An invariant subspace method for large-scale algebraic Riccati equation

Luca Amodei and Jean-Marie Buchot

Institut de Mathématiques de Toulouse
Université Paul Sabatier - Toulouse - France

Sparse Days at CERFACS - June 23-24, 2008
Toulouse - France

Summary

The linear dynamical system

- We consider the LTI dynamical system (S):

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) = Cx(t). \end{cases}$$

- We want to stabilize the system by a feedback control:

$$u(t) = -Kx(t).$$

- The dimension n of the system is large ($n >> 10^5$) and A, B, C are sparse. Examples from semi-discretized evolution PDE.
- Stabilization by the feedback solution of the infinite horizon Linear Quadratic Regulator problem (LQR):

$$\min_u \frac{1}{2} \int_0^{+\infty} [y(t)^T y(t) + u(t)^T u(t)]\, dt.$$

The linear dynamical system

- We consider the LTI dynamical system (S):

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) = Cx(t). \end{cases}$$

- We want to stabilize the system by a feedback control:

$$u(t) = -Kx(t).$$

- The dimension n of the system is large ($n >> 10^5$) and A, B, C are sparse. Examples from semi-discretized evolution PDE.
- Stabilization by the feedback solution of the infinite horizon Linear Quadratic Regulator problem (LQR):

$$\min_u \frac{1}{2} \int_0^{+\infty} [y(t)^T y(t) + u(t)^T u(t)]\, dt.$$

The linear dynamical system

- We consider the LTI dynamical system (S):

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) = Cx(t). \end{cases}$$

- We want to stabilize the system by a feedback control:

$$u(t) = -Kx(t).$$

- The dimension n of the system is large ($n >> 10^5$) and A, B, C are sparse. Examples from semi-discretized evolution PDE.
- Stabilization by the feedback solution of the infinite horizon Linear Quadratic Regulator problem (LQR):

$$\min_u \frac{1}{2} \int_0^{+\infty} [y(t)^T y(t) + u(t)^T u(t)]\, dt.$$

The linear dynamical system

- We consider the LTI dynamical system (S):

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0, \\ y(t) = Cx(t). \end{cases}$$

- We want to stabilize the system by a feedback control:

$$u(t) = -Kx(t).$$

- The dimension n of the system is large $(n >> 10^5)$ and A, B, C are sparse. Examples from semi-discretized evolution PDE.
- Stabilization by the feedback solution of the infinite horizon Linear Quadratic Regulator problem (LQR):

$$\min_u \frac{1}{2} \int_0^{+\infty} [y(t)^T y(t) + u(t)^T u(t)]\, dt.$$

The CARE

The feedback $K = B^T X$, where X is the solution of the Continuous-time Algebraic Riccati Equation (CARE):

$$A^T X + XA - XBB^T X + C^T C = 0.$$

We obtain a stable closed-loop matrix $A - BK$.

Theorem

If (A, B) is stabilizable and (C, A) detectable, then the CARE has a unique positive semi-definite solution X.

The CARE

The feedback $K = B^T X$, where X is the solution of the Continuous-time Algebraic Riccati Equation (CARE):

$$A^T X + XA - XBB^T X + C^T C = 0.$$

We obtain a stable closed-loop matrix $A - BK$.

Theorem

If (A, B) is stabilizable and (C, A) detectable, then the CARE has a unique positive semi-definite solution X.

Numerical methods for solving the CARE

Two classes of methods:

- Newton-Kleinman method.
- Invariant subspace methods.

Related questions:

- Lyapunov equation.
- Model reduction.
- System stabilization.

Numerical methods for solving the CARE

Two classes of methods:

- Newton-Kleinman method.
- Invariant subspace methods.

Related questions:

- Lyapunov equation.
- Model reduction.
- System stabilization.

Stable invariant subspace of the Hamiltonian

- Hamiltonian matrix H associated to the system:

$$H = \begin{pmatrix} A & -BB^T \\ -C^T C & -A^T \end{pmatrix} \in \mathbb{R}^{2n \times 2n}.$$

- Invariant subspace associated to the stable eigenvalues of H:

$$\begin{pmatrix} A & -BB^T \\ -C^T C & -A^T \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda_s,$$

with $Y, Z, \Lambda_s \in \mathbb{C}^{n \times n}$ and Λ_s stable.

- The unique solution X of the CARE is given by

$$X = ZY^{-1} (\Longleftrightarrow XY = Z).$$

Stable invariant subspace of the Hamiltonian

- Hamiltonian matrix H associated to the system:

$$H = \begin{pmatrix} A & -BB^T \\ -C^TC & -A^T \end{pmatrix} \in \mathbb{R}^{2n\times 2n}.$$

- Invariant subspace associated to the stable eigenvalues of H:

$$\begin{pmatrix} A & -BB^T \\ -C^TC & -A^T \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda_s,$$

with $Y, Z, \Lambda_s \in \mathbb{C}^{n\times n}$ and Λ_s stable.

- The unique solution X of the CARE is given by

$$X = ZY^{-1} (\Longleftrightarrow XY = Z).$$

Stable invariant subspace of the Hamiltonian

- Hamiltonian matrix H associated to the system:

$$H = \begin{pmatrix} A & -BB^T \\ -C^T C & -A^T \end{pmatrix} \in \mathbb{R}^{2n \times 2n}.$$

- Invariant subspace associated to the stable eigenvalues of H:

$$\begin{pmatrix} A & -BB^T \\ -C^T C & -A^T \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda_s,$$

with $Y, Z, \Lambda_s \in \mathbb{C}^{n \times n}$ and Λ_s stable.

