

is -1.

Proof. The identity $Z(k)X = \lambda B(k)X$ implies

$$(\lambda + 1)X = B(k)^{-1}(Z(k) + B(k))X$$

which means that if $X \neq 0$, $(\lambda+1)$ is an eigenvalue of the compact operator $B(k)^{-1}(Z(k)+B(k)) : H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{-3/2}(\Gamma) \times H^{-1/2}(\Gamma)$. □

The following figure represents the numerical eigenvalues of the problem

$$Z(k)X = \lambda B(k)X.$$

It shows the accumulation at 1 of the absolute values of the eigenvalues λ.

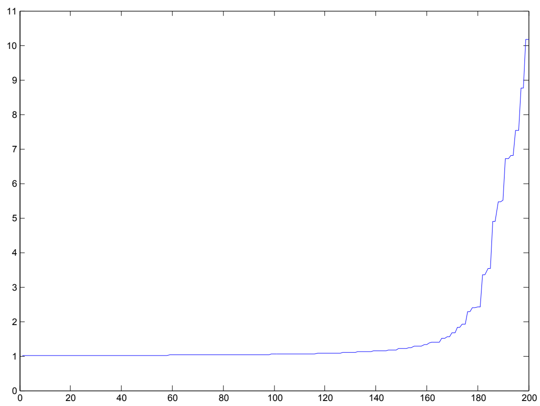


Figure 6.2: Generalized eigenvalues of $Z(k)X = \lambda Z(\mathrm{i}k)X$ for which the absolute values accumulate to 1

6.1.3 Description of the code

This problem is solved using the CESC code which consists in giving a variational formulation of the integral equations and using a $\mathbb{P}_1$ finite elements method. In a variational sense, the problem consists in finding (α, β) and λ such that for all (α', β'), we have

$$\int_\Gamma Z(k) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cdot \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} ds(x) = \lambda \int_\Gamma Z(\mathrm{i}k) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \cdot \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} ds(x). \tag{6.4}$$

Numerically, we shall use the more convenient expression for the normal derivative of the double layer potential given by

$$\int_\Gamma (T_k\beta)(x)\beta'(x)ds(x) = \int_\Gamma \int_\Gamma \Phi_k(x,y)b(x,y)ds(x)ds(y) \tag{6.5}$$

where

$$b(x,y) = (\beta(y)\nu(y)) \cdot (\beta'(x)\nu(x)) - \frac{1}{k^2}\frac{d\beta}{ds(y)}(y)\frac{d\beta'}{ds(x)}(x).$$

First, the boundary Γ is approximated by Γ^h with segments. The currents are then approximated by continuous functions for which the restriction to every segments is a linear function of the curvilinear abscissa. This approximed space of functions is denoted by V_h which is a finite dimensional space of dimension equals to the number of nodes. Indeed, the functions ϕ_n whose support are two consecutive segments, linear on each of them and which equal to 1 on the common node and 0 on the other two nodes form a basis of V_h for $n \in \mathcal{N}$ ($\mathcal{N}$ denotes the set of the nodes of the meshing).

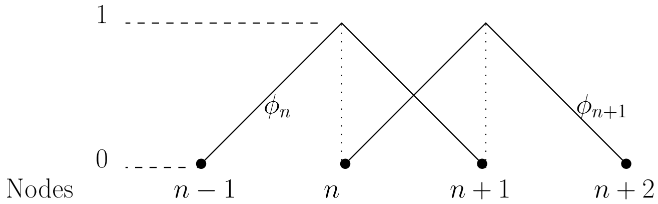


Figure 6.3: Basis functions

The currents can be decomposed in the form

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix}(x_\Gamma) = \sum_{n\in\mathcal{N}} \alpha_n \begin{pmatrix} \phi_n(x_\Gamma) \\ 0 \end{pmatrix} + \sum_{n\in\mathcal{N}} \beta_n \begin{pmatrix} 0 \\ \phi_n(x_\Gamma) \end{pmatrix}.$$

Since the variational formulation is verified for all $(\alpha', \beta') \in V_h \times V_h$ as soon as it is verified for all the basis functions $(\phi_m, 0)$ and $(0, \phi_m)$, $m \in \mathcal{N}$, of $V_h \times V_h$, we finally obtain a $2N$ linear system where N is the number of nodes of the meshing which can be written

$$A_k Y = \lambda A_{\mathrm{i}k} Y$$

where the unknown Y is given by

$$Y_n = \alpha_n, \quad Y_{n+N} = \beta_n, \quad n = 1, ..., N.$$

The matrices of the linear system are given by

$$\begin{aligned}
(A_k)_{n,m} &= \int_\Gamma \int_\Gamma \Phi_k(x,y)\phi_n(x)\phi_m(y)ds(x)ds(y), \\
(A_k)_{n+N,m} &= \int_\Gamma \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial \nu(x)}\phi_n(x)\phi_m(y)ds(x)ds(y), \\
(A_k)_{n,m+N} &= \int_\Gamma \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial \nu(y)}\phi_n(x)\phi_m(y)ds(x)ds(y), \\
(A_k)_{n+N,m+N} &= \int_\Gamma \int_\Gamma \Phi_k(x,y)b_{n,m}(x,y)ds(x)ds(y),
\end{aligned}$$

where

$$b_{n,m}(x,y) = \phi_n(x)\phi_m(y)\nu(x)\nu(y) - \frac{1}{k^2}\frac{d\phi_m}{ds}(y)\frac{d\phi_n}{ds}(x).$$

The matrices A_k and $A_{\mathrm{i}k}$ are assembled as above and finally, the eigenvalues λ_n, $n = 1, ..., N$, such that $0 < |\lambda_0| < |\lambda_1| < ...$, are computed using a function from the *lapack* library which computes the generalized eigenvalues of a system of the form $AX = \lambda BX$. We then plot the inverse of $|\lambda_0|$ against k and the transmission eigenvalues are located at the level of the peaks of this curve.

6.1.4 Computing transmission eigenvalues from far field data

We present the method for the acoustic case in $\mathbb{R}^d$, $d = 2, 3$.

Assume that D is a bounded domain of $\mathbb{R}^d$ with constant index of refraction n and assume that $n = 1$ in the exterior domain $\mathbb{R}^d \backslash \overline{D}$. We consider the scattering of a plane wave incident field $u^i(x, d) = e^{ikx\cdot d}$, for $x \in \mathbb{R}^d$ and direction of propagation $d \in \Omega$, where Ω is the unit sphere. The forward problem has the form

$$\begin{cases} \Delta u + k^2 nu = 0 & \text{in } \mathbb{R}^d \\ u = u^i + u^s & \text{in } \mathbb{R}^d \end{cases}$$

where the scattered field u^s satisfies the Sommerfeld radiating condition

$$\lim_{r\to\infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - \mathrm{i}ku^s \right) = 0.$$

<u>**Generation of the far fields**</u>

The far field pattern is given by

$$u^\infty(\hat{x}) = \gamma \int_\Omega \left(\frac{\partial u^s}{\partial \nu}|_\Gamma(y) e^{-\mathrm{i}k\hat{x}\cdot y} - u^s|_\Gamma(y) \frac{\partial e^{-\mathrm{i}k\hat{x}\cdot y}}{\partial \nu} \right) ds(y)$$

where

$$\gamma = \begin{cases} \dfrac{e^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8\pi k}} & \text{if } n = 2 \\ \frac{1}{4\pi} & \text{if } n = 3. \end{cases}$$

The total field u can be represented by integral equations in D. Let $\beta' := u|_\Gamma$ denote the trace of the total field on the boundary Γ of D and $\alpha' := \dfrac{\partial u}{\partial \nu}|_\Gamma$ the normal derivative of u on Γ. First, we solve the direct scattering problem by finding the boundary values of the total field u using an integral equation method.

Inside D, u solves the Helmholtz equation with wave number k_1 and consequently has the following representation

$$u(x) = \mathrm{SL}_{k_1}\alpha' - \mathrm{DL}_{k_1}\beta' \quad \text{in } D$$

and with the jump properties of the single and double layer potentials, we get on the boundary

$$S_{k_1}\alpha' - K_{k_1}\beta' - \frac{1}{2}\beta' = 0 \tag{6.6}$$

$$K'_{k_1}\alpha' - \frac{1}{2}\alpha' - T_{k_1}\beta' = 0. \tag{6.7}$$

Besides, u^i is a solution to Helmholtz equation in D and consequently on the boundary we have

$$u^i|_\Gamma = S_{k_0}\frac{\partial u^i}{\partial \nu}|_\Gamma - K_{k_0}u^i|_\Gamma + \frac{1}{2}u^i|_\Gamma, \tag{6.8}$$

$$\frac{\partial u^i}{\partial \nu}|_\Gamma = K'_{k_0}\frac{\partial u^i}{\partial \nu}|_\Gamma + \frac{1}{2}\frac{\partial u^i}{\partial \nu}|_\Gamma + T_{k_0}u^i|_\Gamma \tag{6.9}$$

and u^s is a radiating solution to Helmholtz equation and satisfies on the boundary

$$S_{k_0}\frac{\partial u^s}{\partial \nu}|_\Gamma - K_{k_0}u^s|_\Gamma + \frac{1}{2}u^s|_\Gamma = 0, \tag{6.10}$$

$$K'_{k_0}\frac{\partial u^s}{\partial \nu}|_\Gamma + \frac{1}{2}\frac{\partial u^s}{\partial \nu}|_\Gamma - T_{k_0}u^s|_\Gamma = 0. \tag{6.11}$$

Adding (6.8) with (6.10) and (6.9) with (6.11) and using the fact that $\alpha = u^i|_\Gamma + u^s|_\Gamma$ and $\beta = \frac{\partial u^i}{\partial \nu}|_\Gamma + \frac{\partial u^s}{\partial \nu}|_\Gamma$, we get

$$u^i|_\Gamma = S_{k_0}\alpha' - K_{k_0}\beta' + \frac{1}{2}\beta',$$

$$\frac{\partial u^i}{\partial \nu}|_\Gamma = K'_{k_0}\alpha' + \frac{1}{2}\alpha' - T_{k_0}\beta'.$$

Combining the two previous equations with (6.6) and (6.7), we obtain

$$u^i|_\Gamma = (S_{k_0} + S_{k_1})\alpha' - (K_{k_0} + K_{k_1})\beta',$$

$$\frac{\partial u^i}{\partial \nu}|_\Gamma = (K'_{k_0} + K'_{k_1})\alpha' - (T_{k_0} + T_{k_1})\beta'.$$

Remark 6.1.1. *Here, we only formulate the problem for a constant index of refraction n. If the domain D contains an inclusion D_0 with a different index of refraction that can be equal to one, the two previous equations are combined with the values of u on the boundary of D_0 and two more equations that render the continuity of the total field u across the boundary of D_0.*

We solve the forward problem for N incident waves with direction d_ℓ, $\ell = 1, ...N$. The corresponding values α'_ℓ and β'_ℓ are found using a finite element method. To this end, we discretize the density α'_ℓ and β'_ℓ with a $\mathbb{P}_1$-continuous finite element method and the surface Γ is approximated by a triangle meshing.

Remark 6.1.2. *We can also use a $\mathbb{P}_0$ method for α.*

We deduce the data $u^\infty(d_j, d_\ell)$ generated by a plane wave of direction d_ℓ and evaluated on d_j which is given by

$$u^\infty(d_j, d_\ell) = \int_\Gamma \alpha_\ell(y) e^{-\mathrm{i}k d_j \cdot y} - \beta_\ell(y) \frac{\partial e^{-\mathrm{i}k d_j \cdot y}}{\partial \nu} ds(y)$$

where α_ℓ and β_ℓ are the boundary data of u^s found from the computed boundary data α'_ℓ and β'_ℓ of u.

Remark 6.1.3. *On the contrary to the classical use of the linear sampling method where the far field equation is solved for only one k but for a sample of source points z, to compute transmission eigenvalues the source point z is fixed, we need to solve the far field equation for a sample of wave numbers k. Consequently, the direct problem must be solved for each wave number k.*

Discretized far field equation

Given $F^\infty_{\ell,j} := u^\infty(d_j, d_\ell)$ the approximated far field pattern for N incident plane waves with directions d_ℓ and measured on the same directions d_j, we now want to solve the far field equation

$$\mathcal{F} g_z(d_j) := \int_\Omega u^\infty(d_j, d) g(d) ds(d) = \gamma e^{-\mathrm{i}k d_j \cdot z}, \quad j = 1, ..., N,$$

with

$$\gamma = \begin{cases} \dfrac{e^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8\pi k}} & \text{if } n = 2 \\ \frac{1}{4\pi} & \text{if } n = 3 \end{cases}$$

and where the right hand side is the far field generated by a source point located at z. The unit sphere is discretized using a triangle meshing. The unknown g_z can be decomposed with respect to the basis ϕ_j defined on the mesh. Then

$$g_z(d) = \sum_{j=1}^{N} g_j(z) \phi_j(d)$$

and consequently the far field operator becomes

$$\mathcal{F} g_z(d_j) = \sum_{j=1}^{N} F^\infty_{\ell,j} g_j(z) \int_\Omega \phi_j(d) ds(d).$$

Consequently, the far field equation can be transformed at the discrete level into the following system of N equations in N unknowns (g_j)

$$\sum_{j=1}^{N} \omega_j F^\infty_{\ell,j} g_j = \gamma e^{-\mathrm{i}k d_\ell \cdot z}, \quad \ell = 1, ..., N$$

where the weights ω_j depend on the quadrature formulae used in evaluating the integrals over the mesh triangles. For a fixed z and each sampling wave numbers k, we solve a discretized $N \times N$ linear system of the form

$$Fg(z) = b^\infty(z) := \gamma(e^{-ikd_1 \cdot z}, ..., e^{-ikd_N \cdot z})^T$$

where $F := (F^\infty_{\ell,j})$ is a $N \times N$ matrix independent of z and $g(z) := (g_1(z), ..., g_N(z))^T$ is the unknown vector.

Computation of the norm of the solution g_z

The problem is that the system is ill-posed and we need to use a regularization scheme. Precisely, we use a Tikhonov regularization coupled with the Morozov discrepancy principle in order to find the regularization parameter. It consists in solving

$$(\eta + F^*F)g(z) = F^*b^\infty(z)$$

where η is the regularization parameter. As explained in [21] and [33], we consider a singular value decomposition of the matrix F given by

$$F = USV^*$$

where U and V are unitary and Σ is real diagonal with $S_{j,j} := s_j$, $j = 1, ..., N$. This decomposition is possibly truncated to ignore all singular values and vectors of index larger that $p \leq N$. The solution is then given by

$$(V^*g(z))_j = \frac{s_j}{\eta + s_j^2} \left(U^* b^\infty(z)\right)_j, \quad j = 1, ..., N.$$

6.1.5 Numerical examples

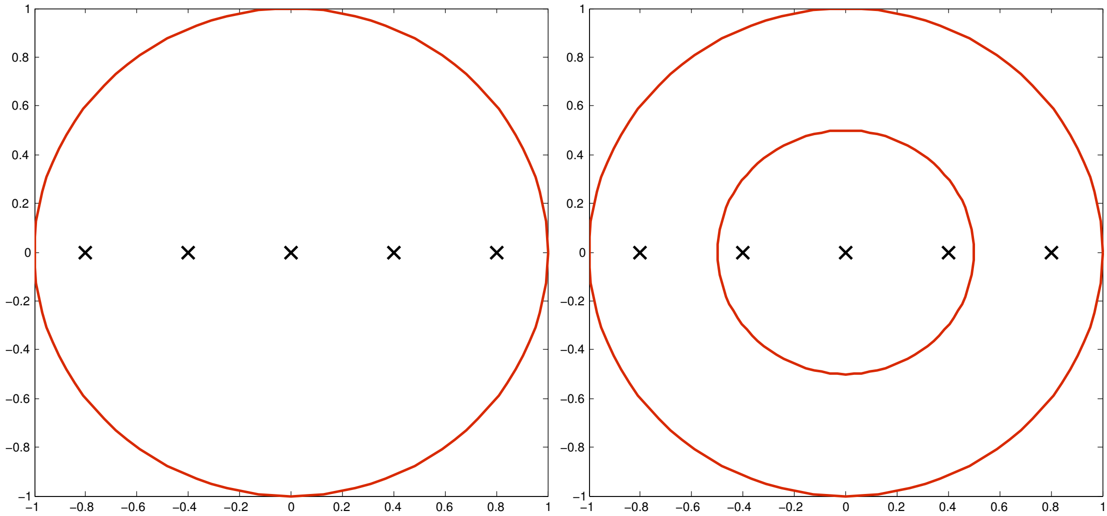
The computations of this section have been done in $\mathbb{R}^2$. We assume that the contrast of the domain D is given by two functions μ and n. The corresponding interior transmission problem is

$$\begin{cases} \nabla \cdot \dfrac{1}{\mu} \nabla w + k^2 n w = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \Gamma \\ \dfrac{1}{\mu}\dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} & \text{on } \Gamma. \end{cases}$$

In the use of the LSM, we compute the norm of the regularized solution g_z for several points z_i, $i \in I$, inside D. The red line corresponds to the sum of the norms: $\sum_{i \in I} ||g_{z_i}||$.

Validation of the methods with the disk

In appendix F, we described the method to compute transmission eigenvalues for spherical geometry. They can be computed from an analytical expression of the solutions to the ITP.



(a) Homogeneous disk of radius 1 with contrasts $\mu = 4$ and $n = 1$.

(b) Disk of radius 1 with contrasts $\mu = 1$ and $n = 4$ containing a cavity of radius 0.5.

Figure 6.4: Geometries and location of the source points z.

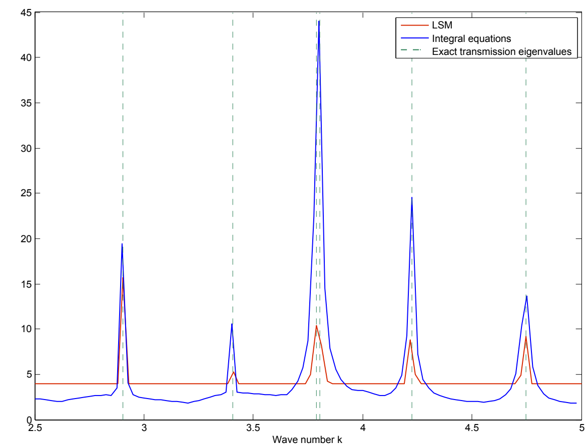


Figure 6.5: Homogeneous disk of radius 1 with contrasts $\mu = 4$ and $n = 1$ (Figure 6.4(a)).

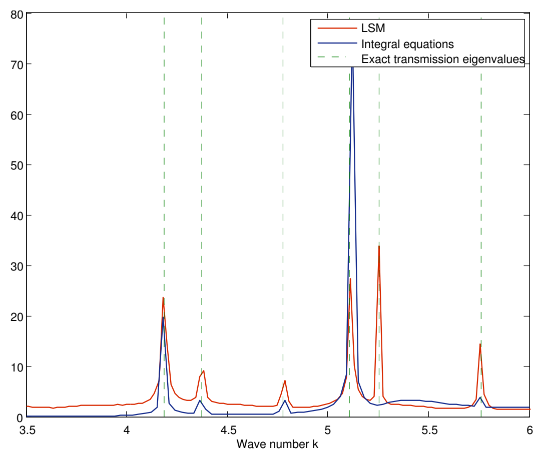


Figure 6.6: Disk of radius 1 with contrasts $\mu = 1$ and $n = 4$ containing a cavity of radius 0.5 (Figure 6.4(b)).

One can observe in Figure 6.5 that the third and the fourth transmission eigenvalues are merged together in only one peak for both methods using the far field data and integral equations due to the proximity of these two transmission eigenvalues.

On Figure 6.6, the method using integral equations failed to find the fifth transmission eigenvalues. However, both methods using LSM and integral equations give accurate values of the transmission eigenvalues and especially the first one.

Elliptic geometry

Now, we look at an another geometry given by the ellipse on Figure 6.7 of equation

$$\begin{cases} x(t) = \cos t \\ y(t) = 0.5 \sin t. \end{cases}$$

Figures 6.8 and 6.9 correspond to the computation of transmission eigenvalues for this geometry with $\mu = 4$, $n = 1$ and $\mu = 1$, $n = 4$ respectively.

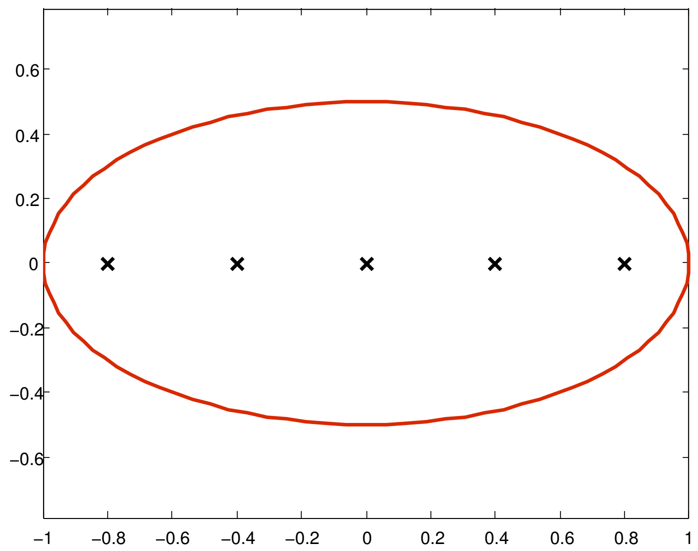


Figure 6.7: Location of the source points z inside the ellipse.

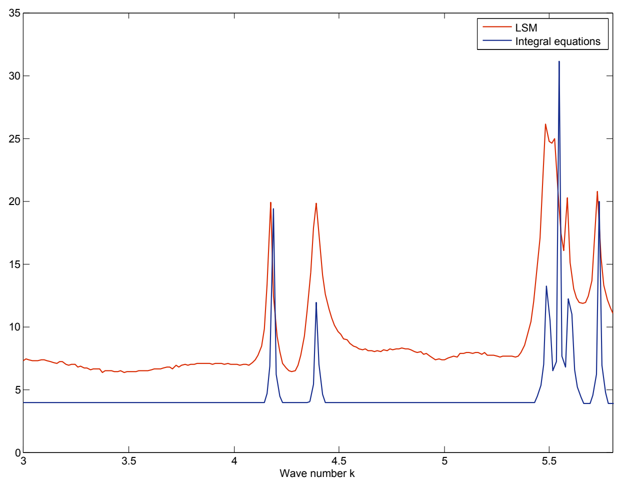


Figure 6.8: Ellipse with contrasts $\mu = 4$ and $n = 1$.

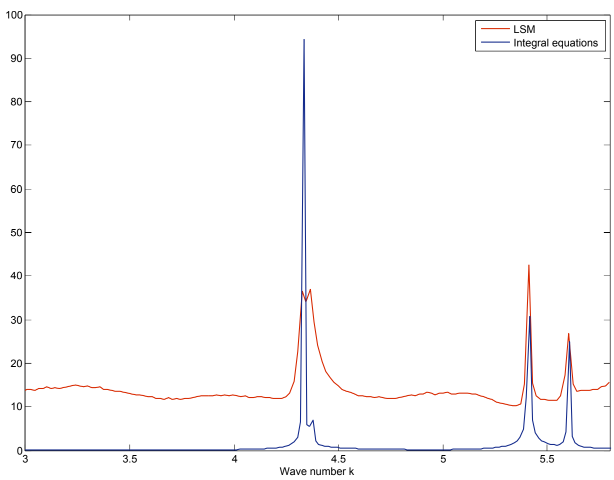
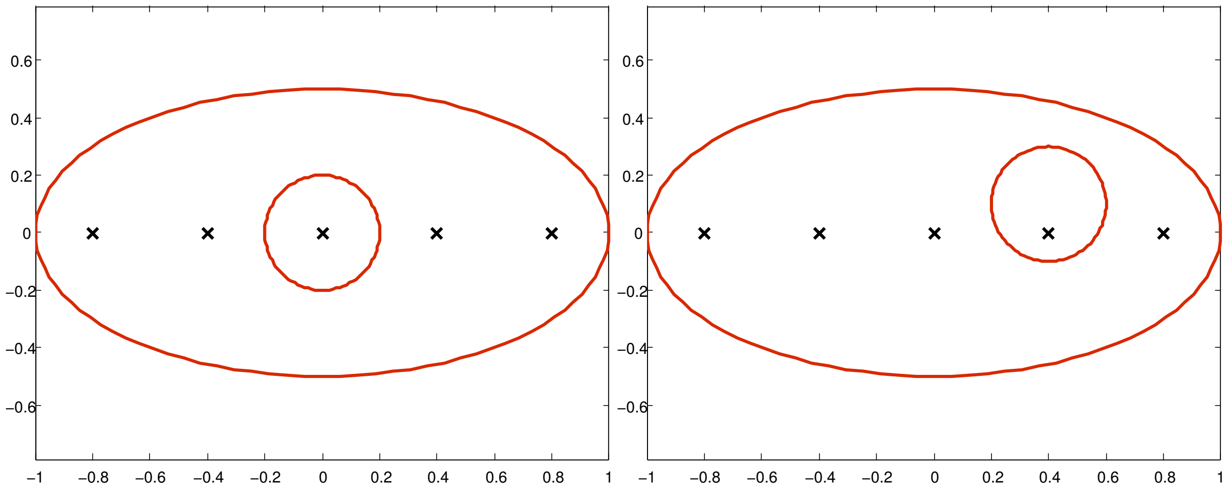


Figure 6.9: Ellipse with contrasts $\mu = 1$ and $n = 4$.