- The unique solution X of the CARE is given by

$$X = ZY^{-1} (\Longleftrightarrow XY = Z).$$

Stable invariant subspace of the Hamiltonian

- Hamiltonian matrix H associated to the system:

$$H = \begin{pmatrix} A & -BB^T \\ -C^TC & -A^T \end{pmatrix} \in \mathbb{R}^{2n\times 2n}.$$

- Invariant subspace associated to the stable eigenvalues of H:

$$\begin{pmatrix} A & -BB^T \\ -C^TC & -A^T \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda_s,$$

with $Y, Z, \Lambda_s \in \mathbb{C}^{n\times n}$ and Λ_s stable.

- The unique solution X of the CARE is given by

$$X = ZY^{-1} (\Longleftrightarrow XY = Z).$$

The problem

- When n is large, only a part of the invariant stable subspace of H is computed.
- Define an approximation $\widehat{X}$ of X from $Y, Z \in \mathbb{C}^{n \times k}$, $k < n$.

Requirements for $\widehat{X}$:

1.
$$\widehat{X} Y = Z$$
$$\implies (A - BB^* \widehat{X}) Y = Y \Lambda_s,$$
i.e. the closed-loop $(A - BB^* \widehat{X})$ is stable on the subspace $\mathrm{Im}(Y)$.

2. $\widehat{X}$ positive semi-definite
$$\implies Y^* Z = Y^* \widehat{X} Y \text{ is positive semi-definite.}$$

The problem

- When n is large, only a part of the invariant stable subspace of H is computed.
- Define an approximation $\widehat{X}$ of X from Y, $Z \in \mathbb{C}^{n\times k}$, $k < n$.

Requirements for $\widehat{X}$:

1. $$\widehat{X}Y = Z$$
$$\implies (A - BB^*\widehat{X})Y = Y\Lambda_s,$$
i.e. the closed-loop $(A - BB^*\widehat{X})$ is stable on the subspace $\mathrm{Im}(Y)$.
2. $\widehat{X}$ positive semi-definite
$$\implies Y^*Z = Y^*\widehat{X}Y \text{ is positive semi-definite.}$$

The problem

- When n is large, only a part of the invariant stable subspace of H is computed.
- Define an approximation $\widehat{X}$ of X from Y, $Z \in \mathbb{C}^{n \times k}$, $k < n$.

Requirements for $\widehat{X}$:

1. $$\widehat{X}Y = Z$$
   $$\Longrightarrow (A - BB^*\widehat{X})Y = Y\Lambda_s,$$
   i.e. the closed-loop $(A - BB^*\widehat{X})$ is stable on the subspace $\mathrm{Im}(Y)$.
2. $\widehat{X}$ positive semi-definite
   $$\Longrightarrow Y^*Z = Y^*\widehat{X}Y \text{ is positive semi-definite.}$$

The problem

- When n is large, only a part of the invariant stable subspace of H is computed.
- Define an approximation $\widehat{X}$ of X from Y, $Z \in \mathbb{C}^{n \times k}$, $k < n$.

Requirements for $\widehat{X}$:

1. $$\widehat{X}Y = Z$$
   $$\implies (A - BB^*\widehat{X})Y = Y\Lambda_s,$$
   i.e. the closed-loop $(A - BB^*\widehat{X})$ is stable on the subspace $\mathrm{Im}(Y)$.
2. $\widehat{X}$ positive semi-definite
   $$\implies Y^*Z = Y^*\widehat{X}Y \text{ is positive semi-definite.}$$

The structure of Y, Z

Lyapunov equation associated to a stable H-invariant subspace

Theorem

If $\mathrm{Im}\begin{pmatrix} Y \\ Z \end{pmatrix}$ *is a stable H-invariant subspace, then* $(Z^*Y) \in \mathbb{C}^{k\times k}$ *is solution of the Lyapunov equation*

$$\Lambda_s^* (Z^*Y) + (Z^*Y)\Lambda_s = -Y^*C^TCY - Z^*BB^TZ,$$

and (Z^*Y) *is positive semi-definite.*

A natural approximation $\widetilde{X}$

$$\widetilde{X} = Z\,Y^+ = Z\,(Y^* Y)^{-1}\,Y^*$$

[Benner, Fassbender, 1997].

- $\widetilde{X}$ verifies the requirement 1: $XY = Z$,
- $\widetilde{X}$ is not positive semi-definite.

The symmetric approximation $\widehat{X}$

- If $(Z^* Y)$ is nonsingular, we take

$$\widehat{X} = Z\,(Z^* Y)^{-1}\,Z^*.$$

$\widehat{X}$ verifies requirements 1 and 2.

- If $(Z^* Y)$ is singular, we take

$$\widehat{X} = Z\,(Z^* Y)^{+}\,Z^*.$$

But now $\widehat{X} Y = Z \Pi_{\mathrm{Im}(Z^* Y)}$.

?

The symmetric approximation $\widehat{X}$

- If (Z^*Y) is nonsingular, we take

$$\widehat{X} = Z\,(Z^*Y)^{-1}\,Z^*.$$

$\widehat{X}$ verifies requirements 1 and 2.

- If (Z^*Y) is singular, we take

$$\widehat{X} = Z\,(Z^*Y)^{+}\,Z^*.$$

But now $\widehat{X}Y = Z\Pi_{\mathrm{Im}(Z^*Y)}$.

?

The symmetric approximation $\widehat{X}$

- If (Z^*Y) is nonsingular, we take

$$\widehat{X} = Z\,(Z^*Y)^{-1}\,Z^*.$$

$\widehat{X}$ verifies requirements 1 and 2.

- If (Z^*Y) is singular, we take

$$\widehat{X} = Z\,(Z^*Y)^{+}\,Z^*.$$

But now $\widehat{X}Y = Z\Pi_{\mathrm{Im}(Z^*Y)}$.

?

The structure of Y, Z

Theorem

If (A, B) stabilizable then $\dim(\mathrm{Im}(Y)) = k$, *and* $\mathrm{Ker}(Z) = \mathrm{Ker}(Z^* Y)$.

$\Longrightarrow$ requirement 2 is verified: $\widehat{X} Y = Z$.

Theorem (general algebraic result)

Let $Y, Z \in \mathbb{C}^{n \times k}$ such that $Z^* Y = Y^* Z$ and $\mathrm{Ker}(Z) = \mathrm{Ker}(Z^* Y)$. *Then*

$$\widehat{X} = Z \, (Z^* Y)^+ \, Z^*$$

is the unique $n \times n$ matrix satisfying

1. $\widehat{X} Y = Z$,
2. $\widehat{X}^* = \widehat{X}$,
3. $\mathrm{Im}(\widehat{X}) = \mathrm{Im}(Z)$.

The structure of Y, Z

Theorem

If (A, B) stabilizable then $\dim(\mathrm{Im}(Y)) = k$, *and* $\mathrm{Ker}(Z) = \mathrm{Ker}(Z^* Y)$.

$\implies$ requirement 2 is verified: $\widehat{X} Y = Z$.

Theorem (general algebraic result)

Let $Y, Z \in \mathbb{C}^{n \times k}$ *such that* $Z^* Y = Y^* Z$ *and* $\mathrm{Ker}(Z) = \mathrm{Ker}(Z^* Y)$. *Then*

$$\widehat{X} = Z\,(Z^* Y)^+\, Z^*$$

is the unique $n \times n$ *matrix satisfying*

1. $\widehat{X} Y = Z$,
2. $\widehat{X}^* = \widehat{X}$,
3. $\mathrm{Im}(\widehat{X}) = \mathrm{Im}(Z)$.

Properties of $\widehat{X}$

- $\widehat{X}$ depends only on the invariant subspace $\mathcal{S}$: we obtain the same approximation by taking YR and ZR with R nonsingular.
- If $\mathcal{S}_1 \subseteq \mathcal{S}_2$ are stable invariant subspaces of H, then $\widehat{X}_1 \preceq \widehat{X}_2$. In particular, $\widehat{X} \preceq X$ for each approximation $\widehat{X}$.
- If $\mathcal{S}$ is conjugate symmetric, i.e. $u \in \mathcal{S}$ implies $\bar{u} \in \mathcal{S}$, then $\widehat{X}$ is real.

Remark: for $k = n$ we have

$$X = ZY^{-1} = Z\left(Z^*Y\right)^{+} Z^*.$$

Conclusion: canonical formulation generalizing the case $k = n$.

Properties of $\widehat{X}$

- $\widehat{X}$ depends only on the invariant subspace $\mathcal{S}$: we obtain the same approximation by taking YR and ZR with R nonsingular.
- If $\mathcal{S}_1 \subseteq \mathcal{S}_2$ are stable invariant subspaces of H, then $\widehat{X}_1 \preceq \widehat{X}_2$. In particular, $\widehat{X} \preceq X$ for each approximation $\widehat{X}$.
- If $\mathcal{S}$ is conjugate symmetric, i.e. $u \in \mathcal{S}$ implies $\bar{u} \in \mathcal{S}$, then $\widehat{X}$ is real.

Remark: for $k = n$ we have

$$X = ZY^{-1} = Z\left(Z^{*}Y\right)^{+}Z^{*}.$$

Conclusion: canonical formulation generalizing the case $k = n$.

Properties of $\widehat{X}$

- $\widehat{X}$ depends only on the invariant subspace $\mathcal{S}$: we obtain the same approximation by taking YR and ZR with R nonsingular.
- If $\mathcal{S}_1 \subseteq \mathcal{S}_2$ are stable invariant subspaces of H, then $\widehat{X}_1 \preceq \widehat{X}_2$. In particular, $\widehat{X} \preceq X$ for each approximation $\widehat{X}$.
- If $\mathcal{S}$ is conjugate symmetric, i.e. $u \in \mathcal{S}$ implies $\bar{u} \in \mathcal{S}$, then $\widehat{X}$ is real.

Remark: for $k = n$ we have

$$X = ZY^{-1} = Z\left(Z^*Y\right)^{+} Z^*.$$

Conclusion: canonical formulation generalizing the case $k = n$.

Properties of $\widehat{X}$

- $\widehat{X}$ depends only on the invariant subspace $\mathcal{S}$: we obtain the same approximation by taking YR and ZR with R nonsingular.
- If $\mathcal{S}_1 \subseteq \mathcal{S}_2$ are stable invariant subspaces of H, then $\widehat{X}_1 \preceq \widehat{X}_2$. In particular, $\widehat{X} \preceq X$ for each approximation $\widehat{X}$.
- If $\mathcal{S}$ is conjugate symmetric, i.e. $u \in \mathcal{S}$ implies $\bar{u} \in \mathcal{S}$, then $\widehat{X}$ is real.

Remark: for $k = n$ we have

$$X = ZY^{-1} = Z\left(Z^* Y\right)^+ Z^*.$$

Conclusion: canonical formulation generalizing the case $k = n$.