Now, we add a cavity of radius 0.2 inside the ellipse, one centered on the origin and one centered on $(0.4, 0.1)$.



(a) Cavity centered on $(0, 0)$ (b) Cavity centered on $(0.4, 0.1)$

Figure 6.10: Geometries and location of the source points z inside the ellipse.

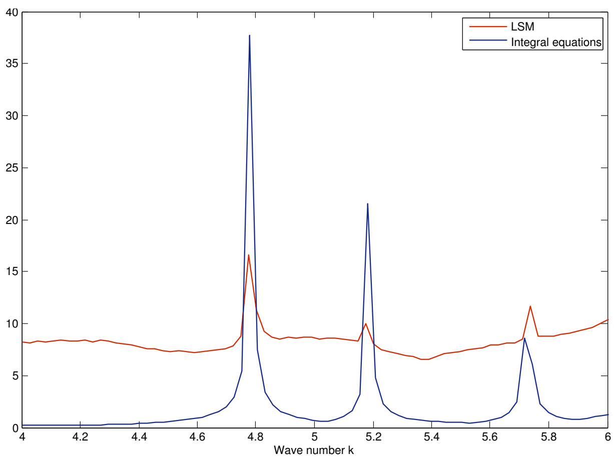


Figure 6.11: Ellipse with contrasts $\mu = 1$ and $n = 4$ containing a cavity of radius 0.2 (Figure 6.10(a)).

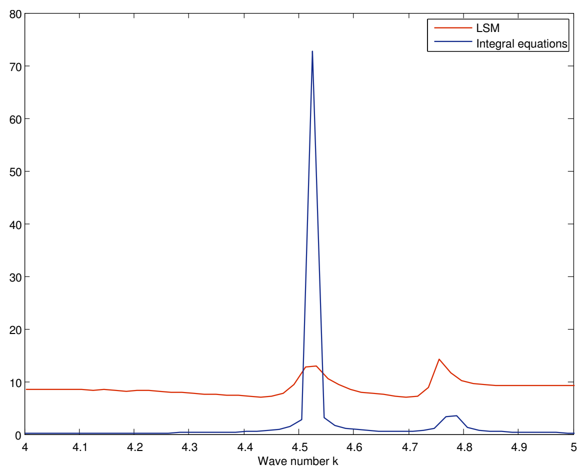


Figure 6.12: Ellipse with contrasts $\mu = 1$ and $n = 4$ with a shifted cavity of radius 0.2 (Figure 6.10(b)).

One can remark that the first transmission eigenvalue has been shifted to the right as predicated in Theorem 3.2.9. Moreover, one can also remark that two similar cavities located at different places inside the domain D do not give the same spectrum of transmission eigenvalues. This shows that it also depends on the location of the cavity.

Parametric geometry

Now, let us look at a domain delimited by a parametric curve of equation

$$\begin{cases} x(t) = 1.2\cos t + 0.3\cos(2t) \\ y(t) = 0.6\sin t \end{cases}$$

and represented in Figure 6.14.

The first figure 6.13 corresponds to a domain with contrasts $\mu = 4$ and $n = 1$.

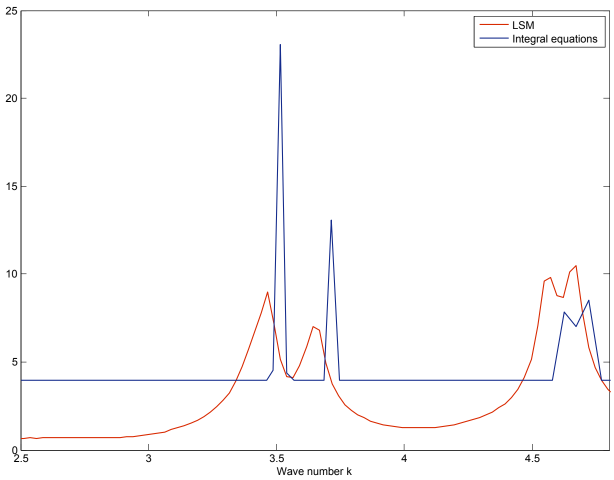
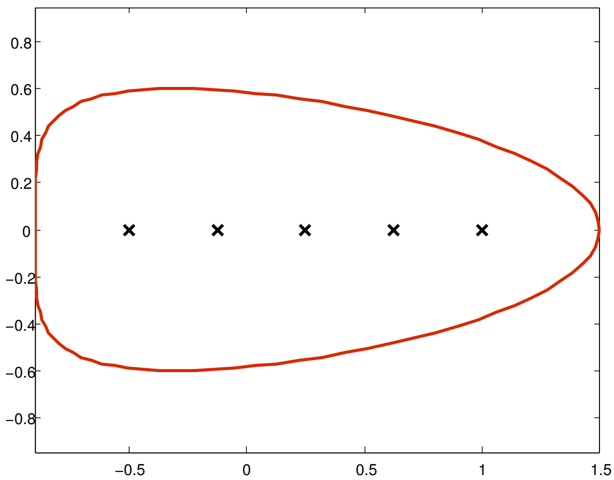
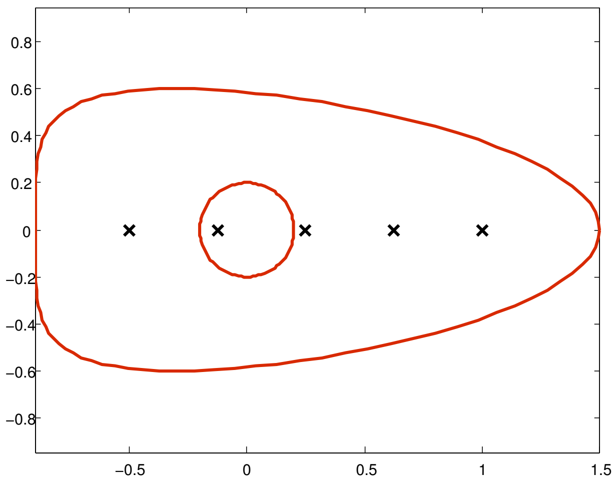


Figure 6.13: With contrasts $\mu = 4$ and $n = 1$, geometry indicated in Figure 6.14(a).

Figures 6.15 and 6.16 correspond to the computation of the transmission eigenvalues for $\mu = 1$ and $n = 4$. One can remark that the first transmission eigenvalue has been shifted to the right when the domain contains a cavity (Figure 6.16).



(a) Homogeneous medium



(b) With a cavity

Figure 6.14: Geometries and location of the source points z.

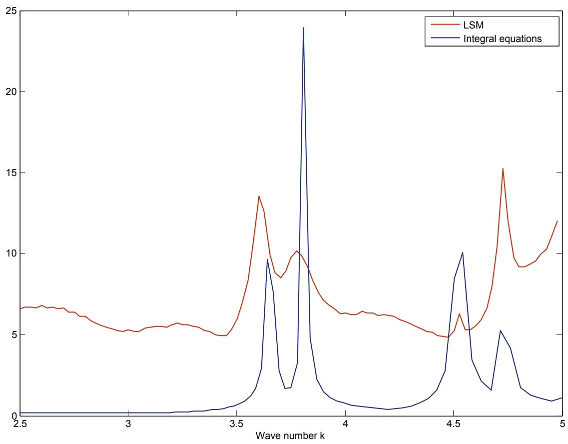


Figure 6.15: With contrasts $\mu = 1$ and $n = 4$, geometry indicated in Figure 6.14(a).

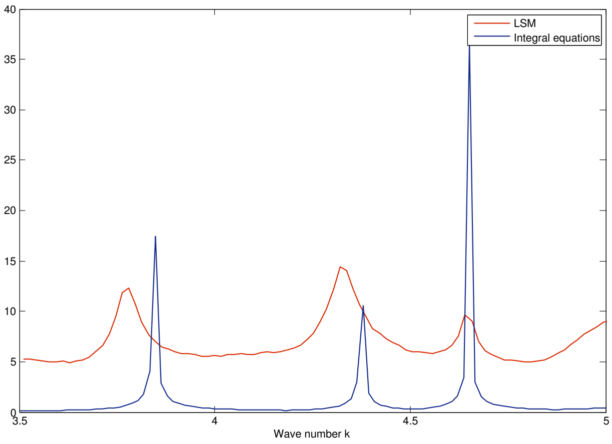


Figure 6.16: With contrasts $\mu = 1$ and $n = 4$ and cavity of radius 0.2, geometry indicated in Figure 6.14(b).

Square geometry

We finally compute the transmission eigenvalues for a square of length 1. One can remark in Table 6.1 that the accuracy between the methods is getting worse when the geometry becomes less regular.

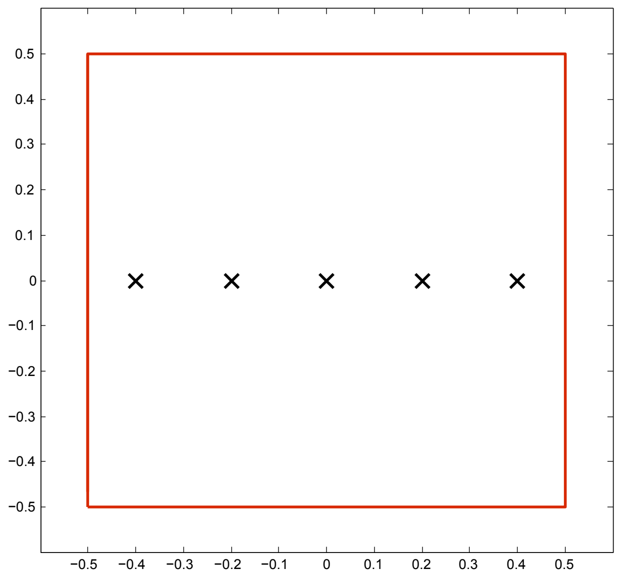


Figure 6.17: Geometry and location of the source points z

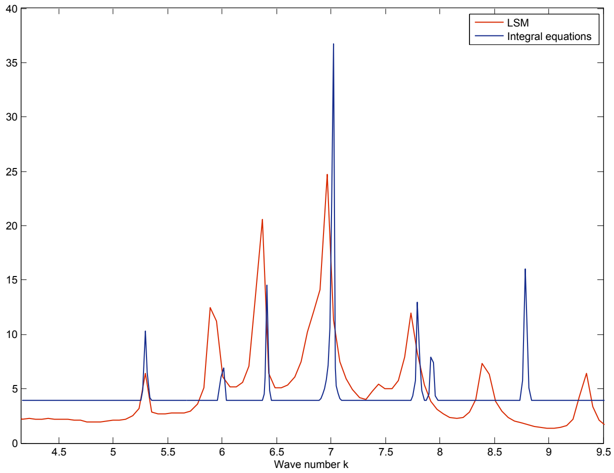


Figure 6.18: Square with contrasts $\mu = 4$ and $n = 1$

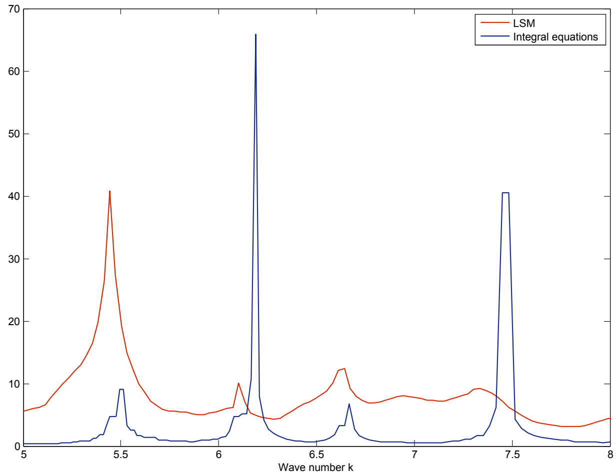


Figure 6.19: Square with contrasts $\mu = 1$ and $n = 4$

One can remark the good match for the first transmission eigenvalues between the two methods, however the higher is the wavenumber, the worst is the accuracy of both methods.

The next table gives the values of the first transmission eigenvalues for each configuration and for both method: k_0(LSM) for the method using the linear sampling method and k_0(INTEQ) for the method using surface integral equations. We can compare the relative error for every geometry. The error is computed with the formula

$$\text{Error} = \frac{|k_0(\text{LSM}) - k_0(\text{INTEQ})|}{\min\{k_0(\text{LSM}), k_0(\text{INTEQ})\}}. \tag{6.12}$$

Configuration	Ellipse		$\begin{cases} x(t) = 1.2\cos t + 0.3\cos(2t) \\ y(t) = 0.6\sin t \end{cases}$		Square	
	$\mu = 4$ $n = 1$	$\mu = 1$ $n = 4$	$\mu = 4$ $n = 1$	$\mu = 1$ $n = 4$	$\mu = 4$ $n = 1$	$\mu = 1$ $n = 4$
k_0(LSM)	4.174	4.325	3.466	3.602	5.296	5.443
k_0(INTEQ)	4.187	4.333	3.514	3.642	5.293	5.495
Error	0.311%	0.184%	1.384%	1.110%	0.056%	0.955%

Table 6.1: Error between the two methods

6.2 Electromagnetic vector case

6.2.1 Properties of $H(\text{curl}, D)$

Let us first recall some properties of the space $H(\text{curl}, D)$ in which solutions to Maxwell's equations are defined. More specifically, we give the trace properties of functions in $H(\text{curl}, D)$. It can be found in [45, 32, 20].

First, we define, for a smooth vector function $\mathbf{v} \in (C^\infty(\overline{D}))^3$, the two traces

$$\begin{aligned}\gamma_t(\mathbf{v}) &= \nu \times \mathbf{v}|_{\partial D},\\ \gamma_T(\mathbf{v}) &= (\nu \times \mathbf{v}|_{\partial D}) \times \nu,\end{aligned}$$

where ν is the unit outward normal to D.

Let us now define the Hilbert space

$$H_{\mathrm{div}}^{-1/2}(\partial D) := \left\{\mathbf{u} \in H^{-1/2}(\partial D)^3 / \mathbf{u}\cdot\nu = 0 \text{ a.e. on } \partial D \text{ and } \nabla_{\partial D}\cdot\mathbf{u} \in H^{-1/2}(\partial D)\right\},$$

and its dual space given by

$$H_{\mathrm{curl}}^{-1/2}(\partial D) := \left\{\mathbf{u} \in H^{-1/2}(\partial D)^3 / \mathbf{u}\cdot\nu = 0 \text{ a.e. on } \partial D \text{ and } \nabla_{\partial D}(\nu\times\mathbf{u}) \in H^{-1/2}(\partial D)\right\}.$$

Now, we can state the main theorem on the trace properties of functions in $H(\mathrm{curl}\,, D)$.

Theorem 6.2.1. *The tangential trace mapping $\gamma_t : H(\mathrm{curl}\,, D) \to H_{\mathrm{div}}^{-1/2}(\partial D)$ and the tangential components trace mapping $\gamma_T : H(\mathrm{curl}\,, D) \to H_{\mathrm{curl}}^{-1/2}(\partial D)$ are continuous linear mappings.*

6.2.2 Integral operators for Maxwell's equations

Let $\mathbf{E}$, $\mathbf{H}$ in $H(\mathrm{curl}\,, D)$ be a solution to Maxwell's equations in D

$$\mathrm{curl}\,\mathbf{E} - \mathrm{i}k\mathbf{H} = 0, \quad \mathrm{curl}\,\mathbf{H} + \mathrm{i}kn\mathbf{E} = 0 \quad \text{in } D.$$

In the following, we denote

$$\begin{cases}\mathbf{J} = -\nu\times\mathbf{H} \in H_{\mathrm{div}}^{-1/2}(\partial D)\\ \mathbf{M} = \nu\times\mathbf{E} \in H_{\mathrm{div}}^{-1/2}(\partial D)\end{cases}$$

the magnetic and electric currents. We define the Green tensor associated with Maxwell's equations by

$$\mathbb{G}_k(x,y) = \Phi_k(x,y)I + \frac{1}{k^2}\nabla_x \mathrm{div}_x\left(\Phi_k(x,y)I\right)$$

where I is the 3×3 identity matrix. Note that

$$\mathrm{curl}_x\,\mathrm{curl}_x\,\mathbb{G}_k(x,y) - k^2\mathbb{G}_k(x,y) = \delta_y I$$

where δ_y is the delta function at the point y.

Let us now introduce the integral operators for Maxwell's equations $\mathcal{T}_k$ and $\mathcal{K}_k$ defined by

$$\begin{aligned}\mathcal{T}_k\mathbf{J}(x) &:= \int_{\partial D}\mathbb{G}_k(x,y)\mathbf{J}(y)ds(y)\\ &= \frac{1}{k^2}\int_{\partial D}\nabla_x\mathrm{div}_x\,\Phi_k(x,y)\mathbf{J}(y)ds(y) + \int_{\partial D}\Phi_k(x,y)\mathbf{J}(y)ds(y)\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}_k\mathbf{J}(x) &:= -\mathrm{curl}_x \int_{\partial D} \mathbb{G}_k(x,y)\mathbf{J}(y)ds(y)\\ &= -\int_{\partial D} \nabla_x \Phi_k(x,y) \times \mathbf{J}(y)ds(y)\end{aligned}$$

where $\nabla_{\partial D}\cdot$ denotes the surface divergence on ∂D. They are continuous from $H^{-1/2}_{\mathrm{div}}(\partial D)$ into $H(\mathrm{curl}\,, D)$.

Theorem 6.2.2. *The Stratton-Chu formula for solutions to Maxwell's equations in D can be written in terms of the integral operators $\mathcal{T}_k$ and $\mathcal{K}_k$*

$$\mathbf{E}(x) = \mathcal{K}_k\mathbf{M}(x) + \mathrm{i}k\mathcal{T}_k\mathbf{J}(x)$$

and

$$\mathbf{H}(x) = \mathrm{i}kn\mathcal{T}_k\mathbf{M}(x) - \mathcal{K}_k\mathbf{J}(x)$$

where $\mathbf{J} = -\nu \times \mathbf{H}$ and $\mathbf{M} = \nu \times \mathbf{E}$ are the magnetic and electric currents.

Proof. Using Theorem B.1.1, we have

$$\mathbf{E}(x) = -\mathrm{curl}_x \int_{\partial D} M(y)\Phi_k(x,y)ds(y) + \frac{\mathrm{i}}{k}\mathrm{curl}_x\,\mathrm{curl}_x \int_{\partial D} J(y)\Phi_k(x,y)ds(y)$$

and

$$\mathbf{E}(x) = \mathrm{curl}_x \int_{\partial D} J(y)\Phi_k(x,y)ds(y) + \frac{in}{k}\mathrm{curl}_x\,\mathrm{curl}_x \int_{\partial D} M(y)\Phi_k(x,y)ds(y).$$

Using the fact that $\mathrm{curl}\,\mathrm{curl} = -\Delta + \nabla\mathrm{div}$ and that $\Delta\Phi_k(x,y) = -k^2\Phi_k(x,y)$, we deduce that

$$\begin{aligned}\mathrm{curl}\,\mathrm{curl} \int_{\partial D} \mathbf{J}(y)\Phi_k(x,y)ds(y) &= \int_{\partial D} \nabla_x \mathrm{div}_x\, \Phi_k(x,y)\mathbf{J}(y)ds(y)\\ &\quad + k^2 \int_{\partial D} \Phi_k(x,y)\mathbf{J}(y)ds(y)\\ &= k^2\mathcal{T}_k\mathbf{J}(x)\end{aligned}$$

and the theorem follows. □

Theorem 6.2.3. *The tangential components trace of $\mathcal{T}_k\mathbf{J}$ is continuous across the boundary ∂D and it is given for all x on ∂D by*

$$\gamma_T(\mathcal{T}_k\mathbf{J})^{\pm}(x) = \mathbf{T}_k\mathbf{J}(x)$$

where

$$\mathbf{T}_k\mathbf{J}(x) = \frac{1}{k^2}\int_{\partial D} \nabla_x \mathrm{div}_x\, \Phi_k(x,y)\mathbf{J}(y)ds(y) + \left(\int_{\partial D} \Phi_k(x,y)\mathbf{J}(y)ds(y)\right)_T.$$

However, the jump of the tangential trace of $\mathcal{K}_k\mathbf{M}$ is given for all x on ∂D by

$$\gamma_T(\mathcal{K}_k\mathbf{M})^{\pm}(x) = \pm\frac{1}{2}\nu \times \mathbf{M} + \mathbf{K}_k\mathbf{M}(x)$$

where

$$\mathbf{K}_k\mathbf{M}(x) = -\mathrm{curl}_x \int_{\partial D} \nabla_y \Phi_k(x,y) \times \mathbf{J}(y) ds(y).$$

From Theorem 6.2.1, the operators $\mathcal{T}_k$ and $\mathbf{K}_k$ are continuous from $H^{-1/2}_{\mathrm{div}}(\partial D)$ into $H^{-1/2}_{\mathrm{curl}}(\partial D)$.

6.2.3 Surface integral equations representation for solutions to the ITP

Integral equations representation can be extended to the case of time-harmonic Maxwell's equations. Consider a scatterer D with index of refraction N that we assume to be of the form $N(x) = n(x)I$. The corresponding interior transmission problem is the interior transmission problem corresponding to the scattering

$$\begin{cases} \mathrm{curl}\,\mathbf{E} - \mathrm{i}k\mathbf{H} = 0, \quad \mathrm{curl}\,\mathbf{H} + \mathrm{i}kn\mathbf{E} = 0 & \text{in } D \\ \mathrm{curl}\,\mathbf{E}_0 - \mathrm{i}k\mathbf{H}_0 = 0, \quad \mathrm{curl}\,\mathbf{H}_0 + \mathrm{i}k\mathbf{E}_0 = 0 & \text{in } D \\ \nu \times \mathbf{E} = \nu \times \mathbf{E}_0, \quad \nu \times \mathbf{H} = \nu \times \mathbf{H}_0 & \text{on } \Gamma. \end{cases} \tag{ITP6.2}$$

As recalled in Appendix A, solutions to (ITP6.2) have an integral equations representation

$$\mathbf{E}(x) = \mathrm{i}k\mathcal{T}_{k_1}\mathbf{J}(x) + \mathcal{K}_{k_1}\mathbf{M}(x),$$

$$\mathbf{H}(x) = \mathrm{i}kn\mathcal{T}_{k_1}\mathbf{M}(x) - \mathcal{K}_{k_1}\mathbf{J}(x),$$
$$\mathbf{E}_0(x) = \mathrm{i}k\mathcal{T}_{k_0}\mathbf{J}(x) + \mathcal{K}_{k_0}\mathbf{M}(x),$$
$$\mathbf{H}_0(x) = \mathrm{i}k\mathcal{T}_{k_0}\mathbf{M}(x) - \mathcal{K}_{k_0}\mathbf{J}(x).$$

On the boundary, the tangential traces are given by

$$\begin{aligned} \mathbf{E}_T &= -\nu \times \mathbf{M} \\ &= \mathrm{i}k\mathbf{T}_{k_1}\mathbf{J} - \frac{1}{2}\nu \times \mathbf{M} + \mathbf{K}_{k_1}\mathbf{M}, \end{aligned}$$

$$\begin{aligned} \mathbf{H}_T &= \nu \times \mathbf{J} \\ &= \mathrm{i}kn\mathbf{T}_{k_1}\mathbf{M} + \frac{1}{2}\nu \times \mathbf{J} - \mathbf{K}_{k_1}\mathbf{J}, \end{aligned}$$

$$\begin{aligned} \mathbf{E}_{0T} &= -\nu \times \mathbf{M} \\ &= \mathrm{i}k\mathbf{T}_{k_0}\mathbf{J} - \frac{1}{2}\nu \times \mathbf{M} + \mathbf{K}_{k_0}\mathbf{M}, \end{aligned}$$

$$\begin{aligned}\mathbf{H}_{0T} &= \nu \times \mathbf{J} \\ &= \mathrm{i}k\mathbf{T}_{k_0}\mathbf{M} + \frac{1}{2}\nu \times \mathbf{J} - \mathbf{K}_{k_0}\mathbf{J}.\end{aligned}$$

Consequently, if $\mathbf{E}$, $\mathbf{E}_0$, $\mathbf{H}$, $\mathbf{H}_0$ are solutions to the interior transmission problem (ITP6.2) then

$$Z(k)\begin{pmatrix}\mathbf{J}\\ \mathbf{M}\end{pmatrix} = 0$$

where the operator $Z(k)$ is defined by

$$Z(k) := \begin{pmatrix}\mathrm{i}k(\mathbf{T}_{k_1} - \mathbf{T}_{k_0}) & \mathbf{K}_{k_1} - \mathbf{K}_{k_0}\\ (\mathbf{K}_{k_1} - \mathbf{K}_{k_0}) & -\mathrm{i}k(n\mathbf{T}_{k_1} - \mathbf{T}_{k_0})\end{pmatrix}.$$

This operator $Z(k)$ is defined from $H^{-1/2}_{\mathrm{div}}(\partial D) \times H^{-1/2}_{\mathrm{div}}(\partial D)$ into $H^{-1/2}_{\mathrm{curl}}(\partial D) \times H^{-1/2}_{\mathrm{curl}}(\partial D)$.

Similarly to the scalar case, the operator $Z(k) : H^{-1/2}_{\mathrm{div}}(\partial D) \times H^{-1/2}_{\mathrm{div}}(\partial D) \to H^{-1/2}_{\mathrm{curl}}(\partial D) \times H^{-1/2}_{\mathrm{curl}}(\partial D)$ is compact. A proof of the compactness of the difference of two volume integral operators for Maxwell's equations in $H(\mathrm{curl}\,, D)$ can be found in [37] and can be adapted to prove the compactness of $Z(k)$. This is one result that still needs to be proven.

Consequently, we again have to use a preconditioner $B(k)$ that we take to be equal to $Z(\mathrm{i}k)$.

6.2.4 Computation using far field data

The procedure described in Section 6.1.4 can be easily adapted for electromagnetic waves. The only important change is that the norm of the regularized solution g_z must be computed for three independent polarization $q_1 = (1, 0, 0)$, $q_2 = (0, 1, 0)$ and $q_3 = (0, 0, 1)$. We then plot

$$\mathcal{G}(z) = ||g_{z,q_1}|| + ||g_{z,q_2}|| + ||g_{z,q_3}||.$$

6.2.5 Numerical examples

We first compute the transmission eigenvalues for a spherical geometry and we compare the values with the values computed from analytical solutions to the ITP (see Appendix F). It is represented in figure 6.2.5. We then add a cavity inside the sphere with the form of a box. One can remark that the first transmission eigenvalue as been shifted to the right.

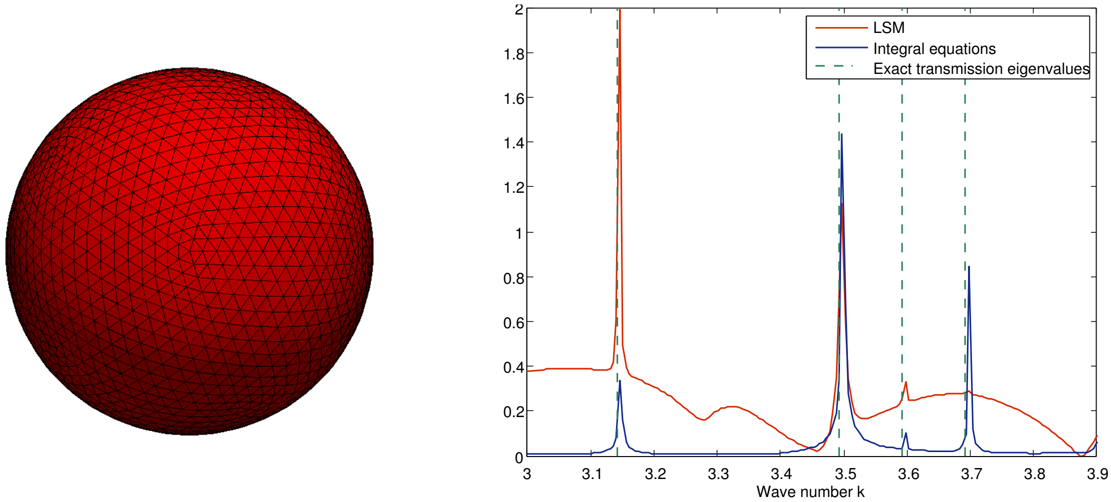


Figure 6.20: Sphere of radius 1 and index of refraction $N = 4$.

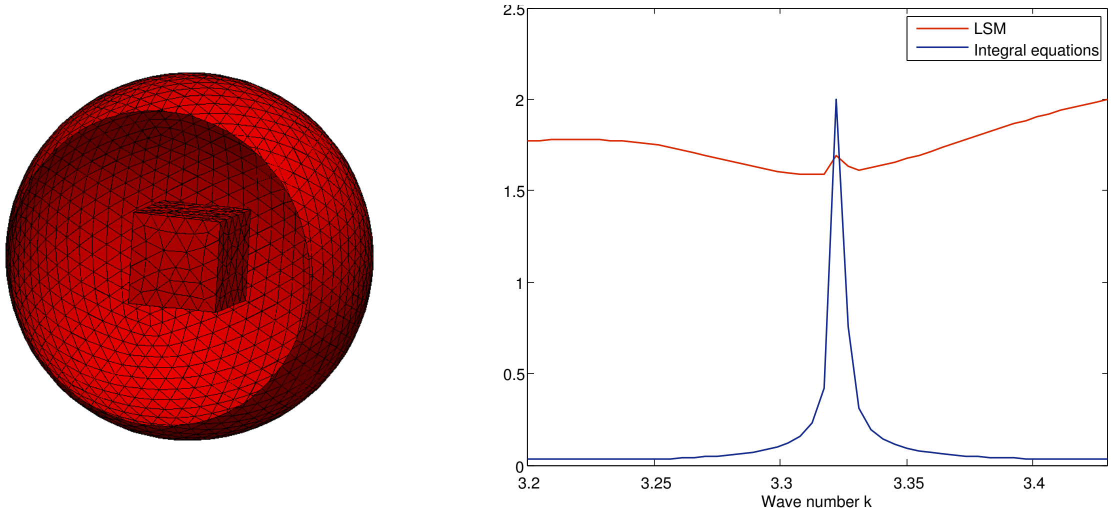


Figure 6.21: Sphere of radius 1 and index of refraction $N = 4$ containing a cavity of the form of a box. We can remark that the first transmission eigenvalues is shifted comparing to the previous example.

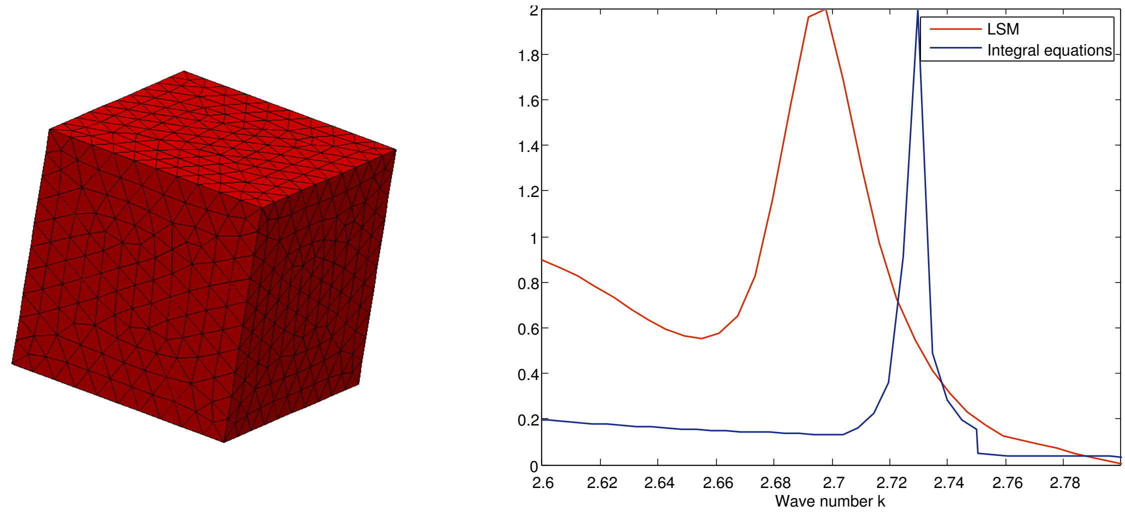


Figure 6.22: Box of length 1 and index of refraction $N = 4$.

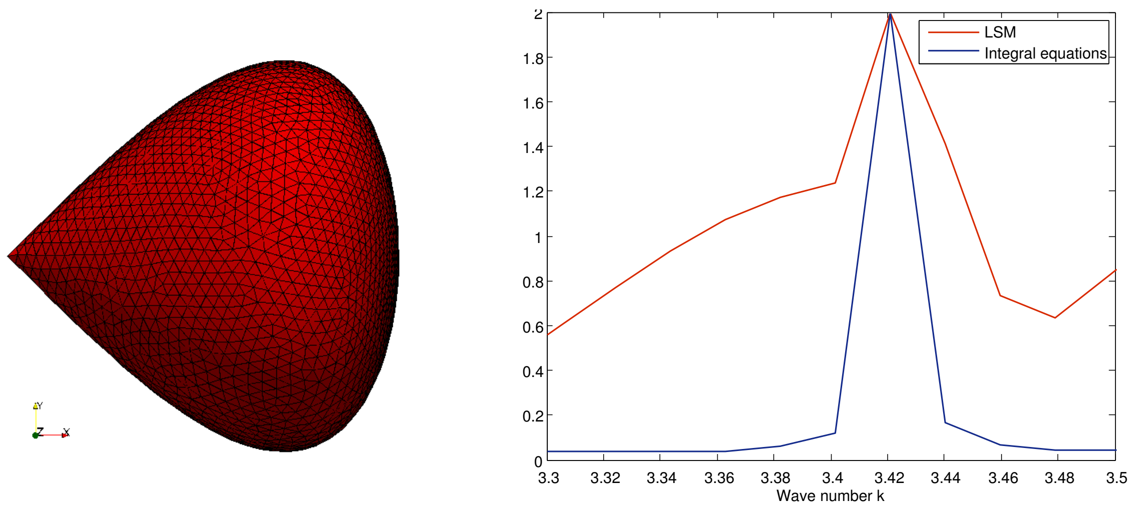


Figure 6.23: Example of the drop for which the reconstruction using the LSM is in Chapter 1. Here we compute the first transmission eigenvalue for the penetrable object of index $N = 4$.

The accuracy between both methods is still good. Even if it seems not satisfactory for the case of the box, the difference between the two values of the first transmission eigenvalues is 0.032 i.e. using the definition of the error in 6.12, the error is only of 1.18%.

Appendix A

Maxwell's equations in 2 dimensions

Maxwell's equations are closely linked to Helmholtz equation that describes the propagation of acoustic waves. In general, the acoustic case is often studied first as it is simpler and to see the difficulties that can occur in the study of electromagnetic scattering problems.

In the case of the scattering by an infinitely long cylinder with axis for instance in the z-direction, we can show that electromagnetic waves satisfy the Helmholtz equation.

A.1 Penetrable obstacles

Assume that the polarization is transverse magnetic that is to say that the magnetic field $\mathbf{H}$ is perpendicular to the cylinder and as a consequence the electric field is parallel to the z-axis. The electric fields have only one non trivial component : $\mathbf{E} = (0, 0, u)$, $\mathbf{E}^i = (0, 0, u^i)$, $\mathbf{E}^s = (0, 0, u^s)$. Since $\mathbf{E}$ and $\mathbf{H}$ are solutions to Maxwell's equations

$$\operatorname{curl} \mathbf{E} - \mathrm{i}k\mathbf{H} = 0, \quad \operatorname{curl} \mathbf{H} + \mathrm{i}k\mathbf{E} = 0$$

for an impenetrable object, u and $\mathbf{H} := (H_1, H_2, H_3)$ satisfy the following system

$$\begin{cases} \dfrac{\partial u}{\partial x_2} - \mathrm{i}kH_1 = 0, \\ -\dfrac{\partial u}{\partial x_1} - \mathrm{i}kH_2 = 0, \\ \dfrac{\partial H_2}{\partial x_1} - \dfrac{\partial H_1}{\partial x_2} + \mathrm{i}ku = 0. \end{cases}$$

Expressing H_1 and H_2 in terms of u, we finally get that u satisfies the Helmholtz equation

$$\Delta u + k^2 u = 0.$$

The direct acoustic scattering problem for impenetrable objects then becomes

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } D, \\ u = u^s + u^i, & \\ u = 0 & \text{on } \partial D, \\ \lim\limits_{r \to \infty} \sqrt{r} \left(\dfrac{\partial u^s}{\partial r} - \mathrm{i}ku^s \right) = 0. & \end{cases}$$

A.2 Impenetrable objects

In this section, we assume that the dielectric cylinder is orthotropic, i.e. the matrices N and μ are of the form

$$N = \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{21} & n_{22} & 0 \\ 0 & 0 & n_{33} \end{pmatrix}$$

and

$$\mu = \begin{pmatrix} \mu_{11} & \mu_{12} & 0 \\ \mu_{21} & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{pmatrix}.$$

Consider first a polarization transverse magnetic, the total fields $\mathbf{E} = (0, 0, u)$ and $\mathbf{H}$ satisfy

$$\operatorname{curl} \mathbf{E} - \mathrm{i}k\mu(x)\mathbf{H} = 0, \quad \operatorname{curl} \mathbf{H} + \mathrm{i}kN(x)\mathbf{E} = 0.$$

Therefore, u and $\mathbf{H} = (H_1, H_2, H_3)$ are solutions to

$$\begin{cases} \dfrac{\partial u}{\partial x_2} - \mathrm{i}k\mu_{11}H_1 - \mathrm{i}k\mu_{12}H_2 = 0, \\ \dfrac{\partial u}{\partial x_1} + \mathrm{i}k\mu_{21}H_1 + \mathrm{i}k\mu_{22}H_2 = 0, \\ \dfrac{\partial H_2}{\partial x_1} - \dfrac{\partial H_1}{\partial x_2} + \mathrm{i}kn_{33}(x)u = 0. \end{cases}$$

In this case, the total field u satisfies

$$\nabla \cdot A\nabla u + k^2 n_{33} u = 0$$

where

$$A = \frac{1}{\mu_{11}\mu_{22} - \mu_{12}\mu_{21}} \begin{pmatrix} \mu_{11} & \mu_{21} \\ \mu_{12} & \mu_{22} \end{pmatrix}.$$

For a transverse electric wave, the magnetic field is parallel to the z-axis so there exists w such that $\mathbf{H} = (0, 0, w)$ and satisfies together with $\mathbf{E} = (E_1, E_2, E_3)$

$$\begin{cases} \dfrac{\partial E_2}{\partial x_1} - \dfrac{\partial E_1}{\partial x_2} - \mathrm{i}k\mu_{33}(x)w = 0, \\ \dfrac{\partial w}{\partial x_2} + \mathrm{i}kn_{11}(x)E_1 - \mathrm{i}kn_{12}(x)E_2 = 0, \\ \dfrac{\partial w}{\partial x_1} - \mathrm{i}kn_{21}(x)E_1 - \mathrm{i}kn_{22}(x)E_2 = 0. \end{cases}$$

We get the same type of equation satisfied by w as in the transverse magnetic polarization

$$\nabla \cdot B\nabla w + k^2 \mu_{33}(x) w = 0$$

where

$$B = \frac{1}{n_{11}n_{22} - n_{12}n_{21}} \begin{pmatrix} n_{11} & n_{21} \\ n_{12} & n_{22} \end{pmatrix}.$$

In both cases, the direct scattreing problem can be written of the form

$$\begin{cases} \nabla \cdot A\nabla u + k^2 n(x) u = 0 & \text{in } D, \\ u = u^s + u^i, & \\ u = 0 & \text{on } \partial D, \\ \lim\limits_{r\to\infty} r \left(\dfrac{\partial u^s}{\partial r} - \mathrm{i}ku^s \right) = 0. & \end{cases}$$

Appendix B

Integral equations representation

It is well know [26, 46] that solutions to Helmholtz equations have an integral equations representation using simple-layer and double-layer potentials. We recall here the Green's representation theorem and basic properties of the single and double layer potentials and also give an equivalent representation theorem for solutions to Maxwell's equations known as the Stratton-Chu formula (see for instance [26, 45]). We will see that we can also define integral operators for Maxwell's equations.

We refer to [26] for representation theorems for solutions to Helmholtz equation in Hölder spaces and to [46] for solutions in Sobolev spaces. Representation theorems for solutions to Maxwell's equations can be found in [45].

B.1 Representation theorems

B.1.1 Solutions to Helmholtz equation

We recall that

$$\Phi_k(x,z) := \begin{cases} \dfrac{e^{\mathrm{i}k|x-z|}}{4\pi|x-z|} & \text{if } n=3 \\ \dfrac{\mathrm{i}}{4}H_0^{(1)}(k|x-y|) & \text{if } n=2 \end{cases}$$

is the fundamental solution to the Helmholtz equation in $\mathbb{R}^n$.

Theorem B.1.1. *Let D be a bounded domain of class C^2 and let ν denote the unit normal vector to the boundary ∂D directed into the exterior of D. Let $u \in H^1(D)$ be a solution to Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad \text{in } D.$$

Then,

$$u(x) = \int_{\partial D} \left(\frac{\partial u}{\partial \nu}(y)\Phi_k(x,y) - u(y)\frac{\partial \Phi_k(x,y)}{\partial \nu(y)} \right) ds(y), \quad x \in D.$$

We have an equivalent theorem for radiating solutions to Helmholtz equations.

Theorem B.1.2. *Assume the bounded set D is the open complement of an unbounded domain of class C^2 and let ν denote the unit normal vector to the boundary ∂D directed* *into the exterior of D. Let $u \in H^1_{loc}(\mathbb{R}^n \backslash \overline{D})$ be a radiating solution to Helmholtz equation*

$$\Delta u + k^2 u = 0 \quad \textit{in } \mathbb{R}^n \backslash \overline{D}.$$

Then,

$$u(x) = \int_{\partial D} \left(u(y) \frac{\partial \Phi_k(x,y)}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y) \Phi_k(x,y) \right) ds(y), \quad x \in \mathbb{R}^n \backslash \overline{D}.$$

Let us define the single and double layer potentials SL_k and DL_k by

$$\mathrm{SL}_k \varphi(x) := \int_{\partial D} \Phi_k(x,y) \varphi(y) ds(y)$$

and

$$\mathrm{DL}_k \varphi(x) := \int_{\partial D} \frac{\partial \Phi_k(x,y)}{\partial \nu(y)} \varphi(y) ds(y)$$

for all $x \in \mathbb{R}^n \backslash \partial D$. With these notations, from Theorem B.1.1, solutions to Helmholtz equation

$$\Delta u + k^2 u = 0 \quad \text{in } D$$

can be written

$$u(x) = \mathrm{SL}_k \frac{\partial u}{\partial \nu}(x) - \mathrm{DL}_k u(x).$$

B.1.2 Solutions to Maxwell's equations

Any solution to the time-harmonic Maxwell's equations can be represented as the electromagnetic field generated by a combination of surface distributions of electric and magnetic dipoles. We only give here the main theorems and more details can be found in [45].

This first theorem concerns solutions to Maxwell's equations inside a bounded domain D with index of refraction n. We recall that

$$\Phi_k(x,z) := \frac{e^{\mathrm{i}k|x-z|}}{4\pi |x-z|}$$

is the fundamental solution to the Helmholtz equation in $\mathbb{R}^3$.

Theorem B.1.3. *Let D be a bounded Lipschitz domain with unit outward normal ν. Let $\mathbf{E}$, $\mathbf{H} \in H(\mathrm{curl}\,, D)$ be a solution to Maxwell's equations*

$$\mathrm{curl}\,\mathbf{E} - \mathrm{i}k\mathbf{H} = 0, \quad \mathrm{curl}\,\mathbf{H} + \mathrm{i}kn\mathbf{E} = 0 \quad \textit{in } D.$$

Then,

$$\begin{aligned} &\mathrm{curl} \int_{\partial D} \nu(y) \times \mathbf{E}(y) \Phi_k(x,y) ds(y) \\ &\qquad - \frac{1}{\mathrm{i}k} \mathrm{curl}\,\mathrm{curl} \int_{\partial D} \nu(y) \times \mathbf{H}(y) \Phi_k(x,y) ds(y) = \begin{cases} -\mathbf{E}(x), & x \in D \\ 0, & x \in \mathbb{R}^3 \backslash \overline{D}, \end{cases} \end{aligned}$$

and

$$\operatorname{curl} \int_{\partial D} \nu(y) \times \mathbf{H}(y)\Phi_k(x,y)ds(y) + \frac{n}{\mathrm{i}k}\operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times \mathbf{E}(y)\Phi_k(x,y)ds(y) = \begin{cases} -\mathbf{H}(x), & x \in D \\ 0, & x \in \mathbb{R}^3 \backslash \overline{D}. \end{cases}$$

We have a similar representation for radiating solutions to Maxwell's equations.

Theorem B.1.4. *Let ν denote the exterior normal to D where D is a bounded Lipschitz domain in $\mathbb{R}^3$ whose complement is connected. Let $\mathbf{E}$, $\mathbf{H} \in H_{loc}(\operatorname{curl}, D)$ be a solution to Maxwell's equations*

$$\operatorname{curl}\mathbf{E} - \mathrm{i}k\mathbf{H} = 0, \quad \operatorname{curl}\mathbf{H} + \mathrm{i}k\mathbf{E} = 0 \quad \text{in } \mathbb{R}^3 \backslash \overline{D}$$

satisfying the Silver-Müller radiation condition

$$\lim_{r \to +\infty} (\mathbf{H}^s \times x - r\mathbf{E}^s) = 0.$$

Then,

$$\operatorname{curl} \int_{\partial D} \nu(y) \times \mathbf{E}(y)\Phi_k(x,y)ds(y) - \frac{1}{\mathrm{i}k}\operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times \mathbf{H}(y)\Phi_k(x,y)ds(y) = \begin{cases} 0, & x \in D \\ \mathbf{E}(x), & x \in \mathbb{R}^3 \backslash \overline{D}, \end{cases}$$

and

$$\operatorname{curl} \int_{\partial D} \nu(y) \times \mathbf{H}(y)\Phi_k(x,y)ds(y) + \frac{1}{\mathrm{i}k}\operatorname{curl}\operatorname{curl} \int_{\partial D} \nu(y) \times \mathbf{E}(y)\Phi_k(x,y)ds(y) = \begin{cases} 0, & x \in D \\ \mathbf{H}(x), & x \in \mathbb{R}^3 \backslash \overline{D}. \end{cases}$$

B.2 Integral equations and electric dipole

Let us recall the definition of an electric dipole

$$\mathbf{E}_e(x,z,q) := \frac{\mathrm{i}}{k}\operatorname{curl}_x \operatorname{curl}_x q\Phi_k(x,z).$$

The following lemma shows two formulas that we used in the proofs of the Theorems 1.2.7 and 2.5.1 valid for all regular function $\mathbf{F}$. This theorem makes the link between the integral operators $\mathcal{T}_k$ and $\mathcal{K}_k$ defined previously and integral equations involving the electric dipole on the boundary.

Lemma B.2.1. *For all $z \in D$, $q \in \mathbb{R}^3$ and for all regular function $\mathbf{F}$, we have*

$$\int_{\partial D} \operatorname{curl} \mathbf{F}(x) \cdot \nu(x) \times \mathbf{E}_e(x, z, q) ds(x) = \mathrm{i}kq \cdot \frac{1}{k^2} \int_{\partial D} \nabla_x \operatorname{div}_x \Phi_k(x, z) \operatorname{curl} \mathbf{F}(x) \times \nu(x) ds(x)$$
$$+ \int_{\partial D} \Phi_k(x, z) \operatorname{curl} \mathbf{F}(x) \times \nu(x) ds(x).$$

$$\int_{\partial D} \nu \times \mathbf{F} \cdot \operatorname{curl}_x \mathbf{E}_e(\cdot, z, q) = \mathrm{i}kq \cdot \operatorname{curl}_z \int_{\partial D} \mathbb{G}_k(x, z) \nu(x) \times \mathbf{F}(x) ds(x).$$

Proof. Let z be in D and q in $\mathbb{R}^3$. First, from the equality $\operatorname{curl} \operatorname{curl} = -\Delta + \nabla \operatorname{div}$, we remark that

$$\begin{aligned} \operatorname{curl}_x \operatorname{curl}_x q\Phi(x, z) &= -\Delta(q\Phi_k(x, z)) + \nabla \operatorname{div} q\Phi_k(x, z) \\ &= k^2 q\Phi_k(x, z) + \nabla \operatorname{div}_x q\Phi_k(x, z) \end{aligned} \tag{B.1}$$

and

$$\operatorname{curl}_x \operatorname{curl}_x \operatorname{curl}_x (q\Phi_k(x, z)) = k^2 \operatorname{curl}_x (q\Phi_k(x, z)). \tag{B.2}$$

From the definition of $\mathbf{E}_e$ and (B.1), we have

$$\begin{aligned} \int_{\partial D} \operatorname{curl} \mathbf{F} \cdot \nu \times \mathbf{E}_e(\cdot, z, q) &= \frac{\mathrm{i}}{k} \int_{\partial D} \operatorname{curl} \mathbf{F}(x) \cdot \nu \times \operatorname{curl}_x \operatorname{curl}_x q\Phi_k(x, z) ds(x) \\ &= \mathrm{i}k \int_{\partial D} \operatorname{curl} \mathbf{F}(x) \cdot \nu(x) \times q\Phi_k(x, z) ds(x) k \\ &\quad + \frac{\mathrm{i}}{k} \int_{\partial D} \operatorname{curl} \mathbf{F}(x) \cdot \nu(x) \times \nabla \operatorname{div} q\Phi_k(x, z) ds(x) \\ &= \mathrm{i}kq \cdot \left(\int_{\partial D} \Phi_k(x, z) \operatorname{curl} \mathbf{F}(x) \times \nu(x) ds(x) \right. \\ &\quad \left. + \frac{1}{k^2} \int_{\partial D} \operatorname{curl} \mathbf{F}(x) \times \nu(x) \nabla \operatorname{div} \Phi_k(x, z) ds(x) \right) \end{aligned}$$

which yields the first equality.