Properties of $\widehat{X}$

- $\widehat{X}$ depends only on the invariant subspace $\mathcal{S}$: we obtain the same approximation by taking YR and ZR with R nonsingular.
- If $\mathcal{S}_1 \subseteq \mathcal{S}_2$ are stable invariant subspaces of H, then $\widehat{X}_1 \preceq \widehat{X}_2$. In particular, $\widehat{X} \preceq X$ for each approximation $\widehat{X}$.
- If $\mathcal{S}$ is conjugate symmetric, i.e. $u \in \mathcal{S}$ implies $\bar{u} \in \mathcal{S}$, then $\widehat{X}$ is real.

Remark: for $k = n$ we have

$$X = ZY^{-1} = Z\left(Z^* Y\right)^+ Z^*.$$

Conclusion: canonical formulation generalizing the case $k = n$.

The selection of an optimal stable invariant subspace

Let $(y,z)^T \in \mathbb{C}^{2n}$ an eigenvector associated to a stable eigenvalue λ of H with y normalized ($y^*y = 1$).

We have $\widehat{X}y = Xy = z$, and $y^*Xy = y^*z = z^*y$.

We are looking for $\widehat{X} = z\,(z^*y)^{-1}\,z^*$ with y such that the Rayleigh quotient

$$y^*Xy = z^*y \quad \text{is hight.}$$

The selection of an optimal stable invariant subspace

Let $(y,z)^T \in \mathbb{C}^{2n}$ an eigenvector associated to a stable eigenvalue λ of H with y normalized ($y^*y = 1$).

We have $\widehat{X}y = Xy = z$, and $y^*Xy = y^*z = z^*y$.

We are looking for $\widehat{X} = z\,(z^*y)^{-1}\,z^*$ with y such that the Rayleigh quotient

$$y^*Xy = z^*y \quad \text{is hight.}$$

The selection of an optimal stable invariant subspace

Let $(y, z)^T \in \mathbb{C}^{2n}$ an eigenvector associated to a stable eigenvalue λ of H with y normalized ($y^* y = 1$).

We have $\widehat{X} y = Xy = z$, and $y^* X y = y^* z = z^* y$.

We are looking for $\widehat{X} = z\,(z^* y)^{-1}\,z^*$ with y such that the Rayleigh quotient

$$y^* X y = z^* y \quad \text{is hight.}$$

The selection of an optimal stable invariant subspace

Let $(y, z)^T \in \mathbb{C}^{2n}$ an eigenvector associated to a stable eigenvalue λ of H with y normalized ($y^* y = 1$).

We have $\widehat{X} y = Xy = z$, and $y^* Xy = y^* z = z^* y$.

We are looking for $\widehat{X} = z\,(z^* y)^{-1}\,z^*$ with y such that the Rayleigh quotient

$$y^* Xy = z^* y \quad \text{is hight.}$$

The selection of an optimal stable invariant subspace

z^*y verifies the Lyapunov equation

$\Longrightarrow$

$$z^*y = \frac{-y^*C^TCy - z^*BB^Tz}{2\Re(\lambda)}.$$

$\Longrightarrow$

- Take (y, z) such that $|\Re(\lambda)|$ is small (easy!).
- Take (y, z) such that $y^*C^TCy + z^*BB^Tz$ is hight (more difficult!).

Conclusion:

- For many systems, good approximations by taking the stable H-invariant subspaces associated to the eigenvalues closest to the imaginary axis.
- True for Riesz systems. Example: the heat equation.

The selection of an optimal stable invariant subspace

z^*y verifies the Lyapunov equation

$\Longrightarrow$

$$z^*y = \frac{-y^*C^TCy - z^*BB^Tz}{2\Re(\lambda)}.$$

$\Longrightarrow$

- Take (y, z) such that $|\Re(\lambda)|$ is small (easy!).
- Take (y, z) such that $y^*C^TCy + z^*BB^Tz$ is hight (more difficult!).

Conclusion:

- For many systems, good approximations by taking the stable H-invariant subspaces associated to the eigenvalues closest to the imaginary axis.
- True for Riesz systems. Example: the heat equation.

The selection of an optimal stable invariant subspace

z^*y verifies the Lyapunov equation

$\Longrightarrow$

$$z^*y = \frac{-y^*C^TCy - z^*BB^Tz}{2\Re(\lambda)}.$$

$\Longrightarrow$

- Take (y, z) such that $|\Re(\lambda)|$ is small (easy!).
- Take (y, z) such that $y^*C^TCy + z^*BB^Tz$ is hight (more difficult!).

Conclusion:

- For many systems, good approximations by taking the stable H-invariant subspaces associated to the eigenvalues closest to the imaginary axis.
- True for Riesz systems. Example: the heat equation.

The selection of an optimal stable invariant subspace

$z^* y$ verifies the Lyapunov equation

$\Longrightarrow$

$$z^* y = \frac{-y^* C^T C y - z^* B B^T z}{2\Re(\lambda)}.$$

$\Longrightarrow$

- Take (y, z) such that $|\Re(\lambda)|$ is small (easy!).
- Take (y, z) such that $y^* C^T C y + z^* B B^T z$ is hight (more difficult!).

Conclusion:

- For many systems, good approximations by taking the stable H-invariant subspaces associated to the eigenvalues closest to the imaginary axis.
- True for Riesz systems. Example: the heat equation.