Now, we have that

$$\begin{aligned} &\int_{\partial D} \nu(x) \times \mathbf{F}(x) \cdot \operatorname{curl}_x \mathbf{E}_e(x, z, q) ds(x) \\ &\qquad = \frac{\mathrm{i}}{k} \int_{\partial D} \operatorname{curl}_x \operatorname{curl}_x \operatorname{curl}_x (q\Phi_k(x, z)) \cdot \nu(x) \times \mathbf{F}(x) ds(x) \end{aligned}$$

Using (B.2) and

$$\nu(x) \times \mathbf{F}(x) \cdot \operatorname{curl}_x (q\Phi_k(x, z)) = q \cdot \operatorname{curl}_z \left(\nu(x) \times \mathbf{F}(x) \Phi_k(x, z) \right),$$

we obtain the second equality

$$\begin{aligned} \int_{\partial D} \nu(x) \times \mathbf{F}(x) \cdot \operatorname{curl}_x \mathbf{E}_e(x, z, q) ds(x) &= \mathrm{i}k \int_{\partial D} q \cdot \operatorname{curl}_z (\nu(x) \times \mathbf{F}(x) \Phi_k(x, z)) ds(x) \\ &= \mathrm{i}kq \cdot \operatorname{curl}_z \int_{\partial D} \nu(x) \times \mathbf{F}(x) \Phi_k(x, z) ds(x). \end{aligned}$$

The results follow from the definitions of the operators $\mathcal{T}_k$ and $\mathcal{K}_k$. $\square$

Appendix C

Approximation of solutions to the interior transmission problem for region containing a cavity

The origin of the results of this appendix comes from the will to show that the first transmission eigenvalue for a domain of index of refraction n containing a cavity ($n = 1$ inside the cavity) can be approximated by the first transmission eigenvalue of an inhomogeneous domain of index n_ε which tends to n as $\varepsilon \to 0$.

To this end, we first started to study the solutions of the interior transmission problem for the domain with cavity and show that it can be approximated by a solution to the interior transmission problem for an approximated domain. However, we only could show a weak convergence result that we have not been able to use to fulfill our original goal.

In the following, we therefore present this weak convergence for both acoustic and electromagnetic cases.

C.1 In acoustics

Let us consider the interior transmission problem studied in Chapter 3 for a domain D containing a cavity D_0.

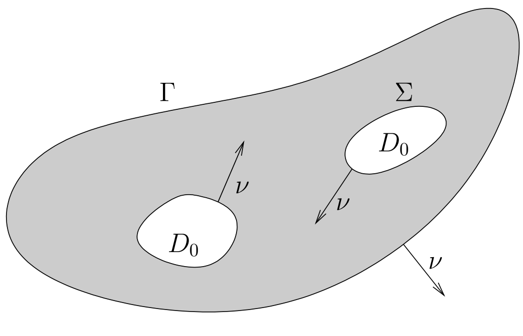


Figure C.1: Geometry and notation

$$
\begin{cases}
\Delta w + k^2 n w = 0 & \text{in } D \\
\Delta v + k^2 v = 0 & \text{in } D \\
w - v = f & \text{on } \Gamma \\
\dfrac{\partial w}{\partial \nu} - \dfrac{\partial v}{\partial \nu} = g & \text{on } \Gamma
\end{cases}
\tag{C.1}
$$

with n an $L^\infty(D)$ complex valued function such that $n = 1$ in D_0 and $\mathrm{Re}(n) \geq c > 0$, $\mathrm{Im}(n) \geq 0$ almost everywhere in $D \backslash \overline{D}_0$ and $f \in H^{3/2}(\Gamma)$ and $g \in H^{1/2}(\Gamma)$.

We recall that this problem is equivalent to finding $u \in V_0(D, D_0, k)$ such that for all $\Psi \in V_0(D, D_0, k)$:

$$
\begin{aligned}
&\int_\Gamma \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{\Psi} + k^2 \overline{\Psi}) dx + k^2 \int_\Gamma (\Delta u + k^2 u)\overline{\Psi} dx \\
&\qquad = -\int_\Gamma \frac{1}{n-1}(\Delta \theta + k^2 \theta)(\Delta \overline{\Psi} + k^2 \overline{\Psi}) dx - k^2 \int_\Gamma (\Delta \theta + k^2 \theta)\overline{\Psi} dx
\end{aligned}
\tag{C.2}
$$

where $\theta \in H^2(D)$ such that $\theta_{|\Gamma} = f$, $\dfrac{\partial \theta}{\partial \nu} = g$ on Γ and $\theta = 0$ in D_θ with $D_0 \subset D_\theta \subset D$. We recall that

$$
V_0(D, D_0, k) := \left\{ u \in \mathbb{H}_0^2(D) / \Delta u - k^2 u = 0 \text{ in } D_0 \right\}.
$$

Let us now introduce the following approximated problem :

$$
\begin{cases}
\Delta w_\varepsilon + k^2 n_\varepsilon w_\varepsilon = 0 & \text{in } D \\
\Delta v_\varepsilon + k^2 v_\varepsilon = 0 & \text{in } D \\
w_\varepsilon - v_\varepsilon = f & \text{on } \Gamma \\
\dfrac{\partial w_\varepsilon}{\partial \nu} - \dfrac{\partial v_\varepsilon}{\partial \nu} = g & \text{on } \Gamma
\end{cases}
\tag{C.3}
$$

where

$$
n_\varepsilon(x) = \begin{cases} n & \text{in } D \backslash \overline{D}_0 \\ 1 + i\varepsilon & \text{in } D_0. \end{cases}
$$

This problem is equivalent to finding $u_\varepsilon \in H_0^2(D)$ such that for all $\Psi \in H_0^2(D)$

$$
\begin{aligned}
&\int_D \frac{1}{n_\varepsilon - 1}(\Delta u_\varepsilon + k^2 u_\varepsilon)(\Delta \overline{\Psi} + k^2 \overline{\Psi}) dx + k^2 \int_D (\Delta u_\varepsilon + k^2 u_\varepsilon)\overline{\Psi} dx \\
&\qquad = -\int_D \frac{1}{n_\varepsilon - 1}(\Delta \theta + k^2 \theta)(\Delta \overline{\Psi} + k^2 \overline{\Psi}) dx - k^2 \int_D (\Delta \theta + k^2 \theta)\overline{\Psi} dx
\end{aligned}
$$

i.e.

$$
\begin{aligned}
&\int_D \frac{1}{n_\varepsilon - 1}(\Delta u_\varepsilon + k^2 u_\varepsilon)(\Delta \overline{\Psi} + k^2 \overline{\Psi}) dx + k^2 \int_D (\Delta u_\varepsilon + k^2 u_\varepsilon)\overline{\Psi} dx \\
&\qquad = -\int_\Gamma \frac{1}{n-1}(\Delta \theta + k^2 \theta)(\Delta \overline{\Psi} + k^2 \overline{\Psi}) dx - k^2 \int_\Gamma (\Delta \theta + k^2 \theta)\overline{\Psi} dx
\end{aligned}
\tag{C.4}
$$

For $u,\, v \in H_0^2(D)$, set

$$\mathcal{A}(u,v) = \int_D \frac{1}{n_\varepsilon - 1}(\Delta u + k^2 u)(\Delta \overline{v} + k^2 \overline{v})dx + k^4 \int_D u\overline{v}dx$$

and

$$\mathcal{B}(u,v) = \int_D \nabla u \nabla \overline{v} dx.$$

Since n_ε is complex, according to Theorem 2.4.2, there are no transmission eigenvalues for (C.3) then for all $k > 0$ there exists $u_\varepsilon \in H_0^2(D)$ verifying (C.4).

Theorem C.1.1. *There exists a subsequence of u_ε that weakly converges to a solution $u \in V_0(D, D_0, k)$ to* C.1.

The proof is decomposed in two parts. First, we show that the sequence u_ε is bounded in $H_0^2(D)$. This leads to the existence of a weakly convergent subsequence. Next, we show that the limit is a solution to the interior transmission problem for the domain containing a cavity.

Lemma C.1.2. *There exists a subsequence to u_ε that weakly converges in $H_0^2(D)$ to some function $u \in H_0^2(D)$.*

Proof. We want to show that u_ε is bounded in $H_0^2(D)$. To this end we assume first that for all $m > 0$ there exists $\varepsilon(m)$ such that $||u_{\varepsilon(m)}||_{H_0^1(D)} > m$. Now set

$$v_m = \frac{u_{\varepsilon(m)}}{||u_{\varepsilon(m)}||_{H_0^1(D)}}.$$

Then $||v_m||_{H_0^1(D)} = 1$ and $||v_m||_{H^2(D)} \geq 1$. Let show now that v_m is uniformly bounded on $H_0^2(D)$. Since $u_{\varepsilon(m)}$ satisfies (C.4) we have

$$\mathcal{A}(v_m, v_m) = -\frac{1}{||u_{\varepsilon(m)}||_{H_0^1(D)}}\big(\mathcal{A}(\theta, v_m) - k^2 \mathcal{B}(\theta, v_m)\big) + k^2 \mathcal{B}(v_m, v_m).$$

$\mathcal{A}$ and $\mathcal{B}$ are bounded in $H_0^2(D)$ and $\mathcal{A}$ is coercive. Therefore we obtain

$$\alpha ||v_m||^2_{H_0^2(D)} \leq C||v_m||_{H_0^2(D)} + k^2 ||v_m||^2_{H_0^1(D)}$$

and then

$$||v_m||_{H_0^2(D)} \leq \frac{C}{\alpha} + \frac{k^2}{\alpha ||v_m||_{H_0^2(D)}} \leq \frac{C + k^2}{\alpha}.$$

We deduce that v_m is bounded in $H_0^2(D)$ and hHence there exists a subsequence still noted v_m which weakly converges to v in $H_0^2(D)$. First, we note that $\varepsilon(m) \longrightarrow 0$ when $m \longrightarrow \infty$. Since $u_{\varepsilon(m)}$ verifies (C.4) then for $\Psi = v$ and multiplying by $\frac{\varepsilon(m)}{||u_{\varepsilon(m)}||_{H_0^1(D)}}$ we have

$$\int_D \frac{\varepsilon(m)}{n_{\varepsilon(m)} - 1}(\Delta v_m + k^2 v_m)(\Delta \overline{v} + k^2 \overline{v})dx + k^2 \varepsilon(m) \int_D (\Delta v_m + k^2 v_m)\overline{v}dx$$
$$= -\frac{\varepsilon(m)}{||u_{\varepsilon(m)}||_{H_0^1(D)}} \int_\Gamma \frac{1}{n-1}(\Delta \theta + k^2 \theta)(\Delta \overline{v} + k^2 \overline{v})dx - k^2 \frac{\varepsilon(m)}{||u_{\varepsilon(m)}||_{H_0^1(D)}} \int_\Gamma (\Delta \theta + k^2 \theta)\overline{v}dx$$

Then we obtain

$$
\begin{aligned}
-i\int_{D_0}(\Delta v_m+k^2v_m)(\Delta\overline{v}+k^2\overline{v})dx+\varepsilon(m)\int_\Gamma\frac{1}{n-1}(\Delta v_m+k^2v_m)(\Delta\overline{v}+k^2\overline{v})dx\\
+k^2\varepsilon(m)\int_D(\Delta v_m+k^2v_m)\overline{v}dx
\end{aligned}
$$

$$
\begin{aligned}
=-\frac{\varepsilon(m)}{||u_{\varepsilon(m)}||_{H_0^1(D)}}\int_\Gamma\frac{1}{n-1}(\Delta\theta+k^2\theta)(\Delta\overline{v}+k^2\overline{v})dx\\
-k^2\frac{\varepsilon(m)}{||u_{\varepsilon(m)}||_{H_0^1(D)}}\int_\Gamma(\Delta\theta+k^2\theta)\overline{v}dx.
\end{aligned}
$$

Since v_m is bounded in $H_0^2(D)$ and $\frac{1}{||u_{\varepsilon(m)}||_{H_0^1(D)}}\leq\frac{1}{m}$ then if $m\longrightarrow\infty$ we obtain

$$
\int_{D_0}|\Delta v+k^2v|^2dx=0
$$

and therefore $\Delta v+k^2v=0$ in D_0. Then $v\in V_0(D,D_0,k)$. Now let us show that v is a weak solution of the homogeneous interior transmission problem. For $\Psi\in V_0(D,D_0,k)$, v_m satisfies

$$
\begin{aligned}
\int_D\frac{1}{n_{\varepsilon(m)}-1}(\Delta v_m+k^2v_m)(\Delta\overline{\Psi}+k^2\overline{\Psi})dx+k^2\int_D(\Delta v_m+k^2v_m)\overline{\Psi}dx\\
=-\frac{1}{||u_{\varepsilon(m)}||_{H_0^1(D)}}\left[\int_\Gamma\frac{1}{n-1}(\Delta\theta+k^2\theta)(\Delta\overline{\Psi}+k^2\overline{\Psi})dx+k^2\int_\Gamma(\Delta\theta+k^2\theta)\overline{\Psi}dx\right]
\end{aligned}
$$

The right hand side converges to 0 when m converges to infinity. Since $\Psi\in V_0(D,D_0,k)$,i.e. $\Delta\Psi+k^2\Psi=0$ in D_0, for the left hand side we have

$$
\begin{aligned}
\int_\Gamma\frac{1}{n-1}(\Delta v_m+k^2v_m)(\Delta\overline{\Psi}+k^2\overline{\Psi})dx+k^2\int_D(\Delta v_m+k^2v_m)\overline{\Psi}dx\\
\underset{m\to+\infty}{\longrightarrow}\int_\Gamma\frac{1}{n-1}(\Delta v+k^2v)(\Delta\overline{\Psi}+k^2\overline{\Psi})dx+k^2\int_\Gamma(\Delta v+k^2v)\overline{\Psi}dx.
\end{aligned}
$$

Then v is a solution of the homogeneous interior transmission problem. If k is not a transmission eigenvalue of (C.1), then $v=0$. Furthermore the embedding of $H^2(D)$ into $H^1(D)$ is compact. Then $||v_m||_{H_0^1(D)}\longrightarrow 0$ as $m\longrightarrow\infty$. This contradicts the fact that $||v_m||_{H_0^1(D)}=1$. Therefore, if k is not a transmission eigenvalue of (C.1), $\{u_\varepsilon\}$ is uniformly bounded in $H_0^1(D)$. Using (C.4) it is now easy to show that u_ε is bounded in $H_0^2(D)$.

Hence there exists a subsequence still noted u_ε that weakly converges to u in $H_0^2(D)$. □

Lemma C.1.3. *The limit u is in $V_0(D,D_0,k)$ and satisfies* (C.2) *for all $\Psi\in V_0(D,D_0,k)$.*

Proof. First let us show that $u \in V_0(D, D_0, k)$ that is $\Delta u + k^2 u = 0$ in D_0. With $\Psi = u$ in (C.4) and multiplying (C.4) by ε we obtain

$$\int_D \frac{\varepsilon}{n_\varepsilon - 1}(\Delta u_\varepsilon + k^2 u_\varepsilon)(\Delta \overline{u} + k^2 \overline{u})dx + k^2 \varepsilon \int_D (\Delta u_\varepsilon + k^2 u_\varepsilon)\overline{u}dx$$
$$= -\int_\Gamma \frac{\varepsilon}{n-1}(\Delta\theta + k^2\theta)(\Delta \overline{u} + k^2 \overline{u})dx - k^2 \varepsilon \int_\Gamma (\Delta\theta + k^2\theta)\overline{u}dx$$

Then we have

$$-i \int_{D_0} (\Delta u_\varepsilon + k^2 u_\varepsilon)(\Delta \overline{u} + k^2 \overline{u})dx + \varepsilon \int_\Gamma \frac{1}{n-1}(\Delta u_\varepsilon + k^2 u_\varepsilon)(\Delta \overline{u} + k^2 \overline{u})dx + k^2 \varepsilon \int_D (\Delta u_\varepsilon$$
$$+ k^2 u_\varepsilon)\overline{u}dx = -\varepsilon \int_\Gamma \frac{1}{n-1}(\Delta\theta + k^2\theta)(\Delta \overline{u} + k^2 \overline{u})dx - k^2 \varepsilon \int_\Gamma (\Delta\theta + k^2\theta)\overline{u}dx.$$

Since u_ε is bounded in $H_0^2(D)$ and $u_\varepsilon \rightharpoonup u$ in $H_0^2(D)$, then making $\varepsilon \longrightarrow 0$ we obtain

$$\int_{D_0} |\Delta u + k^2 u|^2 dx = 0$$

and we conclude that $\Delta u + k^2 u = 0$ in D_0.

Now let show that u satisfies (C.2). To this end we show that the left-hand side of (C.4) converges to the left-hand side of (C.2). Let $\Psi \in V_0(D, D_0, k)$ and let $y_\varepsilon = u_\varepsilon - u$

$$\int_D \frac{1}{n_\varepsilon - 1}(\Delta u_\varepsilon + k^2 u_\varepsilon)(\Delta \overline{\Psi} + k^2 \overline{\Psi})dx + k^2 \int_D (\Delta u_\varepsilon + k^2 u_\varepsilon)\overline{\Psi}dx$$
$$- \int_\Gamma \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{\Psi} - k^2 \overline{\Psi})dx - k^2 \int_\Gamma (\Delta u + k^2 u)\overline{\Psi}dx$$
$$= \int_\Gamma \frac{1}{n-1}(\Delta y_\varepsilon + k^2 y_\varepsilon)(\Delta \overline{\Psi} + k^2 \overline{\Psi})dx + k^2 \int_\Gamma (\Delta y_\varepsilon + k^2 y_\varepsilon)\overline{\Psi}dx$$
$$+ k^2 \int_{D_0} (\Delta u_\varepsilon + k^2 u_\varepsilon)\overline{\Psi}dx \quad \text{(C.5)}$$

Using the fact that y_ε weakly converges to 0 in $H_0^2(D)$ and that the embedding of $H_0^2(D)$ in $L^2(D)$ is compact we deduce that (C.5) converges to 0 when $\varepsilon \longrightarrow 0$. Finally u satisfies (C.2).

□

C.2 For Maxwell's equations

In this section we get the same type of result for the electromagnetic case. We get a weak convergence in the space $\mathcal{U}_0(D)$.

Let us recall the interior transmission problem

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 & \text{in } D \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 & \text{in } D \\ \nu \times \mathbf{E} - \nu \times \mathbf{E}_0 = \mathbf{G} & \text{on } \Gamma \\ \nu \times \operatorname{curl}\mathbf{E} - \nu \times \operatorname{curl}\mathbf{E}_0 = \mathbf{H} & \text{on } \Gamma \end{cases} \quad \text{(C.6)}$$

with $\mathbf{G} \in TH^{3/2}(\Gamma)$ and $\mathbf{H} \in TH^{1/2}(\Gamma)$ and N a 3×3 symmetric matrix whose entries are bounded complex-valued functions in $\mathbb{R}^3$ and such that $N = I$ in D_0. We assume that there exists a constant $\gamma > 0$ such that

$$\mathrm{Re}((N-I)^{-1}\xi, \xi) \geq \gamma |\xi|^2$$

for all ξ in $\mathbb{C}^3$ and almost everywhere in D. We recall that

$$V_0(D, D_0, k) := \left\{ \mathbf{E} \in \mathcal{U}_0(D) / \mathrm{curl}\,\mathrm{curl}\,\mathbf{E} - k^2\mathbf{E} = 0 \text{ in } D_0 \right\}.$$

This problem is equivalent to finding $\mathbf{F} \in V_0(D, D_0, k)$ such that for all $\mathbf{\Psi} \in V_0(D, D_0, k)$

$$\begin{aligned}\int_\Gamma &(N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\,\mathbf{F}_0 - k^2\mathbf{F}_0)\cdot(\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx \\ &\qquad - k^2\int_\Gamma (\mathrm{curl}\,\mathrm{curl}\,\mathbf{F}_0 - k^2\mathbf{F}_0)\cdot\overline{\mathbf{\Psi}}dx \\ &= -\int_\Gamma (N-I)^{-1}(\mathrm{curl}\,\mathrm{curl}\,\mathbf{\Theta} - k^2\mathbf{\Theta})\cdot(\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx \\ &\qquad + k^2\int_\Gamma (\mathrm{curl}\,\mathrm{curl}\,\mathbf{\Theta} - k^2\mathbf{\Theta})\cdot\overline{\mathbf{\Psi}}dx \end{aligned} \tag{C.7}$$

where $\mathbf{\Theta} \in H^2(D)^3$ such that $\nu \times \mathbf{\Theta} = \mathbf{G}$, $\nu \times \mathrm{curl}\,\mathbf{\Theta} = \mathbf{H}$ on Γ and $\mathbf{\Theta} = 0$ in $D_{\mathbf{\Theta}}$ with $D_0 \subset D_{\mathbf{\Theta}} \subset D$.

Let us now introduce the following approximated problem

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\,\mathbf{E}^\varepsilon + k^2 N_\varepsilon \mathbf{E}^\varepsilon = 0 & \text{in } D \\ \mathrm{curl}\,\mathrm{curl}\,\mathbf{E}_0^\varepsilon + k^2\mathbf{E}_0^\varepsilon = 0 & \text{in } D \\ \nu \times \mathbf{E}^\varepsilon - \nu \times \mathbf{E}_0^\varepsilon = \mathbf{G} & \text{on } \Gamma \\ \nu \times \mathrm{curl}\,\mathbf{E}^\varepsilon - \nu \times \mathrm{curl}\,\mathbf{E}_0^\varepsilon = \mathbf{H} & \text{on } \Gamma \end{cases} \tag{C.8}$$

where

$$N_\varepsilon(x) = \begin{cases} N & \text{in } D\backslash\overline{D}_0 \\ (1+i\varepsilon)I & \text{in } D_0. \end{cases}$$

This problem is equivalent to finding $\mathbf{F}_\varepsilon \in \mathcal{U}_0(D)$ such that for all $\mathbf{\Psi} \in \mathcal{U}_0(D)$

$$\begin{aligned}&\int_D (N_\varepsilon - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\,\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot(\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx - k^2\int_D (\mathrm{curl}\,\mathrm{curl}\,\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot\overline{\mathbf{\Psi}}dx \\ &= -\int_D (N_\varepsilon - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\,\mathbf{\Theta} - k^2\mathbf{\Theta})\cdot(\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx + k^2\int_D (\mathrm{curl}\,\mathrm{curl}\,\mathbf{\Theta} - k^2\mathbf{\Theta})\cdot\overline{\mathbf{\Psi}}dx\end{aligned}$$

i.e.

$$\begin{aligned}&\int_D (N_\varepsilon - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\,\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot(\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx - k^2\int_D (\mathrm{curl}\,\mathrm{curl}\,\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot\overline{\mathbf{\Psi}}dx \\ &= -\int_\Gamma (N - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\,\mathbf{\Theta} - k^2\mathbf{\Theta})\cdot(\mathrm{curl}\,\mathrm{curl}\,\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx + k^2\int_\Gamma (\mathrm{curl}\,\mathrm{curl}\,\mathbf{\Theta} - k^2\mathbf{\Theta})\cdot\overline{\mathbf{\Psi}}dx\end{aligned} \tag{C.9}$$

For $\mathbf{u}$, $\mathbf{v} \in \mathcal{U}_0(D)$, set

$$\mathcal{A}(\mathbf{u},\mathbf{v}) = \int_D (N_\varepsilon - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{u} - k^2\mathbf{u}) \cdot (\operatorname{curl}\operatorname{curl}\overline{\mathbf{v}} - k^2\overline{\mathbf{v}})dx + k^4 \int_D \mathbf{u}\cdot\overline{\mathbf{v}}dx$$

and

$$\mathcal{B}(\mathbf{u},\mathbf{v}) = \int_D \operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\overline{\mathbf{v}}dx.$$

Since N_ε is complex, there are no transmission eigenvalues for (C.8), then for all $k > 0$ there exists $\mathbf{F}_\varepsilon \in \mathcal{U}_0(D)$ verifying (C.9).

Theorem C.2.1. *There exists a subsequence of $\mathbf{F}_\varepsilon$ that weakly converges to a solution $\mathbf{F} \in V_0(D, D_0, k)(D)$ to the interior transmission problem (C.6).*

Again we decompose the proof in two parts. First, we show that the sequence $\mathbf{F}_\varepsilon$ is bounded in $\mathcal{U}_0(D)$. This leads to the existence of a weakly convergent subsequence. Next, we show that the limit is a solution to the interior transmission problem for the domain containing a cavity.

Lemma C.2.2. *There exists a subsequence to $\mathbf{F}_\varepsilon$ that weakly converges in $\mathcal{U}_0(D)$ to some function $\mathbf{F} \in \mathcal{U}_0(D)$.*

Proof. We want to show that $\mathbf{F}_\varepsilon$ is bounded in $\mathcal{U}_0(D)$. To this end, we assume first that for all $m > 0$ there exists $\varepsilon(m)$ such that $||\operatorname{curl}\mathbf{F}_{\varepsilon(m)}||_{L^2} > m$. Now set

$$\mathbf{G}_m = \frac{\mathbf{F}_{\varepsilon(m)}}{||\operatorname{curl}\mathbf{F}_{\varepsilon(m)}||_{L^2}}.$$

Then $||\operatorname{curl}\mathbf{G}_m||_{L^2} = 1$ and $||\mathbf{G}_m||_{\mathcal{U}_0} \geq 1$. Let us show now that $\mathbf{G}_m$ is uniformly bounded on $\mathcal{U}_0(D)$. Since $\mathbf{G}_{\varepsilon(m)}$ satisfies (C.9) we have

$$\mathcal{A}(\mathbf{G}_m,\mathbf{G}_m) = -\frac{1}{||\operatorname{curl}\mathbf{F}_{\varepsilon(m)}||_{L^2}}\big(\mathcal{A}(\mathbf{\Theta},\mathbf{G}_m) - k^2\mathcal{B}(\mathbf{\Theta},\mathbf{G}_m)\big) + k^2\mathcal{B}(\mathbf{G}_m,\mathbf{G}_m). \tag{C.10}$$

The operators $\mathcal{A}$ and $\mathcal{B}$ are bounded in $\mathcal{U}_0(D)$ and $\mathcal{A}$ is coercive i.e. there exists $\alpha > 0$ such that for all $\mathbf{G} \in \mathcal{U}_0(D)$,

$$\mathcal{A}(\mathbf{G},\mathbf{G}) \geq \alpha||\mathbf{G}||_{\mathcal{U}_0(D)}.$$

Moreover, since $||\operatorname{curl}\mathbf{F}_{\varepsilon(m)}||_{L^2} > m$, there exists $C > 0$ such that

$$\frac{1}{||\operatorname{curl}\mathbf{F}_{\varepsilon(m)}||_{L^2}} \leq C.$$

Therefore, using (C.10), we obtain

$$\alpha||\mathbf{G}_m||^2_{\mathcal{U}_0} \leq C||\mathbf{G}_m||_{\mathcal{U}_0} + k^2||\operatorname{curl}\mathbf{G}_m||^2_{L^2} \tag{C.11}$$

and then

$$||\mathbf{G}_m||_{\mathcal{U}_0} \leq \frac{C}{\alpha} + \frac{k^2}{\alpha||\mathbf{G}_m||_{\mathcal{U}_0}} \leq \frac{C+k^2}{\alpha}.$$

Therefore $\mathbf{G}_m$ is bounded in $\mathcal{U}_0(D)$. Hence there exists a subsequence still noted $\mathbf{G}_m$ which weakly converges to $\mathbf{G}$ in $\mathcal{U}_0(D)$.

First, we note that $\varepsilon(m) \longrightarrow 0$ when $m \longrightarrow \infty$. Since $\mathbf{F}_{\varepsilon(m)}$ verifies (C.9) then for $\mathbf{\Psi} = \mathbf{G}$ and multiplying by $\dfrac{\varepsilon(m)}{||\mathrm{curl}\, \mathbf{F}_{\varepsilon(m)}||_{\mathcal{U}_0}}$ we have

$$\begin{aligned}\int_D \varepsilon(m)(N_\varepsilon - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, \mathbf{G}_m - k^2\mathbf{G}_m) \cdot (\mathrm{curl}\,\mathrm{curl}\, \overline{\mathbf{G}} - k^2\overline{\mathbf{G}})dx \\ - k^2\varepsilon(m) \int_D (\mathrm{curl}\,\mathrm{curl}\, \mathbf{G}_m - k^2\mathbf{G}_m) \cdot \overline{\mathbf{G}}dx \\ = -\frac{\varepsilon(m)}{||\mathrm{curl}\, \mathbf{F}_{\varepsilon(m)}||_{L^2}} \int_\Gamma (N - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, \mathbf{\Theta} - k^2\mathbf{\Theta}) \cdot (\mathrm{curl}\,\mathrm{curl}\, \overline{\mathbf{G}} + k^2\overline{\mathbf{G}})dx \\ + k^2\frac{\varepsilon(m)}{||\mathrm{curl}\, \mathbf{F}_{\varepsilon(m)}||_{L^2}} \int_\Gamma (\mathrm{curl}\,\mathrm{curl}\, \mathbf{\Theta} - k^2\mathbf{\Theta}) \cdot \overline{\mathbf{G}}dx\end{aligned}$$

Then, we obtain

$$\begin{aligned}\varepsilon(m) \int_\Gamma (N - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, \mathbf{G}_m - k^2\mathbf{G}_m) \cdot (\mathrm{curl}\,\mathrm{curl}\, \overline{\mathbf{G}} - k^2\overline{\mathbf{G}})dx \\ -k^2\varepsilon(m) \int_D (\mathrm{curl}\,\mathrm{curl}\, \mathbf{G}_m - k^2\mathbf{G}_m) \cdot \overline{\mathbf{G}}dx - i \int_{D_0} (\mathrm{curl}\,\mathrm{curl}\, \mathbf{G}_m - k^2\mathbf{G}_m) \cdot (\mathrm{curl}\,\mathrm{curl}\, \overline{\mathbf{G}} - k^2\overline{\mathbf{G}})dx \\ = -\frac{\varepsilon(m)}{||\mathrm{curl}\, \mathbf{F}_{\varepsilon(m)}||_{L^2}} \int_\Gamma (N - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, \mathbf{\Theta} - k^2\mathbf{\Theta}) \cdot (\mathrm{curl}\,\mathrm{curl}\, \overline{\mathbf{G}} - k^2\overline{\mathbf{G}})dx \\ + k^2\frac{\varepsilon(m)}{||\mathrm{curl}\, \mathbf{F}_{\varepsilon(m)}||_{L^2}} \int_\Gamma (\mathrm{curl}\,\mathrm{curl}\, \mathbf{\Theta} - k^2\mathbf{\Theta}) \cdot \overline{\mathbf{G}}dx.\end{aligned}$$

Since $\mathbf{G}_m$ is bounded in $\mathcal{U}_0(D)$ and $\dfrac{1}{||\mathrm{curl}\, \mathbf{F}_{\varepsilon(m)}||_{L^2}} \leq \dfrac{1}{m}$ then if $m \longrightarrow \infty$ we obtain

$$\int_{D_0} |\mathrm{curl}\,\mathrm{curl}\, \mathbf{F} - k^2\mathbf{F}|^2 dx = 0$$

and therefore $\mathrm{curl}\,\mathrm{curl}\, \mathbf{F} - k^2\mathbf{F} = 0$ in D_0 which implies that $\mathbf{F} \in V_0(D, D_0, k)$.

Now let us show that $\mathbf{F}$ is a weak solution of the homogeneous interior transmission problem. For $\mathbf{\Psi} \in V_0(D, D_0, k)$, $\mathbf{G}_m$ satisfies

$$\begin{aligned}\int_D (N_{\varepsilon(m)} - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, \mathbf{G}_m - k^2\mathbf{G}_m) \cdot (\mathrm{curl}\,\mathrm{curl}\, \overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx \\ + k^2 \int_D (\mathrm{curl}\,\mathrm{curl}\, \mathbf{G}_m - k^2\mathbf{G}_m) \cdot \overline{\mathbf{\Psi}}dx \\ = -\frac{1}{||\mathrm{curl}\, \mathbf{F}_{\varepsilon(m)}||_{L^2}} \Big[\int_\Gamma (N - I)^{-1}(\mathrm{curl}\,\mathrm{curl}\, \mathbf{\Theta} - k^2\mathbf{\Theta}) \cdot (\mathrm{curl}\,\mathrm{curl}\, \overline{\mathbf{\Psi}} + k^2\overline{\mathbf{\Psi}})dx \\ -k^2 \int_\Gamma (\mathrm{curl}\,\mathrm{curl}\, \mathbf{\Theta} - k^2\mathbf{\Theta}) \cdot \overline{\mathbf{\Psi}}dx\Big]\end{aligned}$$

The right hand side converges to 0 when m tends to infinity. Since $\mathbf{\Psi} \in V_0(D, D_0, k)$, for the left hand side we have

$$\int_D (N_{\varepsilon(m)} - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{G}_m - k^2\mathbf{G}_m)\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx - k^2\int_D (\operatorname{curl}\operatorname{curl}\mathbf{G}_m - k^2\mathbf{G}_m)\cdot\overline{\mathbf{\Psi}}dx$$

$$= \int_\Gamma (N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{G}_m - k^2\mathbf{G}_m)\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx - k^2\int_D (\operatorname{curl}\operatorname{curl}\mathbf{G}_m - k^2\mathbf{G}_m)\cdot\overline{\mathbf{\Psi}}dx$$

$$\underset{m\to+\infty}{\longrightarrow} \int_\Gamma (N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{G} - k^2\mathbf{G})\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx - k^2\int_D (\operatorname{curl}\operatorname{curl}\mathbf{G} - k^2\mathbf{G})\cdot\overline{\mathbf{\Psi}}dx.$$

We deduce that $\mathbf{G}$ is a solution of the homogeneous interior transmission problem for the domain containing a cavity. If k is not a transmission eigenvalue of (C.8) then $\mathbf{G} = 0$. Furthermore $\mathcal{B}$ is compact. Then $\operatorname{curl}\mathbf{G}_m$ converges in $L^2(D)$ when $m \longrightarrow \infty$. This contradicts the fact that $||\operatorname{curl}\mathbf{G}_m||_{L^2} = 1$. Therefore, if k is not a transmission eigenvalue of (C.8), $\{\operatorname{curl}\mathbf{F}_\varepsilon\}$ is uniformly bounded in $L^2(D)$. Using (C.11), it is now easy to show that $\mathbf{F}_\varepsilon$ is bounded in $\mathcal{U}_0(D)$.

Hence there exists a subsequence still noted $\mathbf{F}_\varepsilon$ that weakly converges to $\mathbf{F}$ in $\mathcal{U}_0(D)$. □

Lemma C.2.3. $\mathbf{F}$ *is in* $V_0(D, D_0, k)$ *and is a solution to* C.8.

Proof. First let show that $\mathbf{F} \in V_0(D, D_0, k)$ that is $\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F} = 0$ in D_0. For $\mathbf{\Psi} = \mathbf{F}$ in (C.9) and multiplying (C.9) by ε, we obtain

$$\int_D \varepsilon(N_\varepsilon - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{F}} - k^2\overline{\mathbf{F}})dx - k^2\varepsilon\int_D (\operatorname{curl}\operatorname{curl}\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot\overline{\mathbf{F}}dx$$
$$= -\int_\Gamma \varepsilon(N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{\Theta} - k^2\mathbf{\Theta})\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{F}} - k^2\overline{\mathbf{F}})dx + k^2\varepsilon\int_\Gamma (\operatorname{curl}\operatorname{curl}\mathbf{\Theta} - k^2\mathbf{\Theta})\cdot\overline{\mathbf{F}}dx$$

Then we have

$$\varepsilon\int_\Gamma (N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{F}} - k^2\overline{\mathbf{F}})dx - k^2\varepsilon\int_D (\operatorname{curl}\operatorname{curl}\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot\overline{\mathbf{F}}dx$$
$$- i\int_{D_0} (\operatorname{curl}\operatorname{curl}\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{F}} - k^2\overline{\mathbf{F}})dx$$
$$= -\varepsilon\int_\Gamma (N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{\Theta} - k^2\mathbf{\Theta})\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{F}} - k^2\overline{\mathbf{F}})dx + k^2\varepsilon\int_\Gamma (\operatorname{curl}\operatorname{curl}\mathbf{\Theta} - k^2\mathbf{\Theta})\cdot\overline{\mathbf{F}}dx.$$

Since $\mathbf{F}_\varepsilon$ is bounded in $\mathcal{U}_0(D)$ and $\mathbf{F}_\varepsilon \rightharpoonup \mathbf{F}$ in $\mathcal{U}_0(D)$, then making $\varepsilon \longrightarrow 0$, we obtain

$$\int_{D_0} |\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F}|^2 dx = 0$$

and we conclude that $\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F} = 0$ in D_0.

Now let show that $\mathbf{F}$ satisfies (C.7). To this end we show that the left-hand side of (C.9) converges to the left-hand side of (C.7). Let $\mathbf{\Psi} \in V_0(D, D_0, k)$ and let $\mathbf{H}_\varepsilon = \mathbf{F}_\varepsilon - \mathbf{F}$

$$\int_D (N_\varepsilon - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx - k^2\int_D(\operatorname{curl}\operatorname{curl}\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot\overline{\mathbf{\Psi}}dx$$
$$-\int_\Gamma (N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F})\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx + k^2\int_\Gamma(\operatorname{curl}\operatorname{curl}\mathbf{F} - k^2\mathbf{F})\cdot\overline{\mathbf{\Psi}}dx$$
$$= \int_\Gamma (N - I)^{-1}(\operatorname{curl}\operatorname{curl}\mathbf{H}_\varepsilon - k^2\mathbf{H}_\varepsilon)\cdot(\operatorname{curl}\operatorname{curl}\overline{\mathbf{\Psi}} - k^2\overline{\mathbf{\Psi}})dx$$
$$- k^2\int_\Gamma(\operatorname{curl}\operatorname{curl}\mathbf{H}_\varepsilon - k^2\mathbf{H}_\varepsilon)\cdot\overline{\mathbf{\Psi}}dx - k^2\int_{D_0}(\operatorname{curl}\mathbf{F}_\varepsilon - k^2\mathbf{F}_\varepsilon)\cdot\overline{\mathbf{\Psi}}dx \quad \text{(C.12)}$$

Using the fact that $\mathbf{H}_\varepsilon$ weakly converges to 0 in $\mathcal{U}_0(D)$ and that the embedding of $\mathcal{U}_0(D)$ in $L^2(D)$ is compact we deduce that (C.12) converges to 0 when $\varepsilon \longrightarrow 0$. Finally $\mathbf{F}$ satisfies (C.7).

□

Appendix D

Far fields and potential operators

Let D be a bounded domain with smooth boundary Γ. We recall the definition of the boundary integral operators

$$
\begin{aligned}
S_k(\varphi)(x) &= \int_\Gamma \Phi_k(x,y)\varphi(y)ds(y), \\
K_k(\psi)(x) &= \int_\Gamma \frac{\partial \Phi_k}{\partial \nu(y)}(x,y)\psi(y)ds(y), \\
K_k'(\varphi)(x) &= \int_\Gamma \frac{\partial \Phi_k}{\partial \nu(x)}(x,y)\varphi(y)ds(y), \\
T_k(\psi)(x) &= \int_\Gamma \frac{\partial^2 \Phi_k}{\partial \nu(y)\nu(x)}(x,y)\psi(y)ds(y)
\end{aligned}
$$

where

$$\Phi_k(y,z) := \frac{e^{\mathrm{i}k|y-z|}}{4\pi|y-z|}$$

is the fundamental solution to Helmholtz equation.

In this appendix, we want to establish a link between the integral operator

$$\begin{pmatrix} S_k & -K_k \\ -K_k' & T_k \end{pmatrix}$$

used in the study of the interior transmission problem using integral equations and the far field patterns. The far field operator is defined by

$$
\begin{array}{rccc}
P^\infty : & H^{-3/2}(\Gamma)\times H^{-1/2}(\Gamma) & \longrightarrow & L^2(\Omega) \\
& (\alpha,\beta) & \longmapsto & P^\infty(\alpha,\beta)
\end{array}
$$

where

$$P^\infty(\alpha,\beta)(\hat{x}) = \frac{1}{4\pi}\int_\Gamma \left(\beta(y)\frac{\partial e^{-\mathrm{i}k\hat{x}\cdot y}}{\partial \nu(y)} - \alpha(y)e^{-\mathrm{i}k\hat{x}\cdot y}\right) ds(y) \quad \forall \hat{x}\in\Omega.$$

We denote by $P^{\infty *}$ the adjoint of P^∞ defined by

$$\left(P^{\infty *}g,(a,b)\right)_{L^2(\Gamma)^2} := \left(g,P^\infty(a,b)\right)_{L^2(\Omega)}$$

for all $g\in L^2(\Omega)$ and $(a,b)\in L^2(\Gamma)\times L^2(\Gamma)$.

The goal of this appendix is to show the following theorem:

Theorem D.0.4.

$$kP^{\infty *}P^{\infty} = \mathrm{Im}\begin{pmatrix} S_k & -K_k \\ -K'_k & T_k \end{pmatrix}.$$

D.1 Computation of the adjoint of the far field operator P^{∞}

$$\begin{array}{rccc} P^{\infty *}: & L^2(\Omega) & \longrightarrow & L^2(\Gamma)\times L^2(\Gamma) \\ & g & \longmapsto & (P_1^{\infty *}g, P_2^{\infty *}g) \end{array}$$

For all $g \in L^2(\Omega)$ and $(a,b) \in L^2(\Gamma)\times L^2(\Gamma)$, $P^{\infty *}$ is defined by :

$$\begin{aligned}
(P^{\infty *}g,(a,b))_{L^2(\Gamma)^2} &:= (g, P^{\infty}(a,b))_{L^2(\Omega)} \\
&= \int_{\Omega} g(\hat{x})\overline{P^{\infty}(a,b)(\hat{x})}\,ds(\hat{x}) \\
&= \frac{1}{4\pi}\int_{\Omega}\int_{\Gamma} g(\hat{x})\left(\overline{b(y)}\frac{\partial e^{\mathrm{i}k\hat{x}\cdot y}}{\partial\nu(y)} - \overline{a(y)}e^{\mathrm{i}k\hat{x}\cdot y}\right)\,dy\,ds(\hat{x}) \\
&= \int_{\Gamma}\left[\overline{b(y)}\int_{\Omega}\frac{1}{4\pi}g(\hat{x})\frac{\partial e^{\mathrm{i}k\hat{x}\cdot y}}{\partial\nu(y)}\,ds(\hat{x}) - \overline{a(y)}\int_{\Omega}\frac{1}{4\pi}g(\hat{x})e^{\mathrm{i}k\hat{x}\cdot y}\,ds(\hat{x})\right]\,dy
\end{aligned}$$

Then we get :

$$\left\{\begin{array}{rcl} P_1^{\infty *}g(y) & := & -\dfrac{1}{4\pi}\displaystyle\int_{\Omega} g(\hat{x})e^{\mathrm{i}k\hat{x}\cdot y}\,ds(\hat{x}) \\ P_2^{\infty *}g(y) & := & \dfrac{1}{4\pi}\displaystyle\int_{\Omega} g(\hat{x})\frac{\partial e^{\mathrm{i}k\hat{x}\cdot y}}{\partial\nu(y)}\,ds(\hat{x}) \end{array}\right.$$

D.2 Computation of $P^{\infty *}P^{\infty}$

Let (a,b) be in $L^2(\Gamma)\times L^2(\Gamma)$.

$$
\begin{aligned}
P^{\infty *}P^{\infty}(a,b)(y) &= \begin{pmatrix} -\dfrac{1}{4\pi}\displaystyle\int_\Omega P^{\infty}(a,b)(\hat{x})e^{\mathrm{i}k\hat{x}\cdot y}\,ds(\hat{x}) \\ \dfrac{1}{4\pi}\displaystyle\int_\Omega P^{\infty}(a,b)(\hat{x})\dfrac{\partial e^{\mathrm{i}k\hat{x}\cdot y}}{\partial\nu(y)}\,ds(\hat{x}) \end{pmatrix} \\
&= \frac{1}{(4\pi)^2}\begin{pmatrix} -\displaystyle\int_\Omega\int_\Gamma \left(b(z)\dfrac{\partial e^{-\mathrm{i}k\hat{x}\cdot z}}{\partial\nu(z)} - a(z)e^{-\mathrm{i}k\hat{x}\cdot z}\right)e^{\mathrm{i}k\hat{x}\cdot y}\,dz\,ds(\hat{x}) \\ \displaystyle\int_\Omega\int_\Gamma \left(b(z)\dfrac{\partial e^{-\mathrm{i}k\hat{x}\cdot z}}{\partial\nu(z)} - a(z)e^{-\mathrm{i}k\hat{x}\cdot z}\right)\dfrac{\partial e^{\mathrm{i}k\hat{x}\cdot y}}{\partial\nu(y)}\,dz\,ds(\hat{x}) \end{pmatrix} \\
&= \frac{1}{(4\pi)^2}\begin{pmatrix} -\displaystyle\int_\Omega\int_\Gamma \left(b(z)\dfrac{\partial e^{\mathrm{i}k(y-z)\cdot\hat{x}}}{\partial\nu(z)} - a(z)e^{\mathrm{i}k(y-z)\cdot\hat{x}}\right)\,dz\,ds(\hat{x}) \\ \displaystyle\int_\Omega\int_\Gamma \left(b(z)\dfrac{\partial^2 e^{\mathrm{i}k(y-z)\cdot\hat{x}}}{\partial\nu(z)\partial\nu(y)} - a(z)\dfrac{\partial e^{\mathrm{i}k(y-z)\cdot\hat{x}}}{\partial\nu(y)}\right)\,dz\,ds(\hat{x}) \end{pmatrix} \\
&= \frac{1}{k}\begin{pmatrix} -\displaystyle\int_\Gamma \left(b(z)\dfrac{\partial G_k(y,z)}{\partial\nu(z)} - a(z)G_k(y,z)\right)dz \\ \displaystyle\int_\Gamma \left(b(z)\dfrac{\partial^2 G_k(y,z)}{\partial\nu(z)\partial\nu(y)} - a(z)\dfrac{\partial G_k(y,z)}{\partial\nu(y)}\right)dz \end{pmatrix}
\end{aligned}
$$

where

$$G_k(y,z) = \frac{k}{4\pi}j_0(k|y-z|) = \frac{1}{4\pi}\frac{\sin(k|y-z|)}{|y-z|}. \tag{D.1}$$

Indeed, this follows from the Funk-Hecke formula (see [26])

$$j_0(k|t|) = \frac{1}{4\pi}\int_\Omega e^{-\mathrm{i}kt\cdot\hat{x}}\,ds(\hat{x}).$$

Moreover, we remark that $G_j(y,z)$ is the imaginary part of the fundamental solution to Helmholtz equation

$$\Phi_k(y,z) := \frac{e^{\mathrm{i}k|y-z|}}{4\pi|y-z|}.$$

and we get the following result

$$kP^{\infty *}P^{\infty} = \mathrm{Im}\begin{pmatrix} S_k & -K_k \\ -K'_k & T_k \end{pmatrix} \tag{D.2}$$

Appendix E

Pseudo-differential operators as integral operators

Regularity properties of the potentials for the Helmholtz equation is a classical result [44] for densities in $H^{-1/2}(\partial D)$ for the single layer potential and in $H^{1/2}(\partial D)$ for the double layer potential. In Chapter 5, since we are confronted to solutions to the Helmholtz equation in $L^2(D)$ with Laplacien in $L^2(D)$, we need to extend the results for densities in $H^{-3/2}(\partial D)$ for the single layer potential and in $H^{-1/2}(\partial D)$ for the double layer potential. This will be done with the use of the theory of pseudo-differential operators.

The first section gives a characterization of pseudo-differential operators that can be written in the form of integral operators with pseudo-homogeneous kernels. The needed definitions are previously given. Then, we give a regularity theorem depending on the order of the pseudo-differential operator.

Then, this theory is used to establish new regularity properties on the potentials by using the asymptotic behaviours of their kernels.

E.1 Pseudo-homogeneous kernels

The definitions and theorems on pseudo-differential operators are extracted from [35].

Definition E.1.1. *A function $k_q(x,z)$ is a $\mathcal{C}^\infty(D\times\mathbb{R}^n\setminus\{0\})$ pseudo-homogeneous function with respect to z of degree $q\in\mathbb{R}$ if*

$$\begin{aligned} k_q(x,tz) &= t^q k_q(x,z) \text{ for all } t>0 \text{ and } z\neq 0 & \text{if } q\notin\mathbb{N}_0;\\ k_q(x,z) &= f_q(x,z)+\log|z|p_q(x,z) & \text{if } q\in\mathbb{N}_0,\end{aligned}$$

where $p_q(x,z)$ is a homogeneous polynomial in z of degree q having $\mathcal{C}^\infty$-coefficients and where the function $f_q(x,z)$ satisfies

$$f_q(x,tz)=t^q f_q(x,z) \text{ for all } t>0 \text{ and } z\neq 0. \tag{E.1}$$

Definition E.1.2. *A kernel function $k(x,x-y)$ with $(x,y)\in D\times D$, $x\neq y$ is said to have a pseudo-homogeneous extension of degree q if there exist pseudo-homogeneous functions*

k_{q+j} *of degree* $q+j$ *for* $j \in \mathbb{N}_0$ *such that*

$$k(x, x-y) - \sum_{j=0}^{J} k_{q+j}(x, x-y) \in \mathcal{C}^{q+J-\delta}(D \times D)$$

for some δ *with* $0 < \delta < 1$. *Such kernels are called pseudo-homogeneous kernel of degree* q.

E.2 Characterization of pseudo-differential operators from their kernels - Main theorems

The next theorem characterizes pseudo-differential operators defined as integral operators. It is only valid for pseudo-differential operators of order m.

Theorem E.2.1. *(Theorem 7.1.1 of [35]). Let* $m < 0$. *An operator* A *is a pseudo-differential operator of order* m *if and only if*

$$(Au)(x) = \int_D k(x, x-y)u(y)dy \text{ for all } u \in \mathcal{C}_0^\infty(D)$$

where k *is a pseudo-homogeneous kernel of degree* $-m-n$.

The class of pseudo-differential operators of order m is closed under the operations of taking the transposed and the adjoint of these operators. The theorem can be found in [35] and the proof is done in [51]. It is used in Chapter 6 to study the regularity of the double-layer potential.

Theorem E.2.2. *(Theorem 6.1.13 of [35]). If* A *is a pseudo-differential operator of order* m, *then its transposed* $A^\top$ *and its adjoint* A^* *are also pseudo-differential operator of order* m.

We define the operator $\tilde{Q}_\Gamma$ by

$$\tilde{Q}_\Gamma u = A(u\delta_\Gamma).$$

The next theorem gives a regularity result on the surface integral operator defined from A.

Theorem E.2.3. *(Theorem 8.5.8 of [35]). Let* A *be a pseudo-differential operator of order* $m \in \mathbb{Z}$. *Then the following linear mapping is continuous*

$$\tilde{Q}_\Gamma : H^s(\Gamma) \longrightarrow H^{s-m-1/2}(D)$$

for $s \in \mathbb{R}$.