The rank of X

Kalman canonical decomposition: the system is equivalent via a transformation T to

$$\begin{cases} \begin{pmatrix} \dot{x}_{c\bar{o}} \\ \dot{x}_{co} \\ \dot{x}_{\bar{c}\bar{o}} \\ \dot{x}_{\bar{c}o} \end{pmatrix} = \begin{pmatrix} A_{c\bar{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{co} & 0 & A_{24} \\ 0 & 0 & A_{\bar{c}\bar{o}} & A_{34} \\ 0 & 0 & 0 & A_{\bar{c}o} \end{pmatrix} \begin{pmatrix} x_{c\bar{o}} \\ x_{co} \\ x_{\bar{c}\bar{o}} \\ x_{\bar{c}o} \end{pmatrix} + \begin{pmatrix} B_{c\bar{o}} \\ B_{co} \\ 0 \\ 0 \end{pmatrix} u, \\ y = \begin{pmatrix} 0 & C_{co} & 0 & C_{\bar{c}o} \end{pmatrix} \begin{pmatrix} x_{c\bar{o}} \\ x_{co} \\ x_{\bar{c}\bar{o}} \\ x_{\bar{c}o} \end{pmatrix}. \end{cases}$$

The observable vector $\begin{pmatrix} x_{co} \\ x_{\bar{c}o} \end{pmatrix}$ is independent of the other components.

The rank of X

Kalman canonical decomposition: the system is equivalent via a transformation T to

$$\left\{\begin{array}{l} \begin{pmatrix} \dot{x}_{c\bar{o}} \\ \dot{x}_{co} \\ \dot{x}_{\bar{c}\bar{o}} \\ \dot{x}_{\bar{c}o} \end{pmatrix} = \begin{pmatrix} A_{c\bar{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{co} & 0 & A_{24} \\ 0 & 0 & A_{\bar{c}\bar{o}} & A_{34} \\ 0 & 0 & 0 & A_{\bar{c}o} \end{pmatrix} \begin{pmatrix} x_{c\bar{o}} \\ x_{co} \\ x_{\bar{c}\bar{o}} \\ x_{\bar{c}o} \end{pmatrix} + \begin{pmatrix} B_{c\bar{o}} \\ B_{co} \\ 0 \\ 0 \end{pmatrix} u, \\ y = \begin{pmatrix} 0 & C_{co} & 0 & C_{\bar{c}o} \end{pmatrix} \begin{pmatrix} x_{c\bar{o}} \\ x_{co} \\ x_{\bar{c}\bar{o}} \\ x_{\bar{c}o} \end{pmatrix}. \end{array}\right.$$

The observable vector $\begin{pmatrix} x_{co} \\ x_{\bar{c}o} \end{pmatrix}$ is independent of the other components.

Subsystem (S_o)

Subsystem (S_o) is observable and stabilizable:

$$\left\{ \begin{array}{l} \begin{pmatrix} \dot{x}_{co} \\ \dot{x}_{\bar{c}o} \end{pmatrix} = \begin{pmatrix} A_{co} & A_{24} \\ 0 & A_{\bar{c}o} \end{pmatrix} \begin{pmatrix} x_{co} \\ x_{\bar{c}o} \end{pmatrix} + \begin{pmatrix} B_{co} \\ 0 \end{pmatrix} u, \\ y = \begin{pmatrix} C_{co} & C_{\bar{c}o} \end{pmatrix} \begin{pmatrix} x_{co} \\ x_{\bar{c}o} \end{pmatrix}. \end{array} \right.$$

$$\dim(S_o) = \dim\left(\text{observable subspace of } (S)\right).$$

The solution X_o of the CARE associated to (S_o) is positive definite. We get the solution X of the CARE: $X = T^T X_T T$ with

$$X_T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (X_o)_{11} & 0 & (X_o)_{12} \\ 0 & 0 & 0 & 0 \\ 0 & (X_o)_{12}^T & 0 & (X_o)_{22} \end{pmatrix}.$$

Eigenvalues modified by the feedback

$$
T(A - BB^T X)T^{-1} = \begin{pmatrix} A_{c\bar{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{co} & 0 & A_{24} \\ 0 & 0 & A_{\bar{c}\bar{o}} & A_{34} \\ 0 & 0 & 0 & A_{\bar{c}o} \end{pmatrix}
- \begin{pmatrix} B_{c\bar{o}} B_{c\bar{o}}^T & B_{c\bar{o}} B_{co}{}^T & 0 & 0 \\ B_{co} B_{c\bar{o}}^T & B_{co} B_{co}{}^T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (X_o)_{11} & 0 & (X_o)_{12} \\ 0 & 0 & 0 & 0 \\ 0 & (X_o)_{12}^T & 0 & (X_o)_{22} \end{pmatrix}.
$$

Only the controllable and observable eigenvalues are modified by the feedback:

$$
\begin{aligned} \sigma(A - BB^T X) = \ & \sigma(A_{co} - B_{co} B_{co}^T (X_o)_{11}) \\ & \bigcup \sigma(A_{c\bar{o}}) \bigcup \sigma(A_{\bar{c}\bar{o}}) \bigcup \sigma(A_{\bar{c}o}). \end{aligned}
$$

Eigenvalues modified by the feedback

$$T(A - BB^TX)T^{-1} = \begin{pmatrix} A_{c\bar{o}} & A_{12} & A_{13} & A_{14} \\ 0 & A_{co} & 0 & A_{24} \\ 0 & 0 & A_{\bar{c}\bar{o}} & A_{34} \\ 0 & 0 & 0 & A_{\bar{c}o} \end{pmatrix}$$

$$- \begin{pmatrix} B_{c\bar{o}}B_{c\bar{o}}^T & B_{c\bar{o}}B_{co}{}^T & 0 & 0 \\ B_{co}B_{c\bar{o}}^T & B_{co}B_{co}{}^T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (X_o)_{11} & 0 & (X_o)_{12} \\ 0 & 0 & 0 & 0 \\ 0 & (X_o)_{12}^T & 0 & (X_o)_{22} \end{pmatrix}.$$

Only the controllable and observable eigenvalues are modified by the feedback:

$$\begin{aligned} \sigma(A - BB^TX) = \; & \sigma(A_{co} - B_{co}\,B_{co}^T(X_o)_{11}) \\ & \bigcup \sigma(A_{c\bar{o}}) \bigcup \sigma(A_{\bar{c}\bar{o}}) \bigcup \sigma(A_{\bar{c}o}). \end{aligned}$$

A simple example

System (S):

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}.$$

- 2 controllable modes : $\{1, -3\}$.
- 2 observable modes : $\{1, -2\}$.