Appendix F

Spherical harmonics and spherical Bessel functions - Computation of transmission eigenvalues for spherical geometries

In this appendix, we recall the definitions of spherical harmonics, Bessel and Hankel functions and some properties that are useful in Chapter 6 for the computation of transmission eigenvalues for spherical geometries. For the electromagnetic case, we also need to define the vector spherical harmonics. For the proof of the following theorems, one can refer to [26].

F.1 Spherical harmonics

A *spherical harmonic of order* n is a restriction of a homogeneous harmonic polynomial of degree n to the unit sphere Ω of $\mathbb{R}^3$. In the following, (Y_n^m), for $m = -n, ..., n$ and $n = 0, 1, 2, ...$ is a family of spherical harmonics defined by

$$Y_n^m(\theta, \varphi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-|m|)!}{(n+|m|)!}} P_n^m(\cos\theta)e^{\mathrm{i}m\varphi}$$

where P_n^m are the associated Legendre functions.

The following theorem justifies the decomposition of square-integrable functions on the unit sphere.

Theorem F.1.1. *The spherical harmonics Y_n^m for $m = -n, ..., n$ and $n = 0, 1, 2, ...$ form a complete orthonormal system in $L^2(\Omega)$.*

We now just recall a special case of the addition theorem that will be useful in the next calculations.

Theorem F.1.2. *Let Y_n^m for $m = -n, ..., n$ be any system of $2n+1$ orthonormal spherical harmonics of order n. Then for all $\hat{x} \in \Omega$, we have*

$$\sum_{m=-n}^{n} Y_n^m(\hat{x})\overline{Y_n^m(\hat{x})} = \frac{2n+1}{4\pi}.$$

Again, the proof of this theorem can be found in [26].

F.2 Bessel functions

We denote by j_n and y_n the *spherical Bessel functions* and the s*pherical Neumann functions* of order n, respectively. The *Hankel functions of the first and second kind of order n* are defined by

$$h_n^{(1,2)} := j_n \pm \mathrm{i} y_n.$$

Remark F.2.1. *From Theorem F.1.1, we know that every function u in $L^2(B_R)$, where $B_R \subset \mathbb{R}^3$ is a sphere of radius R, have the following expansion*

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m(|x|) Y_n^m(\hat{x})$$

where

$$a_n^m(|x|) = \int_{\Omega} u(\hat{x}) \overline{Y_n^m(\hat{x})} ds(\hat{x})$$

and $\hat{x} = x/|x|$.

Moreover, if u is a solution to Helmholtz equation, the a_n^m are solutions to the spherical Bessel equation and consequently, from the definitions of Bessel and Hankel functions, the coefficients a_n^m can be written

$$a_n^m(r) = \alpha_n^m h_n^{(1)}(kr) + \beta_n^m h_n^{(2)}(kr).$$

From the Funk-Hecke formula

$$\int_{\Omega} e^{-\mathrm{i}kr\hat{x}\cdot\hat{z}} Y_n^m(\hat{z}) ds(\hat{z}) = \frac{4\pi}{\mathrm{i}^n} j_n(kr) Y_n^m(\hat{x}), \quad \hat{x} \in \Omega, \quad r > 0,$$

the expression of the plane wave with incident direction d is given by

$$e^{\mathrm{i}kx\cdot d} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} 4\pi \mathrm{i}^n j_n(k|x|) \overline{Y_n^m(\hat{x})} Y_n^m(d).$$

We can state the following theorem that gives the expansion of radiating solutions to the Helmholtz equation and its corresponding far field pattern.

Theorem F.2.1. *Let u be a radiating solution to the Helmholtz equation in the exterior $|x| > R > 0$ of a sphere. Then u has an expansion with respect to the spherical wave functions of the form*

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_n^m h_n^{(1)}(k|x|)(|x|)Y_n^m(\hat{x})$$

that converges absolutely and uniformly on compact subsets of $|x| > R$. The corresponding far field pattern is given by the uniformly convergent series

$$u_\infty(\hat{x}) = \frac{1}{k} \sum_{n=0}^{\infty} \frac{1}{\mathrm{i}^{n+1}} \sum_{m=-n}^{n} a_n^m Y_n^m(\hat{x}).$$

Remark F.2.2. *We have equivalent theorems in $\mathbb{R}^2$ considering the Bessel functions J_n and the Neumann functions Y_n. We also define the Hankel functions of the first and second kind of order n by*

$$H_n^{(1,2)} := J_n \pm \mathrm{i}Y_n.$$

In $\mathbb{R}^2$, a function $u \in L^2(B_R)$ where B_R is a disk of radius R have the Fourier expansion

$$u(x) = \sum_{n=-\infty}^{\infty} a_n(r)e^{\mathrm{i}n\theta}$$

where

$$a_n(r) = \frac{1}{2\pi} \int_0^{2\pi} u(r,\theta)e^{-\mathrm{i}n\theta} d\theta$$

and $x = re^{\mathrm{i}\theta}$. Moreover, if u is a solution to Helmholtz equation the coefficients a_n can be written

$$a_n(r) = \alpha_n H_n^{(1)}(kr) + \beta_n H_n^{(2)}(kr)$$

with $\beta_n = 0$ when u is a radiating solution.

For more details of the $\mathbb{R}^2$ case, one can refer to [26, 8].

F.3 Vector wave functions

In the electromagnetic case, we need to define the vector spherical harmonics that will play the role of the spherical harmonics Y_n^m in the scalar case.

For any orthonormal system Y_n^m, $m = -n, ..., n$, of spherical harmonics of order $n > 0$, the tangential fields on the unit sphere

$$U_n^m := \frac{1}{\sqrt{n(n+1)}} \operatorname{Grad} Y_n^m, \quad V_n^m(\hat{x}) = \frac{1}{\sqrt{n(n+1)}} \hat{x} \times \operatorname{Grad} Y_n^m(\hat{x})$$

are called *vector spherical harmonics of order n*.

Similarly to the scalar case, we also have a completeness theorem for the vector spherical harmonics.

Theorem F.3.1. *The vector spherical harmonics U_n^m and V_n^m for $m = -n, ..., n$ and $n = 0, 1, 2, ...$ form a complete orthonormal system in*

$$L_t^2(\Omega) := \left\{ u : \Omega \to \mathbb{C}^3 / u \in L^2(\Omega), u \cdot \nu = 0 \right\}.$$

Remark F.3.1. *An electric field solutions to Maxwell's equations has the following expansion*

$$\mathbf{E}(x, d, p) = \operatorname{curl}(xv(x, d, p)) + \frac{\mathrm{i}}{k} \operatorname{curl}\operatorname{curl}(xu(x, d, p))$$

where u and v are both solutions to the Helmholtz equation

$$\Delta u + k^2 n u = 0.$$

The first term in the decomposition correspond to a transverse electric polarization and the second term a transverse magnetic polarization.

Using spherical harmonics, the general form of $\mathbf{E}$ is

$$\begin{aligned}\mathbf{E}(x, d, p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \operatorname{curl} \left(x \left(a_n^m h_n^{(1)}(k|x|) + b_n^m h_n^{(2)}(k|x|) \right) Y_n^m(\hat{x}) \right) \\ + \frac{\mathrm{i}}{k} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \operatorname{curl}\operatorname{curl} \left(x \left(\tilde{a}_n^m h_n^{(1)}(k|x|) + \tilde{b}_n^m h_n^{(2)}(k|x|) \right) Y_n^m(\hat{x}) \right)\end{aligned}$$

In particular, we can characterize the radiating solutions to Maxwell's equations and its far field pattern.

Theorem F.3.2. *Let $\mathbf{E}$, $\mathbf{H}$ be a radiating solution to Maxwell's equations for $|x| > R > 0$. Then $\mathbf{E}$ has an expansion with respect to the spherical vector wave functions of the form*

$$\begin{aligned}\mathbf{E}(x, d, p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} a_n^m \operatorname{curl} \left(x h_n^{(1)}(k|x|) Y_n^m(\hat{x}) \right) \\ + \frac{\mathrm{i}}{k} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} b_n^m \operatorname{curl}\operatorname{curl} \left(x h_n^{(1)}(k|x|) Y_n^m(\hat{x}) \right)\end{aligned}$$

that together with its derivatives converges uniformly on compact subsets of $|x| > R$. In this case, the far field expansion is given by

$$\mathbf{E}_\infty = -\frac{1}{k} \sum_{n=1}^{\infty} \frac{1}{\mathrm{i}^{n+1}} \sum_{m=-n}^{n} (b_n^m \operatorname{Grad} Y_n^m + a_n^m \hat{x} \times \operatorname{Grad} Y_n^m).$$

We now give the expansion of the far field pattern of an electric dipole

$$\mathbf{E}_e(x, z, q) = \operatorname{curl}_x (q \Phi(x, z))$$

that will be needed to solve the far field equation. It is given by

$$\mathbf{E}_{e,\infty}(\hat{x}, z, q) := \frac{\mathrm{i}k}{4\pi} (\hat{x} \times q) e^{-\mathrm{i}k \hat{x} \cdot z}$$

and have the expansion

$$\mathbf{E}_{e,\infty}(\hat{x}, z, q) = \sum_{n=1}^{\infty} \frac{-(\mathrm{i})^{n+1}}{\sqrt{n(n+1)}} \sum_{m=-n}^{n} \left(\mathrm{i}k\overline{M_n^m(z)} \cdot q\right) U_n^m(\hat{x}) + \sum_{n=1}^{\infty} \frac{-(\mathrm{i})^{n+1}}{\sqrt{n(n+1)}} \sum_{m=-n}^{n} \left(\mathrm{curl}\,\overline{M_n^m(z)} \cdot q\right) V_n^m(\hat{x}) \quad (\mathrm{F.1})$$

where M_n^m is a radiating solutions of Maxwell's equation defined by

$$M_n^m(x, k) = \mathrm{curl}\,\left(x h_n^{(1)}(k|x|) Y_n^m(\hat{x})\right).$$

F.4 Spherical geometry

In the case of spherical geometry, using separation of variables, solutions to Helmholtz equation or Maxwell's equations have an analytical expansion and the computation of transmission eigenvalues leads to compute the zero of some determinant. After recalling the definitions and the properties of Bessel's and Hankel's functions, we will compute the transmission eigenvalues of a sphere with homogeneous or stratified index of refraction for both the scalar case and the electromagnetic case in 3 dimensions.

F.4.1 Scalar case

Assume that $D \subset \mathbb{R}^3$ is a sphere of radius R and index of refraction $n(x)$. The corresponding interior transmission problem is

$$\begin{cases} \Delta w + k^2 n w = 0 & \text{in } D \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \Gamma \\ \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} & \text{on } \Gamma. \end{cases}$$

Direct method by solving analytically ITP

First assume that n is constant in D. Then, since D contains the origin and $y_n(x)$ is singular when $x = 0$, v and w have the following expansion

$$v(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_n^m j_n(k|x|) Y_n^m(\hat{x})$$

and

$$w(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \beta_n^m j_n(k\sqrt{n}|x|) Y_n^m(\hat{x}).$$

Therefore, from the boundary conditions satisfied by v and w, the problem of finding a non trivial solution to ITP is equivalent to finding $m \geq 0$ such that

$$\det \begin{pmatrix} j_m(kR) & -j_m(k\sqrt{n}R) \\ j_m'(kR) & -\sqrt{n} j_m'(k\sqrt{n}R) \end{pmatrix} = 0.$$

Nevertheless, we would like to compute transmission eigenvalues also when the domain is not homogeneous or when it contains a cavity.

Consequently, we assume now that D is a double layer sphere i.e. there exists $0 < r < R$ such that

$$n(x) = \begin{cases} n_2 & \text{if } r < |x| < R \\ n_1 & \text{if } |x| < r \end{cases}$$

where n_1 can be equal to 1 and we denote by $k_1 := k\sqrt{n_1}$ and $k_2 := k\sqrt{n_2}$.

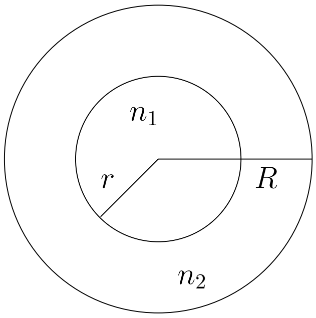


Figure F.1: Spherical geometry

In this way, v and w can be written as

$$v(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_n^m j_n(k|x|) Y_n^m(\hat{x})$$

and

$$w(x) = \begin{cases} \displaystyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} (a_n^m h_n^{(1)}(k|x|\sqrt{n_2}) + b_n^m h_n^{(2)}(k|x|\sqrt{n_2})) Y_n^m(\hat{x}), & r < |x| \leq R \\ \displaystyle\sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_n^m j_n(k|x|\sqrt{n_1}) Y_n^m(\hat{x},) & 0 \leq |x| \leq r. \end{cases}$$

Consequently, from the boundary conditions that must be satisfied by the solutions of the interior transmission problem, v and w are non trivial solutions to the interior transmission problem if and only if there exists $m \geq 0$ such that the determinant of the matrix A_m defined by

$$A_m = \begin{pmatrix} j_m(kR) & -h_m^{(1)}(kR\sqrt{n_2}) & -h_m^{(2)}(kR\sqrt{n_2}) & 0 \\ j_m'(kR) & -\sqrt{n_2}h_m^{(1)'}(kR\sqrt{n_2}) & -\sqrt{n_2}h_m^{(2)'}(kR\sqrt{n_2}) & 0 \\ 0 & h_m^{(1)}(kr\sqrt{n_2}) & h_m^{(2)}(kr\sqrt{n_2}) & -j_m(kr\sqrt{n_1}) \\ 0 & \sqrt{n_2}h_m^{(1)'}(kr\sqrt{n_2}) & \sqrt{n_2}h_m^{(2)'}(kr\sqrt{n_2}) & -\sqrt{n_1}j_m'(kr\sqrt{n_1}) \end{pmatrix}.$$

Transmission eigenvalues are values of k for which the there exists a positive m such that

$$\det A_m = 0.$$

The code to solve the determinant is done using Fortran 90. With an existing code on the computation of Bessel functions, we first build the matrices A_m for the first modes. The zeros of the determinants has been computed using a simple dichotomy method.

From the linear sampling method

It has been shown that transmission eigenvalues can be computed from far field data. More precisely, transmission eigenvalues are characterized by the behavior of the regularized solution to the far field equation

$$\mathcal{F}g_z := \int_\Omega u_\infty(\hat{x}, d)g(d)ds(d) = \Phi_\infty(\cdot, z).$$

The right hand side $\Phi_\infty(\cdot, z)$ denotes the far field pattern of a source point $\Phi_k(\cdot, z)$ defined by

$$\Phi_k(x, z) = \frac{1}{4\pi}\frac{e^{\mathrm{i}k|x-z|}}{|x-z|}, \quad x \neq z.$$

Explicitly,

$$\Phi_\infty(\hat{x}, z) = \frac{1}{4\pi}e^{-\mathrm{i}k\hat{x}\cdot z}.$$

We consider here the scattering problem for a non homogeneous medium :

$$\begin{cases} \Delta u + k^2 n(x)u = 0 & \text{in } \mathbb{R}^3 \\ u = u^i + u^s \\ \lim\limits_{r\to+\infty} r\left(\dfrac{\partial u^s}{\partial r} - iku^s\right) = 0 \end{cases}$$

where u^i is a plane wave which can be developed in terms of spherical harmonics

$$u^i = e^{\mathrm{i}kx\cdot d} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} 4\pi \mathrm{i}^n j_n(k|x|)\overline{Y_n^m(\hat{x})}Y_n^m(d). \tag{F.2}$$

Since u^s is a radiating solution to Helmholtz equation, it has the expansion

$$u^s(x) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n^m h_n^{(1)}(k|x|)Y_n^m(\hat{x})$$

when $x \in \mathbb{R}^3\backslash\overline{D}$ i.e. $|x| > R$. Its far field pattern is consequently

$$u_\infty(\hat{x}) = \frac{1}{k}\sum_{n=0}^{\infty}\frac{1}{\mathrm{i}^{n+1}}\sum_{m=-n}^{n} a_n^m Y_n^m(\hat{x}).$$

In order to solve the far field equation, we first need to compute the coefficients a_n^m for all $n \geq 0$ and $m = -n, ..., n$. To this end, we need to solve the direct scattering problem. Solutions can be expanded thanks to spherical harmonics :

- $|x| > R$:

$$u^s(x) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_n^m(d)h_n^{(1)}(k|x|)Y_n^m(\hat{x}).$$

To simplify the next calculus, we set $a_n^m(d) = 4\pi \tilde{a}_n^m(d)\overline{Y_n^m(d)}$. Consequently, using (F.2) :

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} 4\pi \left(\tilde{a}_n^m(d) h_n^{(1)}(k|x|) + i^n j_n(k|x|) \right) \overline{Y_n^m(d)} Y_n^m(\hat{x}).$$

- $r < |x| < R$: the total wave is of the form

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(b_n^m(d) h_n^{(1)}(k\sqrt{n_2}|x|) + c_n^m(d) h_n^{(2)}(k\sqrt{n_2}|x|) \right) \overline{Y_n^m(d)} Y_n^m(\hat{x}).$$

- $|x| < r$: in this case, the singularity of y_n at the origin needs to be taken into account

$$u(x) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} d_n^m(d) j_n(k\sqrt{n_1}|x|) \overline{Y_n^m(d)} Y_n^m(\hat{x}).$$

In order to get an expression of the far field pattern we just need to know the coefficient a_n^m. They are given by solving the system consequent to the boundary conditions on Γ and the continuity of the total field through σ. The problem to solve is of the form :

$$A_n X = b_n$$

where

$$A_n = \begin{pmatrix} h_n^{(1)}(kR) & -h_n^{(1)}(kR\sqrt{n_2}) & -h_n^{(2)}(kR\sqrt{n_2}) & 0 \\ h_n^{(1)'}(kR) & -\sqrt{n_2} h_n^{(1)'}(kR\sqrt{n_2}) & -\sqrt{n_2} h_n^{(2)'}(kR\sqrt{n_2}) & 0 \\ 0 & h_n^{(1)}(kr\sqrt{n_2}) & h_n^{(2)}(kr\sqrt{n_2}) & -j_n(kr\sqrt{n_1}) \\ 0 & \sqrt{n_2} h_n^{(1)'}(kr\sqrt{n_2}) & \sqrt{n_2} h_n^{(2)'}(kr\sqrt{n_2}) & -\sqrt{n_1} j_n'(kr\sqrt{n_1}) \end{pmatrix}$$

$$b_n = \begin{pmatrix} -i^n j_n(kR) \\ -i^n j_n'(kR) \\ 0 \\ 0 \end{pmatrix}$$

$$X = \begin{pmatrix} \tilde{a}_n^m(d) \\ b_n^m(d) \\ c_n^m(d) \\ d_n^m(d) \end{pmatrix}.$$

However we note that A_n and b_n do not depend on m and d so the coefficients are just depending on n and we denote by $\tilde{a}_n := \tilde{a}_n^m(d)$.

Now, the coefficients $\tilde{a}_n$ are computed by inverting the matrix A_n.

If g_z is of the form

$$g_z(d) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} g_n^m(z) Y_n^m(d)$$

where

$$g_n^m(z) = \int_\Omega g_z(d)\overline{Y_n^m(d)}ds(d)$$

and using the orthogonality of the spherical harmonics,

$$\int_\Omega \overline{Y_n^m(d)}Y_i^j(d)ds(d) = \delta_{ni}\delta_{mj},$$

we can have an explicit expression of $\mathcal{F}g_z$

$$\mathcal{F}g_z(\hat{x}) = \int_\Omega \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{m=-n}^{n}\sum_{j=-i}^{i} \frac{4\pi}{k\mathrm{i}^{n+1}}\tilde{a}_n g_i^j(z)\overline{Y_n^m(d)}Y_i^j(d)Y_n^m(\hat{x})ds(d) \tag{F.3}$$

$$= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{4\pi}{k\mathrm{i}^{n+1}}\tilde{a}_n g_n^m(z)Y_n^m(\hat{x}). \tag{F.4}$$

Finally, using the expansion of the far field pattern of a source point $\Phi_k(\cdot, z)$ with spherical harmonics,

$$\Phi_\infty(\hat{x}, z) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} (-\mathrm{i})^n j_n(k|z|)\overline{Y_n^m(z)}Y_n^m(\hat{x}) \tag{F.5}$$

and identifying the coefficients in (F.4) and (F.5), we deduce the coefficients of g_z

$$g_n^m(z) = \frac{\mathrm{i}k}{4\pi}\frac{j_n(k|z|)}{\tilde{a}_n}\overline{Y_n^m(z)}.$$

Furthermore, using the identity of Theorem F.1.2

$$\sum_{m=-n}^{n} Y_n^m(z)\overline{Y_n^m(z)} = \frac{2n+1}{4\pi},$$

we get an explicit expression of the norm of g_z

$$||g_z||^2 = \frac{k^2}{(4\pi)^3}\sum_{n=0}^{\infty}\frac{j_n(k|z|)^2}{|\tilde{a}_n|^2}(2n+1). \tag{F.6}$$

Remark F.4.1. *From* (F.4)*, we can see that* $\frac{4\pi}{k\mathrm{i}^{n+1}}\tilde{a}_n$ *are the eigenvalues of the compact operator* $\mathcal{F}$ *associated with the eigenfunctions* Y_n^m*. We can observe that these eigenvalues tends to zero exponentially.*

Consequently, the norm of g_z *blows up and we need to use a regularization scheme to compute the norm of* g_z*. We choose to use Tikhonov regularization with parameter* η *which is determined using Morozov discrepancy principle.*

Tikhonov regularization consists in solving the regularized equation

$$(\eta + \mathcal{F}^*\mathcal{F})g_z = \mathcal{F}^*\Phi_\infty(\cdot, z). \tag{F.7}$$

Let us denote

$$\mathcal{F}g_z = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_n^m g_n^m Y_n^m(\hat{x})$$

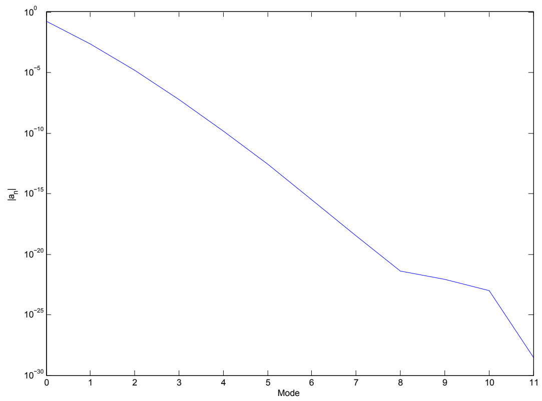


Figure F.2: Norm of the coefficients a_n for the 11-th first modes

where for all $n \geq 0$ and $m = -n..n$,

$$f_n^m = \frac{4\pi \tilde{a}_n}{k \mathrm{i}^{n+1}}.$$

Consequently,

$$\mathcal{F}^* \mathcal{F} g_z = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} |f_n^m|^2 g_n^m Y_n^m(\hat{x})$$

and

$$\mathcal{F}^* \Phi_\infty(\hat{x}, z) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \overline{f_n^m} \Phi_{\infty,n}^m Y_n^m(\hat{x})$$

with

$$\Phi_{\infty,n}^m = (-\mathrm{i})^n j_n(k|z|\overline{Y_n^m(z)}.$$

Solving (F.7), we finally we get that

$$g_n^m = \frac{\overline{f_n^m}}{\eta + |f_n^m|^2} \Phi_{\infty,n}^m.$$

Therefore,

$$\begin{aligned} ||g_z||^2 &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \overline{g_n^m} g_n^m \\ &= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{4\pi}{k^2} \frac{|\tilde{a}_n|^2}{\left(\eta + \left(\frac{4\pi}{k}\right)^2 |\tilde{a}_n|^2\right)^2} j_n(k|z|)^2 (2n+1). \end{aligned}$$

Remark F.4.2. *When the parameter η is equal to zero, we get the same expression as* (F.6) *computed previously without regularization.*

The code we developed takes as entries the radius of the sphere R and the included sphere r, the index of refraction n_1 and n_2. We also choose the interval of k in which we search the transmission eigenvalues. Since the Theorem 2.5.1 is only valid for almost every k, we compute the norm of g_z for a sample of points z in the scatterer D to make sure to get all the peaks corresponding to the eigenvalues. Then, we take an average of the values of $||g_z||$ computed for each k.

Numerical results

We compute the eigenvalues for a sphere of radius $R = 1$. First, we consider an homogeneous sphere with index of refraction $n = 4$, then we consider a double layer sphere with $n_1 = 2$ in the sphere of radius r centered at the origin and $n_2 = 4$ in the annulus. Finally, we compute the eigenvalues for a sphere of index of refraction $n = 4$ containing a cavity of radius r.

The next figures represent the norm of the regularized solution $g_{z,k}$ against k. The rounds correspond to the zeros of the determinants of A_m.

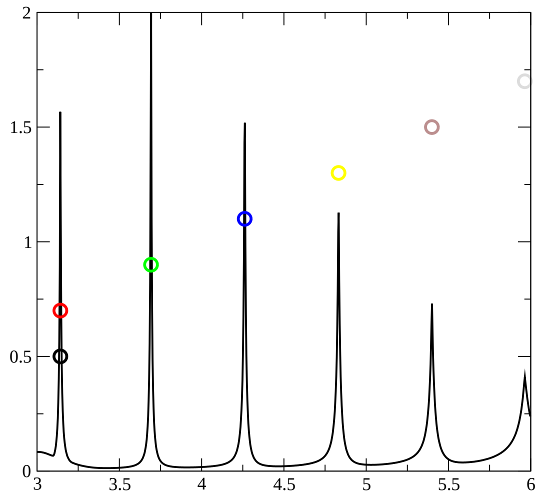


Figure F.3: Homogeneous sphere of radius 1 and index of refraction $n = 4$. The first transmission eigenvalue is equal to 3.14.

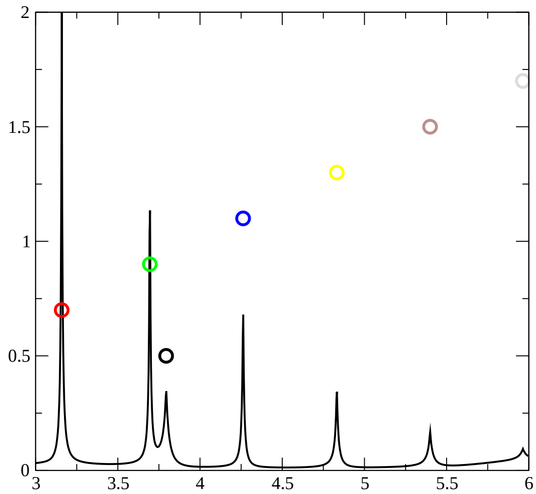


Figure F.4: The index of refraction is equal to $n_1 = 2$ in a sphere of radius $r = 0.2$. The value of the first transmission eigenvalue is $k_0 = 3.15$

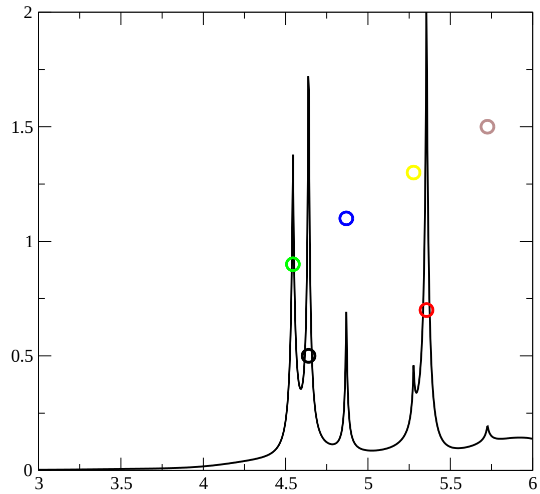


Figure F.5: The index of refraction is equal to $n_1 = 2$ in a sphere of radius $r = 0.6$. The value of the first transmission eigenvalue is $k_0 = 4.54$

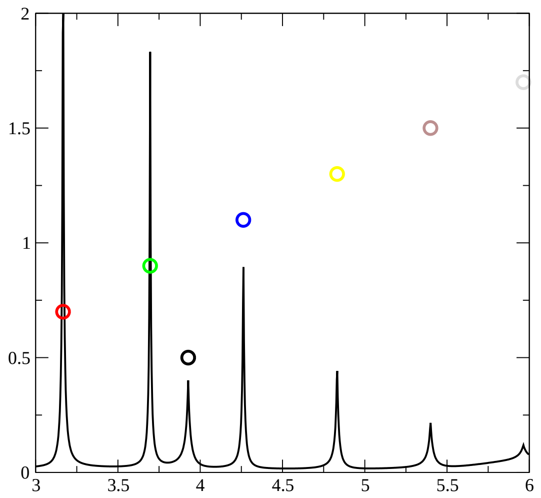


Figure F.6: The sphere of radius $R = 1$ contains a cavity of radius $r = 0.2$. The first transmission eigenvalue is equal to 3.16

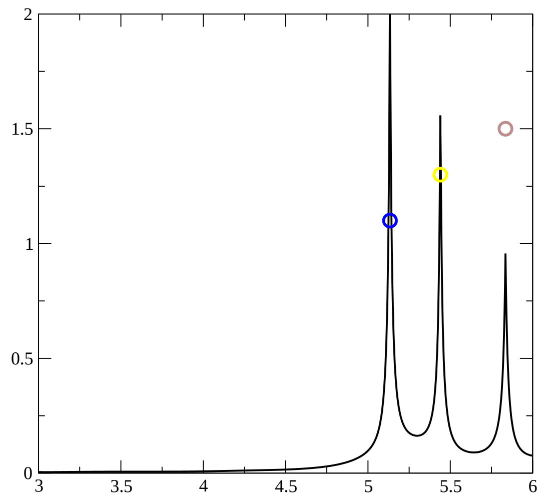


Figure F.7: The sphere of radius $R = 1$ contains a cavity of radius $r = 0.6$. In this case, $k_0 = 5.13$

The peaks of the norm of the regularized solution to the far field equation perfectly match with the exact values of the transmission eigenvalues. We also remark that in the case of a cavity, we verified the monotonous property of the first transmission eigenvalue with respect to the size of the cavity. Here the index of refraction is greater than one and the first transmission eigenvalue is shifted to the right as the cavity grows.

Remark F.4.3. *The computation of transmission eigenvalues for circular geometry can be easily adapt in $\mathbb{R}^2$ by only replacing the Bessel functions j_n and y_n of $\mathbb{R}^3$ by the Bessel functions J_n and Y_n of $\mathbb{R}^2$.*

F.4.2 Electromagnetic case

Now let us consider the computation of the transmission eigenvalues for electromagnetic waves. The method is exactly the same as for the acoustic case. The difference is of course the expression of the solutions in terms of spherical harmonics. We will see that transmission eigenvalues for the acoustic case are also transmission eigenvalues for the electromagnetic case due to the decomposition of solutions in the case of a spherical geometry.

We consider here directly the case of a double layer layer sphere or radius R. Assume that

$$n(x) = \begin{cases} n_2 & \text{if } r < |x| < R \\ n_1 & \text{if } |x| < r \end{cases}$$

with $n_1 > 0$ and $n_2 > 0$. We denote in the following $k_1 = k\sqrt{n_1}$ and $k_2 = k\sqrt{n_2}$. We also recall the definitions of Ricatti-Bessel functions

$$\psi_n(t) := tj_n(t), \quad \zeta^{(1)}(t) := th_n^{(1)}(t), \quad \zeta^{(2)}(t) := th_n^{(2)}(t).$$

From the ITP

Similarly to the scalar case, the first method consists in solving the interior transmission problem. This will lead to find the zeros of some determinant. We consider the following interior transmission problem :

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 n\mathbf{E} = 0 & \text{in } D \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 & \text{in } D \\ \nu \times \mathbf{E} = \nu \times \mathbf{E}_0 & \text{on } \Gamma \\ \nu \times \operatorname{curl}\mathbf{E} = \nu \times \operatorname{curl}\mathbf{E}_0 & \text{on } \Gamma \end{cases} \tag{F.8}$$

Solutions of the previous problem have the following form

$$\begin{cases} \mathbf{E} = \operatorname{curl}(xu) + \frac{\mathrm{i}}{k}\operatorname{curl}\operatorname{curl}(xv) \\ \mathbf{E}_0 = \operatorname{curl}(xu_0) + \frac{\mathrm{i}}{k}\operatorname{curl}\operatorname{curl}(xv_0) \end{cases} \tag{F.9}$$

where

$$\begin{cases} \Delta u + k^2 n u = 0 & \text{in } D \\ \Delta u_0 + k^2 u_0 = 0 & \text{in } D \\ u = u_0 & \text{on } \partial D \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial u_0}{\partial \nu} & \text{on } \partial D \end{cases} \tag{F.10}$$

$$\begin{cases} \Delta v + k^2 n v = 0 & \text{in } D \\ \Delta v_0 + k^2 v_0 = 0 & \text{in } D \\ nv = v_0 & \text{on } \Gamma \\ \dfrac{\partial (rv)}{\partial r} = \dfrac{\partial (rv_0)}{\partial r} & \text{on } \Gamma \end{cases} \tag{F.11}$$

u, u_0, v and v_0 have the following expansions :

$$u_0(x) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} u_{0,n}^m j_n(k|x|) Y_n^m(\hat{x}), \tag{F.12}$$

$$v_0(x) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} v_{0,n}^m j_n(k|x|) Y_n^m(\hat{x}), \tag{F.13}$$

$$u(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty}\sum_{m=-n}^{n} \left(u_n^{m,1} h_n^{(1)}(k_2|x|) + u_n^{m,2} h_n^{(2)}(k_2|x|)\right) Y_n^m(\hat{x}) & \text{if } r < |x| < R \\ \displaystyle\sum_{n=1}^{\infty}\sum_{m=-n}^{n} u_n^m j_n(k_1|x|) Y_n^m(\hat{x}) & \text{if } |x| < r, \end{cases} \tag{F.14}$$

$$v(x) = \begin{cases} \displaystyle\sum_{n=1}^{\infty}\sum_{m=-n}^{n} \left(v_n^{m,1} h_n^{(1)}(k_2|x|) + v_n^{m,2} h_n^{(2)}(k_2|x|)\right) Y_n^m(\hat{x}), & \text{if } r < |x| < R \\ \displaystyle\sum_{n=1}^{\infty}\sum_{m=-n}^{n} v_n^m j_n(k_1|x|) Y_n^m(\hat{x}) & \text{if } |x| < r. \end{cases} \tag{F.15}$$

By the boundary conditions and the continuity of solutions of Helmholtz equation, the problem is equivalent to finding k such that $\det A_m = 0$ or $\det B_m = 0$ for some $m \geq 0$ where A_m is the same matrix as for the scalar case and

$$B_m = \begin{pmatrix} -j_m(kR) & n_2 h_m^{(1)}(k_2R) & n_2 h_m^{(2)}(k_2R) & 0 \\ -\psi'_m(kR) & \zeta_m^{(1)'}(k_2R) & \zeta_m^{(2)'}(k_2R) & 0 \\ 0 & n_2 h_m^{(1)}(k_2r) & n_2 h_m^{(2)}(k_2r) & -n_1 j_m(k_1r) \\ 0 & \zeta_m^{(1)'}(k_2r) & \zeta_m^{(2)'}(k_2r) & -\psi'_m(k_1r) \end{pmatrix}.$$

Remark F.4.4. *For spherical geometry, the transmission eigenvalues of the scalar case are also transmission eigenvalues for the electromagnetic case.*

From the LSM

We consider the scattering problem by a plane wave in $\mathbb{R}^3$:

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 n\mathbf{E} = 0 \quad \text{in } \mathbb{R}^3 \\ \mathbf{E} = \mathbf{E}^i + \mathbf{E}^s \end{cases}$$

where $\mathbf{E}^s$ satisfies the Silver-Müller radiation condition and the incident field is defined by

$$\begin{aligned} \mathbf{E}^i(x; d, p) &:= \mathrm{i}k(d \times p) \times d e^{\mathrm{i}kx\cdot d} \\ &= \operatorname{curl}(xv^i(x; d, p)) + \frac{\mathrm{i}}{k}\operatorname{curl}\operatorname{curl}(xu^i(x; d, p)). \end{aligned}$$

where

$$\begin{aligned} u^i(x; d, p) &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_n^m(d, p) j_n(k|x|) Y_n^m(\hat{x}) \\ v^i(x; d, p) &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n} b_n^m(d, p) j_n(k|x|) Y_n^m(\hat{x}) \end{aligned}$$

with

$$\begin{aligned} a_n^m(d, p) &= -\mathrm{i}^n \frac{4\pi \mathrm{i}k}{n(n+1)} \operatorname{Grad}\overline{Y_n^m(d)} \cdot p, \\ b_n^m(d, p) &= -\mathrm{i}^n \frac{4\pi \mathrm{i}k}{n(n+1)} (d \times \operatorname{Grad}\overline{Y_n^m(d)}) \cdot p. \end{aligned}$$

The linear sampling method consists in solving the far field equation

$$\mathcal{F}g_z(\hat{x}) := \int_\Omega \mathbf{E}_\infty(\hat{x}, d, g_z(d)) ds(d) = \mathbf{E}_{e,\infty}(\hat{x}, z, q).$$

We now want to write the total field in a similar way i.e. :

$$E(x; d, p) = \operatorname{curl}(xv(x; d, p)) + \frac{\mathrm{i}}{k}\operatorname{curl}\operatorname{curl}(xu(x; d, p))$$

where the total fields $u = u^i + u^s$ and $v = v^i + v^s$ with incident waves as defined previously, satisfy

$$\Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^3,$$
$$\Delta v + k^2 n v = 0 \text{ in } \mathbb{R}^3.$$

- If $|x| > R$, from Theorem F.3.2, the radiating solutions have the form

$$u^s(x; d, p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} u_n a_n^m(d, p) h_n^{(1)}(k|x|) Y_n^m(\hat{x})$$
$$v^s(x; d, p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} v_n b_n^m(d, p) h_n^{(1)}(k|x|) Y_n^m(\hat{x}).$$

- If $r < |x| < R$, the total fields have the general expansion

$$u(x; d, p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} a_n^m(d, p) \left(u_n^1 h_n^{(1)}(k\sqrt{n_2}|x|) + u_n^2 h_n^{(2)}(k\sqrt{n_2}|x|) \right) Y_n^m(\hat{x})$$
$$v(x; d, p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} b_n^m(d, p) \left(v_n^1 h_n^{(1)}(k\sqrt{n_2}|x|) + v_n^2 h_n^{(2)}(k\sqrt{n_2}|x|) \right) Y_n^m(\hat{x}).$$

- Finally, if $|x| < r$, from the singularity of y_n at the origin, the expansions of the total fields are

$$u(x; d, p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \tilde{u}_n a_n^m(d, p) j_n(k\sqrt{n_1}|x|) Y_n^m(\hat{x})$$
$$v(x; d, p) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \tilde{v}_n b_n^m(d, p) j_n(k\sqrt{n_1}|x|) Y_n^m(\hat{x}).$$

In order to get an explicit expansion of the far field pattern $\mathbf{E}_\infty$ generated by the plane wave, we need to compute the coefficients u_n and v_n of the scattered fields. To this end, we use the boundary conditions on Γ and Σ. The continuity of $\nu \times \mathbf{E}$ and $\nu \times \operatorname{curl} \mathbf{E}$ through the obstacle is equivalent to the continuity of v, $\dfrac{\partial v}{\partial \nu}$, nu and $\dfrac{\partial (ru)}{\partial r}$. The continuity of the function v which corresponds to a transverse magnetic polarization leads to the same system as the scalar case

$$AX = b$$

where
$$A = \begin{pmatrix} -h_n^{(1)}(kR) & h_n^{(1)}(k_2 R) & h_n^{(2)}(k_2 R) & 0 \\ -h_n^{(1)'}(kR) & \sqrt{n_2} h_n^{(1)'}(k_2 R) & \sqrt{n_2} h_n^{(2)'}(k_2 R) & 0 \\ 0 & h_n^{(1)}(k_2 r) & h_n^{(2)}(k_2 r) & -j_n(k_1 r) \\ 0 & \sqrt{n_2} h_n^{(1)'}(k_2 r) & \sqrt{n_2} h_n^{(2)'}(k_2 r) & -\sqrt{n_1} j_n'(k_1 r) \end{pmatrix}$$

$$X = \begin{pmatrix} v_n \\ v_n^1 \\ v_n^2 \\ \tilde{v}_n \end{pmatrix}, \qquad b = \begin{pmatrix} j_n(kR) \\ j_n'(kR) \\ 0 \\ 0 \end{pmatrix}.$$

This confirms the fact that the transmission eigenvalues for the scalar case are also transmission eigenvalues for the electromagnetic case. Furthermore, we also need to solve

$$BY = c$$

where

$$B = \begin{pmatrix} -h_n^{(1)}(kR) & n_2 h_n^{(1)}(k_2R) & n_2 h_n^{(2)}(k_2R) & 0 \\ -\zeta_n^{(1)'}(kR) & \zeta_n^{(1)'}(k_2R) & \zeta_n^{(2)'}(k_2R) & 0 \\ 0 & n_2 h_n^{(1)}(k_2r) & n_2 h_n^{(2)}(k_2r) & -n_1 j_n(k_1r) \\ 0 & \zeta_n^{(1)'}(k_2r) & \zeta_n^{(2)'}(k_2r) & -\psi_n'(k_1r) \end{pmatrix},$$

$$Y = \begin{pmatrix} u_n \\ u_n^1 \\ u_n^2 \\ \tilde{u}_n \end{pmatrix}, \qquad c = \begin{pmatrix} j_n(kR) \\ \psi_n'(kR) \\ 0 \\ 0 \end{pmatrix}$$

By inverting the matrices A and B, we deduce the coefficients u_n and v_n. From Theorem F.3.2 and the expressions of a_n^m and b_n^m, the far field pattern becomes

$$\begin{aligned} \mathbf{E}_\infty(\hat{x}, d, p) &= -\frac{1}{k}\sum_{n=1}^{\infty}\frac{1}{\mathrm{i}^{n+1}}\sum_{m=-n}^{n}(u_n a_n^m \mathrm{Grad}\, Y_n^m + v_n b_n^m \hat{x}\times \mathrm{Grad}\, Y_n^m) \\ &= \sum_{n=1}^{\infty}\sum_{m=-n}^{n} 4\pi u_n(\overline{U_n^m(d)}\cdot p)U_n^m(\hat{x}) + 4\pi v_n(\overline{V_n^m(d)}\cdot p)V_n^m(\hat{x}) \end{aligned}$$

We now have all we need to solve the far field equation. Since the solution g_z is in $L_t^2(\Omega)$, we can write

$$g(d) = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} g_{u,n}^m U_n^m(d) + g_{v,n}^m V_n^m(d)$$

where

$$g_{u,n}^m = \int_\Omega g(d)\cdot\overline{U_n^m(d)}ds(d), \quad g_{v,n}^m = \int_\Omega g(d)\cdot\overline{V_n^m(d)}ds(d), \quad d\in\Omega.$$

Since the U_n^m and V_n^m are orthonormal, we get that

$$\begin{aligned} \mathcal{F}g_z(\hat{x}) &= \int_\Omega \mathbf{E}_\infty(\hat{x}, d, g(d))ds(d) \\ &= 4\pi\sum_{n=1}^{\infty}\sum_{m=-n}^{n} u_n g_{u,n}^m U_n^m(\hat{x}) + v_n g_{v,n}^m V_n^m(\hat{x}). \end{aligned}$$

From the expression (F.1) of the right-hand side of the far field equation that is the far field pattern of an electric dipole, we deduce the coefficients of g_z

$$g_{u,n}^m = \frac{\alpha_n^m(z,q)}{4\pi u_n} \quad ; \quad g_{v,n}^m = \frac{\beta_n^m(z,q)}{4\pi v_n}$$

where

$$\alpha_n^m(z,q) = \frac{(-i)^n}{\sqrt{n(n+1)}} ik\overline{M_n^m(v)}\cdot q$$

and

$$\beta_n^m(z,q) = -\frac{(-i)^n}{\sqrt{n(n+1)}}\mathrm{curl}\,\overline{M_n^m(v)} \cdot q.$$

The norm of the solution g_z is equal to

$$||g_z||_{L_t^2(\Omega)} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{|\alpha_n^m|^2}{(4\pi)^2|u_n|^2} + \frac{|\beta_n^m|^2}{(4\pi)^2|v_n|^2}.$$

Remark F.4.5. *Similarly to the scalar case, the eigenvalues $4\pi u_n$ and $4\pi v_n$ of $\mathcal{F}$ corresponding to the eigenfunctions U_n^m and V_n^m tends quickly to zero as n tends to infinity. We use again Tikhonov regularization to compute the norm of g_z*

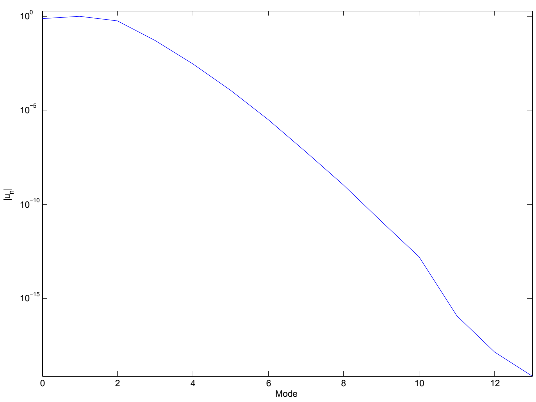


Figure F.8: Norm of the coefficients u_n for the 13-th first modes

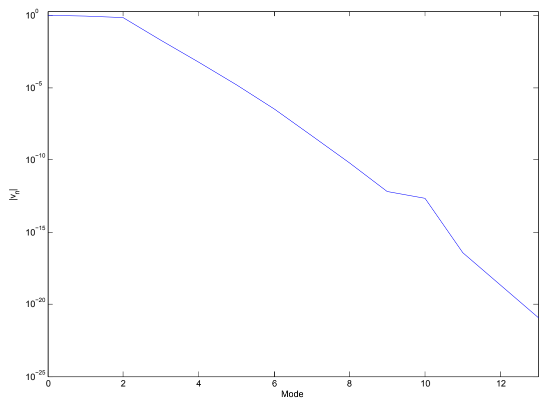


Figure F.9: Norm of the coefficients v_n for the 13-th first modes

The problem to solve is now

$$(\eta + \mathcal{F}^*\mathcal{F})g_z = \mathcal{F}^*\Phi_\infty(\cdot, z). \tag{F.16}$$

Using the properties of the vector spherical harmonics, the coefficients of the regularized solution are

$$g_{u,n}^m = \frac{4\pi\overline{u}_n\alpha_n^m}{\eta + (4\pi)^2|u_n|^2}, \quad g_{v,n}^m = \frac{4\pi\overline{v}_n\beta_n^m}{\eta + (4\pi)^2|v_n|^2}$$

and consequently

$$||g_z||^2 = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} \frac{(4\pi)^2|u_n|^2}{(\eta + (4\pi)^2|u_n|^2)^2}|\alpha_n^m|^2 + \frac{(4\pi)^2|v_n|^2}{(\eta + (4\pi)^2|v_n|^2)^2}|\beta_n^m|^2.$$

Some numerical results

In the following figures, the solid line represents the norm of g_z against k. The rounds indicate the zeros of the two determinants A_m and B_m for different modes. They represent the exact transmission eigenvalues. Again the monotonicity of the first transmission eigenvalue with the size of the cavity is verified. We test the same examples as in the scalar case.

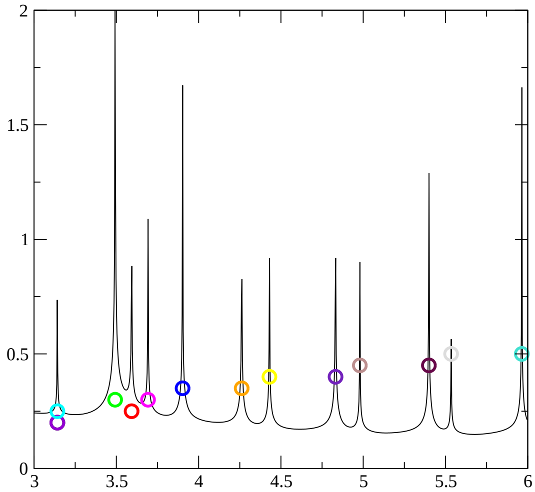


Figure F.10: Transmission eigenvalues for a homogeneous sphere of radius $R = 1$ and index of refraction $n = 4$.