The system is stabilizable and detectable.

A simple example

System (S):

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}.$$

- 2 controllable modes : $\{1, -3\}$.
- 2 observable modes : $\{1, -2\}$.

The system is stabilizable and detectable.

Solution X of the CARE

$$X = \begin{pmatrix} 2.4142 & 0.2929 & 0 \\ 0.2929 & 0.2286 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$\mathrm{rank}(X) = 2 =$ dimension of the observable subspace.

$$K = B^T X = \begin{pmatrix} 2.4142 & 0.2929 & 0 \end{pmatrix},$$

$$\sigma\left(A - BB^T X\right) = \{-1.4142, -2, -3\}.$$

The mode 1 is the unique controllable and observable mode $\Longrightarrow$ 1 is modified by the feedback.

Approximation $\widehat{X} = z\,(z^* y)^{-1}\,z^*$

$$\widehat{X} = \begin{pmatrix} 2.4142 & 0.2929 & 0 \\ 0.2929 & 0.0355 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$K = B^T \widehat{X} = \begin{pmatrix} 2.4142 & 0.2929 & 0 \end{pmatrix},$$

$$\sigma\left(A - BB^T \widehat{X}\right) = \{-1.4142, -2, -3\}.$$

Approximation $\widetilde{X} = z\,(y^* y)^{-1}\, y^*$

$$\widetilde{X} = \begin{pmatrix} 0.7277 & 0 & -1.1078 \\ 0.0883 & 0 & -0.1344 \\ 0 & 0 & 0 \end{pmatrix},$$

$$K = B^T \widetilde{X} = \begin{pmatrix} 0.7277 & 0 & -1.1078 \end{pmatrix},$$

$$\sigma\,(A - BB^T \widetilde{X}) = \{-0.2056, -1.4142, -2\}.$$

The DARE

Kalman filter, ...

Discrete Algebraic Riccati Equation (DARE):

$$A^T XA - X - A^T XB(I_n + B^T XB)^{-1} B^T XA + C^T C = 0.$$

Stable invariant subspace of the pencil

$$\begin{pmatrix} A & 0 \\ -C^T C & I_n \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} I_n & BB^T \\ 0 & A^T \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda_s$$

with $\Lambda_s \in \mathbb{C}^{n \times n}$ stable.

The DARE

Kalman filter, ...
Discrete Algebraic Riccati Equation (DARE):

$$A^T XA - X - A^T XB(I_n + B^T XB)^{-1} B^T XA + C^T C = 0.$$

Stable invariant subspace of the pencil

$$\begin{pmatrix} A & 0 \\ -C^T C & I_n \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} I_n & BB^T \\ 0 & A^T \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda_s$$

with $\Lambda_s \in \mathbb{C}^{n \times n}$ stable.

The Lyapunov equation

A is stable.

- $$\begin{pmatrix} A & 0 \\ -C^T C & -A^T \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda,$$
  $\Longrightarrow$ observability Gramian $A^T X + XA + C^T C = 0.$
- $$\begin{pmatrix} A^T & 0 \\ -BB^T & -A \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda,$$
  $\Longrightarrow$ controllability Gramian $AX + XA^T + BB^T = 0.$

The Lyapunov equation

A is stable.

- $$\begin{pmatrix} A & 0 \\ -C^T C & -A^T \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda,$$

  $\Longrightarrow$ observability Gramian $A^T X + XA + C^T C = 0$.

- $$\begin{pmatrix} A^T & 0 \\ -BB^T & -A \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda,$$

  $\Longrightarrow$ controllability Gramian $AX + XA^T + BB^T = 0$.

The algebraic Bernoulli equation

If $C = 0$, we obtain the ABE

$$A^T X + XA - XBB^T X = 0.$$

Theorem (Benner 2007)

If (A, B) is stabilizable and $\sigma(A) \cap i\mathbb{R} = \{\emptyset\}$, then the ABE has a unique stabilizing positive semi-definite solution X, and $\mathrm{rank}(X) = k$, where k is the number of unstable modes of A.

The algebraic Bernoulli equation

If $C = 0$, we obtain the ABE

$$A^T X + XA - XBB^T X = 0.$$

Theorem (Benner 2007)

If (A, B) is stabilizable and $\sigma(A) \cap i\mathbb{R} = \{\emptyset\}$, then the ABE has a unique stabilizing positive semi-definite solution X, and $\mathrm{rank}(X) = k$, where k is the number of unstable modes of A.

The algebraic Bernoulli equation

$$\begin{pmatrix} A & -BB^T \\ 0 & -A^T \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda$$

with $\sigma(-\Lambda)$ the set of unstable eigenvalues of A.

Theorem

The unique stabilizing positive semi-definite solution X of the ABE is given by

$$X = Z \left(Z^* Y \right)^{+} Z^*.$$

The algebraic Bernoulli equation

$$\begin{pmatrix} A & -BB^T \\ 0 & -A^T \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} Y \\ Z \end{pmatrix} \Lambda$$

with $\sigma(-\Lambda)$ the set of unstable eigenvalues of A.

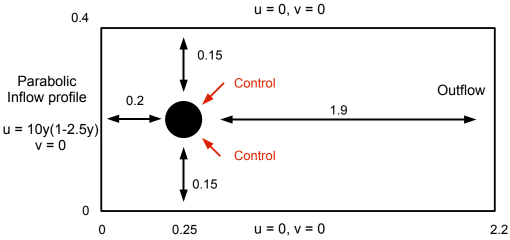
Theorem

The unique stabilizing positive semi-definite solution X of the ABE is given by

$$X = Z\,(Z^* Y)^+\, Z^*.$$

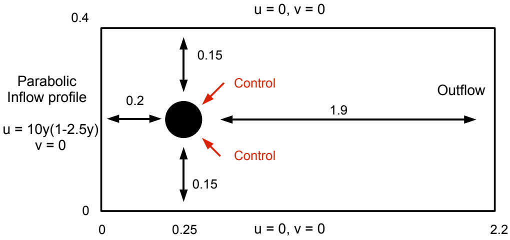
Control of the 2-D incompressible Navier-Stokes equations

- Flow around a cylinder in a channel.
- Left side: parabolic inflow. Right side: natural boundary conditions at outflow.
- No slip conditions at the upper and lower walls and around the cylinder excepted on two slots where we act by suction and blowing (the control).
- Reynolds number: $Re = 80$.



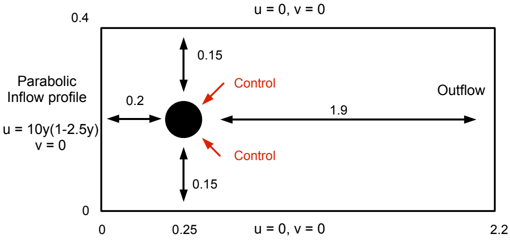
Control of the 2-D incompressible Navier-Stokes equations

- Flow around a cylinder in a channel.
- Left side: parabolic inflow. Right side: natural boundary conditions at outflow.
- No slip conditions at the upper and lower walls and around the cylinder excepted on two slots where we act by suction and blowing (the control).
- Reynolds number: $Re = 80$.



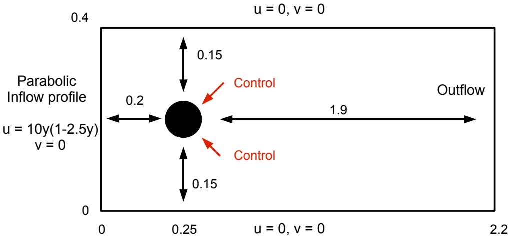
Control of the 2-D incompressible Navier-Stokes equations

- Flow around a cylinder in a channel.
- Left side: parabolic inflow. Right side: natural boundary conditions at outflow.
- No slip conditions at the upper and lower walls and around the cylinder excepted on two slips where we act by suction and blowing (the control).
- Reynolds number: $Re = 80$.



Control of the 2-D incompressible Navier-Stokes equations

- Flow around a cylinder in a channel.
- Left side: parabolic inflow. Right side: natural boundary conditions at outflow.
- No slip conditions at the upper and lower walls and around the cylinder excepted on two slots where we act by suction and blowing (the control).
- Reynolds number: $Re = 80$.



Control of the 2-D incompressible Navier-Stokes equations

- Equation is linearized around a stationary solution.
- Destabilized at $t = 1s$ during 0.1 s.
- At time $t = 7.5$ s the feedback control is applied on the non-linear equation.
- Divergence equation: $\varepsilon p + div(\mathbf{u}) = 0, \quad \varepsilon > 0.$
- Stabilization of the linearized equation via the generalized ABE.
- Discretization: finite elements T-H $P_4 - P_5$. Dimension: $n \approx 10^5$.
- Unstable eigenvalues of A^T computed with implicitly restarted Arnoldi method (Matlab *eigs*).

Control of the 2-D incompressible Navier-Stokes equations

- Equation is linearized around a stationary solution.
- Destabilized at $t = 1s$ during 0.1 s.
- At time $t = 7.5$ s the feedback control is applied on the non-linear equation.
- Divergence equation: $\varepsilon p + div(\mathbf{u}) = 0, \quad \varepsilon > 0.$
- Stabilization of the linearized equation via the generalized ABE.
- Discretization: finite elements T-H $P_4 - P_5$. Dimension: $n \approx 10^5$.
- Unstable eigenvalues of A^T computed with implicitly restarted Arnoldi method (Matlab *eigs*).

Control of the 2-D incompressible Navier-Stokes equations

- Equation is linearized around a stationary solution.
- Destabilized at $t = 1s$ during 0.1 s.
- At time $t = 7.5$ s the feedback control is applied on the non-linear equation.
- Divergence equation: $\varepsilon p + div(\mathbf{u}) = 0, \quad \varepsilon > 0.$
- Stabilization of the linearized equation via the generalized ABE.
- Discretization: finite elements T-H $P_4 - P_5$. Dimension: $n \approx 10^5$.
- Unstable eigenvalues of A^T computed with implicitly restarted Arnoldi method (Matlab *eigs*).