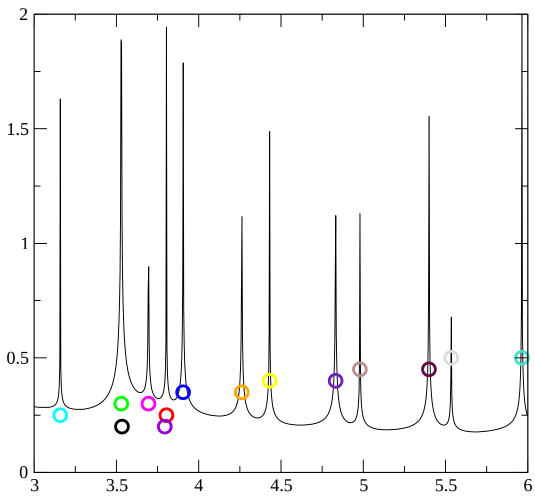


Figure F.11: Double layer sphere with $r = 0.2$ and index of refraction $n_1 = 2$ and $n_2 = 4$. The first transmission eigenvalue is equal to $k_0 = 3.15$

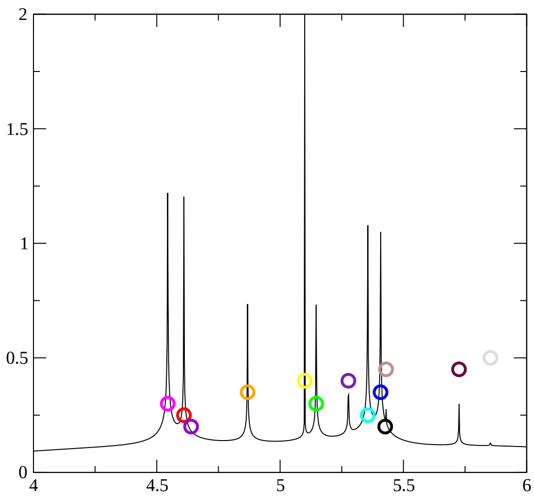


Figure F.12: Double layer sphere with $r = 0.6$ and index of refraction $n_1 = 2$ and $n_2 = 4$. The first transmission eigenvalue is equal to $k_0 = 4.54$

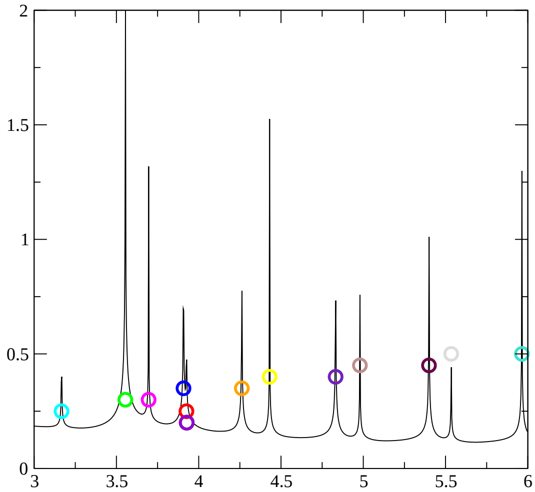


Figure F.13: Homogeneous sphere with index of refraction $n = 4$ containing a cavity of radius $r = 0.2$. $k_0 = 3.16$

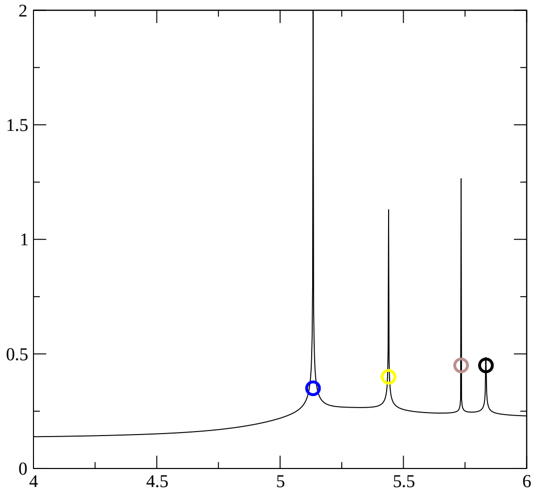


Figure F.14: Homogeneous sphere with index of refraction $n = 4$ containing a cavity of radius $r = 0.6$. $k_0 = 5.13$

Résumé en français

La théorie des problèmes de diffraction inverses pour les ondes acoustiques et électromagnétiques est un domaine de recherche très actif avec des avancées significatives ces dernières années. Les problèmes inverses consistent à retrouver des informations sur un objet à partir de données mesurées. Plus précisement, le problème de diffraction inverse revient à trouver les caractéristiques d'un objet diffractant (localisation, forme, propriété du matériau, ...) à partir de mesures des ondes acoustiques ou électromagnétiques diffractées par cet objet. La question n'est donc pas seulement de détecter des objets comme le radar ou le sonar peuvent le faire, mais aussi de les identifier.

Les problèmes inverses ne sont pas faciles à résoudre puisqu'ils appartiennent à la classe des problèmes dit mal posés selon la définition de Hadamard. En effet, une solution peut ne pas exister, et même si c'est effectivement le cas, la solution ne dépend pas continûment des données. De tels problèmes nécessitent alors le recours à des schémas de régularisation afin d'être résolus numériquement. Les premiers algorithmes permettant de résoudre les problèmes inverses d'identification sont basés soit sur une approximation de diffraction faible (weak-scattering approximation) soit sur des techniques d'optimisation non linéaire. Le principal problème de la "weak-scattering approximation" est qu'elle ne prend pas en compte les effets de polarisation et ne peut ainsi être utilisée dans des environnements complexes. Les deux méthodes citées ci-dessus reposent également sur une connaissance a priori des propriétés physiques de l'objet (par exemple si l'objet est penetrable ou non), informations qui ne sont en général pas disponibles. De plus, les techniques d'optimisation non linéaire sont très couteuses numériquement. Un aperçu de ces méthodes peut être trouvé dans [22].

Ces inconvénients ont naturallement amené à chercher de nouveaux algorithmes sur les problèmes d'identification à la fois facile à implémenter et nécessitant le moins d'information possible sur l'objet diffractant. De nouvelles méthodes ont alors été développées et ont été regroupées dans une classe appelée méthodes qualitatives dans la théorie du problème de diffraction inverse [8] pour résoudre des problèmes harmoniques en temps pour des ondes acoustiques et électromagnétiques. Les méthodes les plus représentatives de cette classe sont la Linear Sampling Method (LSM) [24, 29], le méthode de factorisation [40, 36] et la méthode des sources singulières [47, 48]. Ces méthodes permettent la reconstruction de la forme d'un objet à partir de la connaissance de données multi-statiques à fréquence fixée. Elles sont basées sur la résolution d'une équation intégrale linéaire mal posée, connue sous le nom d'équation du champ lointain. Un autre avantage de ces méthodes, par exemple en comparaison des méthodes itératives, est qu'elles évitent de résoudre le problème de diffraction direct et qu'elles n'utilisent aucune information a priori sur la géométrie ou les propriétés physiques de l'objet diffractant.

L'étude théorique de la LSM pour les obstacles impénétrables dont la frontière admet une condition de conducteur parfait amène à étudier le problème de valeurs propres pour le Laplacien en acoustique et pour l'opérateur rot rot en électromagnétisme à l'intérieur de l'objet avec une condition de Dirichlet sur le bord. Il est montré que cette méthode échoue lorsque le carré du nombre d'onde est une valeur propre de Dirichlet i.e. k^2 est une valeur propre de Dirichlet ou une valeur propre de Maxwell. Les propriétés d'existence et le caractère discret de ces valeurs propres étant bien connues, il est alors facile d'éviter ces valeurs afin d'utiliser la LSM. Dans le cas d'objets pénétrables, Colton et Kirsch [23, 37] ont montré que la LSM amène à étudier un nouveau type de problème appelé *problème de transmission intérieur*. Les valeurs propres de ce problème intérieur sont appelées *valeurs propres de transmission* et la théorie sur la LSM suggère de les exclure tout comme les valeurs propres de Dirichlet ou de Maxwell dans le cas d'obstacles impénétrables. L'étude du problème de transmission intérieur est alors naturellement devenu un sujet de grand intérêt, tout d'abord pour clarifier le rôle des valeurs propres de transmission dans la LSM mais il s'est également avéré ensuite intéressant pour le problème d'identification consistant à obtenir des informations sur les propriétés physiques de l'objet diffractant [14, 11]. Ici, la principale particularité des valeurs propres de transmission est que non seulement elles permettent d'obtenir des informations qualitatives sur les propriétés physiques de l'objet [7, 15, 11] mais elles peuvent aussi être calculées à partir du champ lointain [13].

Ainsi, trois questions essentielles sur les valeurs propres de transmission peuvent être posées. Les deux premières sont reliées à leur impact sur la LSM. Il est important de savoir si ces valeurs propres de transmission dans un premier temps existent, mais aussi si elles se comportent comme les valeurs propres de Dirichlet ou de Maxwell c'est à dire si elles forment un ensemble discret. Enfin, on peut se demander si on ne peut pas tirer profit de ces valeurs pour obtenir des estimations sur les caractéristiques du matériau telles que son indice de réfraction ou si l'objet contient des défauts comme par exemple des trous, ce qui pourrait être utile en contrôle non destructif.

Bien que la formulation du problème de transmission intérieur parait simple à première vue, il ne fait cependant pas partie des équations aux dérivées partielles elliptiques et n'est pas non plus auto-adjoint. Deux principales méthodes pour étudier ce problème ont alors vu le jour : des méthodes d'équations intégrales [25, 37] et des méthodes variationnelles [10, 14, 49]. Alors que le caractère discret des valeurs propres de transmission a été assez rapidement montré grâce au théorème de Fredholm analytique, le preuve de l'existence des valeurs propres de transmission a été plus laborieuse. L'existence a été montrée en premier lieu pour des milieux sphériques stratifiés dans [26], et beaucoup plus tard par Päivärinta et Sylvester [43] dans le cas scalaire général de milieux isotropes sous la condition que l'indice de réfraction est grand devant un. Plusieurs résultats sur l'existence d'un ensemble infini discret de valeurs propres de transmission ont ensuite été établis dans des cas plus généraux pour à la fois les ondes acoustiques et électromagnétiques et avec des hypothèses moins restrictives sur l'indice de réfraction n [17]. Ce résultat a également été montré dans le cas d'un milieu contenant une cavité [12] i.e. des régions dont l'indice est le même que le milieu extérieur. Dans tous les cas, une hypothèse sur l'indice de réfraction persiste : $n-1$ ne doit pas changer de signe. Cependant, Sylvester [50] et Bonnet-Ben Dhia Chesnel and Haddar [5] ont récemment montré que les valeurs propres de transmission forment un ensemble discret seulement si cette hypothèse sur n est vérifiée sur un voisinage de la frontière de l'objet étudié.

Le but de cette thèse est de contribuer à l'étude du problème de transmission intérieur et de répondre à des questions toujours ouvertes.

Chapitre 1 :

Le premier chapitre est consacré à introduire les notions de problème de transmission intérieur et de valeurs propres de transmission en expliquant comment ils apparaissent dans la théorie des problèmes de diffraction inverses pour les ondes électromagnétiques. Lors de l'étude de la LSM, il apparait que pour certaines fréquences, on peut trouver des ondes incidentes qui ne rayonnent pas. Ce sont ces fréquences qui sont appelées valeurs propres de transmission.

Dans ce premier chapitre, nous rappelons tout d'abord le contexte du problème direct de diffraction pour les objets pénétrables et impénétrables. Nous donnons ensuite les propriétés principales de solutions entières particulières des équations de Maxwell appelées paires de Herglotz.

Nous donnons ensuite les grandes lignes de la méthode permettant de retrouver la forme d'un objet à partir des mesures du champ lointain appelée Linear Sampling Method décrite pour la première fois par Colton et Kirsch en 1996 [24]. Nous allons montrer que cette méthode échoue pour des fréquences particulières : les valeurs propres de Maxwell dans le cas d'obstacles impénétrables et les valeurs propres de transmission dans le cas d'objets pénétrables.

Les valeurs propres de transmission sont définies à partir d'un problème de transmission singulier où deux champs possédant les mêmes données de Cauchy au bord satisfont deux équations de Maxwell avec des nombres d'ondes différents à l'intérieur d'un même objet. Il s'écrit sous la forme

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E} - k^2 N\mathbf{E} = 0 & \text{in } D \\ \operatorname{curl}\operatorname{curl}\mathbf{E}_0 - k^2\mathbf{E}_0 = 0 & \text{in } D \\ \nu \times \mathbf{E} - \nu \times \mathbf{E}_0 = 0 & \text{on } \partial D \\ \nu \times \operatorname{curl}\mathbf{E} - \nu \times \operatorname{curl}\mathbf{E}_0 = 0 & \text{on } \partial D. \end{cases}$$

Si le problème de transmission intérieur précédent possède au moins une solution non triviale alors k est une valeur propre de transmission. Il est particulièrement intéressant d'étudier l'existence de telles valeurs propres mais aussi d'étudier la répartition du spectre afin d'être sûr d'éviter facilement ces fréquences dans la LSM. Il apparait également qu'elles procurent des informations qualitatives sur l'indice de réfraction du milieu.

Dans la dernière section de ce chapitre, nous mettons en évidence que ce problème de transmission n'est pas classique et donc difficile à résoudre dans le sens que les formulations variationnelles habituelles ne sont pas appropriées ici. En effet, ce problème n'est ni elliptique ni auto-adjoint. Pour finir, un état de l'art concernant le problème de transmission intérieur est présenté.

Chapitre 2 :

Le chapitre 2 étudie le problème de transmission intérieur pour un milieu non homogène. C'est le premier cas à avoir été étudié et le plus simple. Il sert également de base pour la résolution du problème de transmission intérieur pour d'autres configurations. Les premiers résultats d'existence d'un ensemble infini discret de valeurs propres de transmission a été prouvé dans [26] mais seulement pour des milieux sphériques stratifiés. Il a fallu attendre 2008 avec [43] pour que Päivärinta et Sylvester montrent le même résultat pour des géométries plus générales.

Ce chapitre s'ouvre tout d'abord sur les principaux théorèmes utilisés dans la théorie des problèmes de transmission intérieur. Nous rappelons dans un premier temps les théorèmes principaux de la théorie de Fredholm : l'alternative de Fredholm qui permet d'étudier le caractère bien posé du problème de transmission intérieur et le théorème de Fredholm analytique, principal outil pour montrer le caractère discret des valeurs propres de transmission. Enfin, le dernier résultat est basé sur la théorie des problèmes aux valeurs propres généralisés et donne, grâce à un théorème des valeurs intermédiaires, une méthode puissante pour montrer l'existence des valeurs propres de transmission mais aussi pour trouver des estimations sur les valeurs propres.

Nous reprenons ensuite les résultats prouvés dans [34] et [18]. Nous considérons tout d'abord le caractère bien posé du problème de transmission intérieur. En particulier, en utilisant une formulation du quatrième ordre et une approche variationnelle, il est possible de montrer l'existence de solutions dans L^2 à condition que le nombre d'onde k ne soit pas une valeur propre de transmission.

Les dernières sections du chapitre 2 sont consacrées à l'etude des propriétés des valeurs propres de transmission. L'existence d'un ensemble infini discret de valeurs propres de transmission est prouvé en utilisant le théorème de Fredholm analytique et un problème aux valeurs propres auxiliaire. Un nouveau résultat sur la continuité de la première valeur propre de transmission par rapport à l'indice de réfraction est ensuite montré dans le cas d'un milieu isotrope. Pour finir, nous établissons le parallèle entre les valeurs propres de Maxwell si l'objet est impénétrable et les valeurs propres de transmission si l'objet est pénétrable en montrant que la norme de la solution régularisée de l'équation de LSM explose quand la fréquence est une valeur propre de transmission. Cette propriété généralise le résultat prouvé dans [13] pour l'acoustique.

Chapitre 3 :

Ce chapitre concerne l'étude du problème de transmission intérieur pour des inclusions anisotropes pouvant contenir des cavités c'est à dire des régions où l'indice de réfraction est le même que le milieu extérieur. Mathématiquement, la cavité correspond à une forme dégénérée du problème de transmission intérieur (les deux champs vérifient la même équation) ce qui génère des difficultés pour adapter les techniques pour les formes régulières étudiées dans le chapitre 2. Une première étude de cette configuration a été initiée dans [12] dans le cas scalaire. L'étude développée ici généralise ces travaux pour le cas électromagnétique.

Le problème de transmission intérieur est reformulé en une équation aux dérivées partielles du quatrième ordre à l'extérieur de la cavité et cette dernière est prise en compte en tant que contrainte dans l'espace variationnel. En plus de la technicité inhérente aux équations de Maxwell, la principale difficulté ici repose sur l'équivalence entre les solutions faibles et les solutions du problème variationnel et sur la décomposition de la formulation variationnelle en une partie compacte et une partie coercive. Dans un second temps et inspiré par des travaux récents [18, 17], cette formulation est utilisée afin de prouver l'existence d'un ensemble infini discret de valeurs propres de transmission et afin de prouver une propriété de monotonicité par rapport à la taille de la cavité de la première valeur propre de transmission. De plus, le fait que l'espace variationnel dépende de la fréquence rend l'étude plus difficile et nécessite l'introduction d'un opérateur de projection dont les propriétés de continuité permettront la résolution du problème.

Les rÃľsultats exposés ci-dessus sont extraits de [30] et sont complétés par un théorème caractérisant les valeurs propres de transmission à partir du champ lointain.

Chapitre 4 :

Ce chapitre est dédié à l'étude du problème de transmission intérieur correspondant au problème de diffraction d'un milieu non homogène (anisotrope ou non) de $\mathbb{R}^d$ ($d = 2$ ou $d = 3$) contenant un conducteur parfait. Le contraste du milieu est supposé donné par deux fonctions. D'un point de vue pratique, l'importance de ce problème (tout comme le problème du chapitre précédent) repose sur la possiblité d'utiliser les valeurs propres de transmission pour détecter des anomalies à l'intérieur de milieux non homogènes en contrôle non destructif. Ce type de problème a été étudié dans [41] dans lequel les auteurs retrouvent un obstacle inclus dans un milieu non homogène.

Ce problème n'ayant jamais été étudié avant, ce chapitre se concentre uniquement sur le cas scalaire. Le problème de transmission intérieur correspondant est de la forme

$$\begin{cases} \nabla \cdot A\nabla w + k^2 n w = 0 & \text{in } D\backslash\overline{D}_0 \\ \Delta v + k^2 v = 0 & \text{in } D \\ w = v & \text{on } \partial D \\ \nu \cdot A\nabla w = \nu \cdot \nabla v & \text{on } \partial D \\ w = 0 & \text{on } \partial D_0. \end{cases}$$

Ce chapiptre se concentre sur l'étude de l'existence et du caractère discret des valeurs propres de transmission, valeurs pour lesquelles le problème précédent admet une solution non triviale. L'étude est divisée en deux parties : la première concerne le cas isotrope ($A = I$) et la seconde le cas anisotrope ($A \neq I$). Pour chaque cas, la difficulté repose sur le fait que le champ w n'est pas défini à l'intérieur dand $D\backslash\overline{D}_0$.

Dans le cas isotrope, la première difficulté revient à définir le bon espace dans lequel le problème est bien posé. La méthode habituelle consiste à étudier l'équation satisfaite par la différence $u := w - v$. Cependant dans ce cas, la différence n'est définie que dans $D\backslash\overline{D}_0$ ce qui pose un problème de régularité pour cette fonction et elle doit être définie dans un espace plus faible. Le problème est alors reformulé en un problème du quatrième ordre pour u, couplé avec l'équation de Helmholtz satisfaite par v à l'intérieur de D_0. Afin d'obtenir l'existence d'un ensemble discret de valeurs propres de transmission, la formulation variationnelle est divisée en deux parties : une partie coercive et une partie compacte. Cependant, contrairement aux cas étudiés précédemment, l'espace faible dans lequel u est définie ne permet d'avoir la compacité seulement sur les termes d'ordres les plus faibles. Ainsi, seul le cas n plus petit que un peut être traité.

Pour le cas anisotrope, la difficulté ne repose pas sur la définition des espaces de solutions mais dans la reformulation du problème de transmission intérieur en un problème de type Fredholm. La méthode utilisée est adaptée d'une approche développée dans [38] et [19] qui permet de traiter à la fois l'existence et le caractère discret. Cependant, une méthode alternative inspirée de l'étude des métamatériaux qui utilise la T-coercivité est préférée afin de montrer que l'ensemble des valeurs propres de transmission est discret. Dans le cas où $A-I$ est une matrice définie positive, l'existence d'un ensemble fini discret de valeurs propres de transmission est établie alors que dans le cas où $I-A$ est définie positive, il est possible de montrer seulement le caractère discret pour n plus petit que un, l'existence étant encore une question ouverte.

Chapitre 5 :

Ce chapitre est dédié à l'étude du problème de transmission intérieur par l'utilisation d'une formulation en équations intégrales de surface. La principale motivation à l'origine ce cette étude était le développement d'une méthode numérique permettant de résoudre le problème de transmission intérieur dans le cas où l'indice de réfraction est constant par morceaux et de calculer les valeurs propres de transmission pour des géométries générales. Cette étude numérique est effectuée dans le chapitre 6. La méthode des équations intégrales a été naturellement adoptée puisqu'un solveur efficace appelé CESC pour les problèmes directs en électromagnétisme utilisant cette technique a été développé au CERFACS.

Il s'est ensuite avéré que la formulation en équations intégrales présentait également des intérêts théoriques. Par exemple, établir l'équivalence entre cette formulation et le problème original dans le cas d'ondes transverse magnétique nécessite l'utilisation de résultats non standards sur les potentiels. Ceci est dû au fait que l'espace variationnel des solutions est $L^2(D)$ avec le Laplacien appartenant également à $L^2(D)$, où D est le domaine étudié. Ainsi, l'espace naturel pour les solutions des équations intégrales serait $H^{-1/2}(\partial D) \times H^{-3/2}(\partial D)$, puisque les inconnues correspondent aux traces et aux traces normales des solutions (variationnelles). Les propriétés de régularité, de continuité et de coercivité des potentiels utilisés font parties des nouveaux ingrédients de cette études. Le principal outil permettant d'obtenir ces propriétés est la théorie des opérateurs pseudo-différentiels. Ainsi, en utilisant des arguments de densité convenables, les formules classiques de traces sont généralisées aux potentiels dont les densités sont moins régulières. Les propriétés de coercivité des potentiels sont analysées dans le cas de nombres d'ondes imaginaires purs. Soulignons dès à présent qu'une méthode alternative aurait été de considérer des potentiels dont les noyaux correspondent à la solution fondamentale du bilaplacien. Cependant cette méthode est moins intuitive et moins bien appropriée pour l'étude numérique du chapitre suivant.

Le second intérêt et sûrement le plus important de cette formulation en équations intégrales est lié à l'étude du problème de transmission intérieur pour des hypothèses moins restrictive sur le signe des contrastes. Plus présicément, le signe de la différence entre l'indice de l'inclusion et du milieu extérieur peut changer à l'intérieur de D. Contrairement à la méthode variationnelle utilisée dans le chapitre 3 pour traiter le cas de cavités, l'approche des équations intégrales permet de montrer que le problème de transmission intérieur est de type Fredholm si le contraste est constant et positif (ou négatif) seulement dans un voisinage de la frontière de D. On en déduit en particuler que l'ensemble des valeurs propres de transmission est discret. Le principal inconvénient de cette méthode est qu'il permet de traiter seulement le caractère discret et non l'existence des valeurs propres de transmission. Ce type de résultat est similaire à celui établi récemment par Sylvester [50] dans le cas transverse magnétique et par Bonnet-Ben Dhia-Chesnel-Haddar [5] pour le cas scalaire anisotrope. La méthode utilisée dans [50] est basée sur la notion d'opérateurs triangulaires supérieurs compacts, mais le résultats peut également être montré en utilisant la théorie classique de Fredholm analytique et l'utilisation d'une condition inf-sup appropriée, comme montré par Kirsch [39]. La technique utilisée dans [5] est basée sur la notion de T-coercivité, déjà utilisée dans le chapitre 4. Nous précisons également que les résultats sur le caractère discret dans le cas scalaire anisotrope ont été obtenus dans [42] avec des conditions plus faibles. Rapidement, dans cet article, le signe du contraste doit seulement être positif (ou négatif) dans un voisinage d'un point de la frontière, mais la partie imaginaire de l'indice de réfraction ne doit pas être nul.

Dans ce chapitre n'est considéré que le cas scalaire. Cependant la technique peut être étendue au cas du problème de Maxwell. Ce dernier sera seulement présenté formellement dans le chapitre 6 afin de mettre en place la méthode numérique.

On considère une région bornée simplement connexe $D \subset \mathbb{R}^d$, ($d = 2$ ou $d = 3$) dont la frontière $\Gamma := \partial D$ est régulière. Le forme générale du problème de transmission intérieur isotrope scalaire est

$$\begin{cases} \nabla \cdot \dfrac{1}{\mu(x)} \nabla w + k^2 n(x) w = 0 & \text{in } D, \\ \Delta v + k^2 v = 0 & \text{in } D, \\ w = v & \text{on } \Gamma, \\ \dfrac{1}{\mu} \dfrac{\partial w}{\partial \nu} = \dfrac{\partial v}{\partial \nu} & \text{on } \Gamma, \end{cases}$$

où $v, w \in H^1(D)$ si $\mu \neq 1$ et $v, w \in L^2(D)$ tels que $u := w - v \in H^2(D)$ si $\mu = 1$.

Afin de présenter la méthode des équations intégrales de surface, la première partie du chapitre traite le cas où n et μ sont constants. Le cas $\mu \neq 1$ est traité à part, les propriétés classiques des potentiels pouvant être utilisés. Le cas $\mu = 1$ nécessite plus de travail, notamment l'extension des propriétés des potentiels pour des densités définies dans des espaces faibles. Dans une dernière partie, on traite le cas plus général où le contraste peut changer de signe.

Chapitre 6 :

Ce chapitre présente différentes méthodes permettant de calculer numériquement les valeurs propres de transmission. La première approche utilise les solutions du problème de transmission intérieur et est basée sur la théorie développée dans le chapitre précédent. Le problème de transmission intérieur est reformulé en une équation intégrale de surface. Calculer les valeurs propres de transmission revient alors à résoudre un système de la forme :

$$Z(k)X = 0$$

où $Z(k)$ est un opérateur intégral. Numériquement, l'idée est de calculer les valeurs propres de $Z(k)$ et identifier les valeurs de k pour lesquelles $Z(k)$ admet la plus petite valeur propre est proche de zéro. Cependant, quand $\mu = 1$, l'opérateur $Z(k)$ est compact et par conséquent ses valeurs propres s'accumulent en zéro. Il est donc impossible de discerner si $Z(k)$ possède vraiment la valeur propre zéro ou non à cause des erreurs numériques. Pour contourner cette difficulté, on utilise un préconditionneur $B(k)$ et on résout le problème aux valeurs propres généralisé suivant

$$Z(k)X = \lambda B(k)X.$$

Si $B(k)$ est injectif, alors la valeur propre $\lambda = 0$ implique que le k correspondant est bien une valeur propre de transmission. Si de plus la partie principale de $B(k)$ coïncide avec la partie principale de $Z(k)$ alors l'accumulation des valeurs propres généralisées sera bien décalée en dehors de zéro.

Enfin, la dernière méthode est inspirée de la caractérisation des valeurs propres par le champ lointain. Contrairement à l'utilisation habituelle de la LSM à fréquence k fixée avec un échantillon de points z où l'objet est supposé se trouver, ici, le point source z est fixé à l'intérieur de l'objet (supposé connu) et on fait varier la fréquence. Si la norme de la solution régularisée de la LSM explose alors la fréquence correspond à une valeur propre de transmission.

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