Control of the 2-D incompressible Navier-Stokes equations

- Equation is linearized around a stationary solution.
- Destabilized at $t = 1s$ during 0.1 s.
- At time $t = 7.5$ s the feedback control is applied on the non-linear equation.
- Divergence equation: $\varepsilon p + div(\mathbf{u}) = 0, \quad \varepsilon > 0.$
- Stabilization of the linearized equation via the generalized ABE.
- Discretization: finite elements T-H $P_4 - P_5$. Dimension: $n \approx 10^5$.
- Unstable eigenvalues of A^T computed with implicitly restarted Arnoldi method (Matlab *eigs*).

Control of the 2-D incompressible Navier-Stokes equations

- Equation is linearized around a stationary solution.
- Destabilized at $t = 1s$ during 0.1 s.
- At time $t = 7.5$ s the feedback control is applied on the non-linear equation.
- Divergence equation: $\varepsilon p + div(\mathbf{u}) = 0, \quad \varepsilon > 0.$
- Stabilization of the linearized equation via the generalized ABE.
- Discretization: finite elements T-H $P_4 - P_5$. Dimension: $n \approx 10^5$.
- Unstable eigenvalues of A^T computed with implicitly restarted Arnoldi method (Matlab *eigs*).

Control of the 2-D incompressible Navier-Stokes equations

- Equation is linearized around a stationary solution.
- Destabilized at $t = 1s$ during 0.1 s.
- At time $t = 7.5$ s the feedback control is applied on the non-linear equation.
- Divergence equation: $\varepsilon p + div(\mathbf{u}) = 0, \quad \varepsilon > 0$.
- Stabilization of the linearized equation via the generalized ABE.
- Discretization: finite elements T-H $P_4 - P_5$. Dimension: $n \approx 10^5$.
- Unstable eigenvalues of A^T computed with implicitly restarted Arnoldi method (Matlab *eigs*).

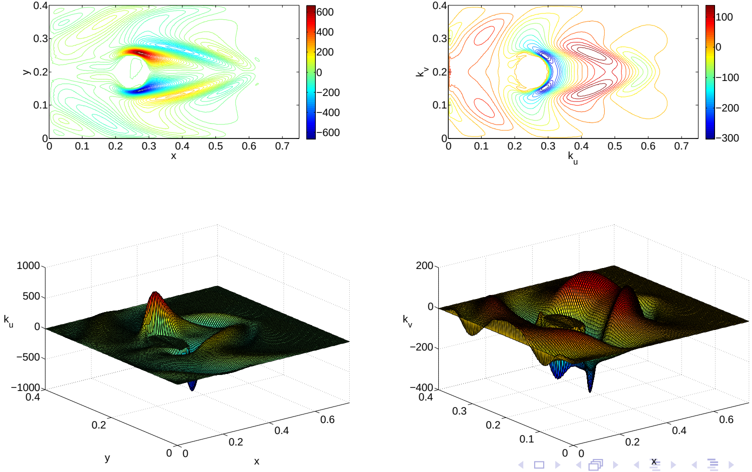
Control of the 2-D incompressible Navier-Stokes equations

- Equation is linearized around a stationary solution.
- Destabilized at $t = 1s$ during 0.1 s.
- At time $t = 7.5$ s the feedback control is applied on the non-linear equation.
- Divergence equation: $\varepsilon p + div(\mathbf{u}) = 0, \quad \varepsilon > 0.$
- Stabilization of the linearized equation via the generalized ABE.
- Discretization: finite elements T-H $P_4 - P_5$. Dimension: $n \approx 10^5$.
- Unstable eigenvalues of A^T computed with implicitly restarted Arnoldi method (Matlab *eigs*).

Kernels $k_u(x,y)$ and $k_v(x,y)$

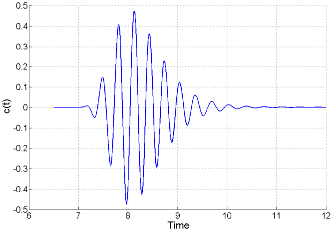
Control $c(t)$:

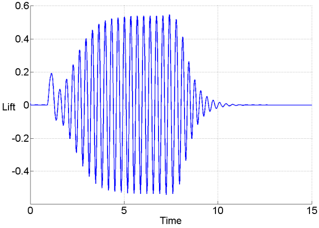
$$c(t) = -\int_{\Omega} \left(k_u(x,y)u(t,x,y) + k_v(x,y)v(t,x,y) \right) dxdy.$$

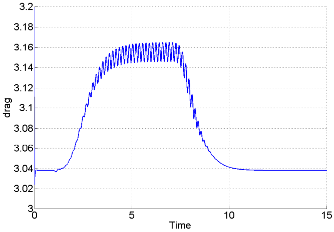


Control of the Navier-Stokes equations

Control $c(t)$:

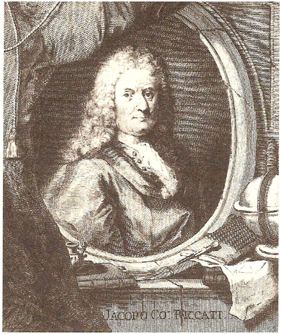






Conclusion

- General consistent approximations related to stable invariant subspaces of the Hamiltonian.
- Relevance of the symmetry.
- Same approach for the DARE, the GARE (Generalized ARE), the ABE, the Lyapunov equation.
- Cheap method to get the solution of the ABE and stabilize a system with few unstable modes.
- Limitations: the choice of the stable invariant subspace!
  $\longrightarrow$ Guidelines: use the stable modes of the Hamiltonian near the imaginary axis in order to approximate well the transfer function of low frequencies.



Count Jacopo Riccati, 1676-1754

Thanks for your attention!