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Iterative Methods for Symmetric Quasi-Definite Linear Systems

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Overview of talk

- Symmetric Quasi Positive Definite matrices

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- G-K bidiagonalization
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- G-K bidiagonalization
- Generalized LSQR and Craig (Stopping criteria)
- Numerical examples

Symmetric Quasi-Definite Systems

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \qquad \text{where} \qquad \mathbf{M} = \mathbf{M}^T \succ 0, \ \mathbf{N} = \mathbf{N}^T \succ 0.$$

- Interior-point methods for LP, QP, NLP, SOCP, SDP, ...
- Regularized/stabilized PDE problems
- Regularized least squares
- How to best take advantage of the structure?

Main Property

Theorem (Vanderbei, 1995)
If $\mathbf{K}$ *is SQD, it is* **strongly factorizable**, *i.e., for any permutation matrix* $\mathbf{P}$, *there exists a unit lower triangular* $\mathbf{L}$ *and a diagonal* $\mathbf{D}$ *such that* $\mathbf{P}^T\mathbf{K}\mathbf{P} = \mathbf{L}\mathbf{D}\mathbf{L}^T$.

- Cholesky-factorizable
- Used to speed up factorization in regularized least-squares (Saunders) and interior-point methods (Friedlander and O.)
- Stability analysis by Gill, Saunders, Shinnerl (1996).

Centered preconditioning

$$\begin{bmatrix} \mathbf{M}^{-\frac{1}{2}} & \\ & \mathbf{N}^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{M}^{-\frac{1}{2}} & \\ & \mathbf{N}^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{-\frac{1}{2}}\mathbf{f} \\ \mathbf{N}^{-\frac{1}{2}}\mathbf{g} \end{bmatrix}$$

which is equivalent to

$$\overbrace{\begin{bmatrix} \mathbf{I}_m & \mathbf{M}^{-\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}} \\ \mathbf{N}^{-\frac{1}{2}}\mathbf{A}^T\mathbf{M}^{-\frac{1}{2}} & -\mathbf{I}_n \end{bmatrix}}^{\mathcal{A}} \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} \mathbf{M}^{-\frac{1}{2}}\mathbf{f} \\ \mathbf{N}^{-\frac{1}{2}}\mathbf{g} \end{bmatrix}$$

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Theorem (Saunders (1995))

Suppose $\tilde{\mathbf{A}} = \mathbf{M}^{-\frac{1}{2}}\mathbf{A}\mathbf{N}^{-\frac{1}{2}}$ *has rank* $p \le m$ *with nonzero singular values* $\sigma_1, \ldots, \sigma_p$. *The eigenvalues of* $\mathcal{A}$ *are* $+1$, -1 *and* $\pm\sqrt{1+\sigma_k}$, $k = 1, \ldots, p$.

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Symmetric spectrum and Iterative methods

A symmetric matrix with a symmetric spectrum can be transform preserving the symmetry of the spectrum in a SQD one. Moreover, Fischer (Theorem 6.9.9 in "Polynomial based iteration methods for symmetric linear systems") Freund (1983), Freund Golub Nachtigal (1992), and Ramage Silvester Wathen (1995) give different poofs that MINRES and CG perform redundant iterations.

Iterative Methods I

Facts: SQD systems are symmetric, non-singular, square and indefinite.

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- SYMMLQ
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Fact: ... none exploits the SQD structure and they are doing redundant iterations

Related Problems: an example

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}$$

Related Problems: an example

$$\begin{bmatrix} \mathbf{M} & \mathbf{A} \\ \mathbf{A}^T & -\mathbf{N} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}$$

are the optimality conditions of

$$\min_{\mathbf{y}\in\mathbb{R}^m} \frac{1}{2} \left\| \begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix} \mathbf{y} - \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} \right\|^2_{E_+^{-1}} \equiv \min_{y\in\mathbb{R}^m} \frac{1}{2} \left\| \begin{bmatrix} \mathbf{M}^{-\frac{1}{2}} & 0 \\ 0 & \mathbf{N}^{\frac{1}{2}} \end{bmatrix} \left(\begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix} \mathbf{y} - \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} \right) \right\|^2_2$$

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Normal equations for SQD

Let assume that

$$\mathbf{M} = \mathbf{R}^T\mathbf{R} \qquad \text{and} \qquad \mathbf{N} = \mathbf{U}^T\mathbf{U}$$

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We observe that the normal equations

$$\mathcal{A}^2 \begin{bmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \end{bmatrix} = \mathcal{A} \begin{bmatrix} \mathbf{R}^{-1}\mathbf{b} \\ 0 \end{bmatrix}$$

have a very interesting structure because

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have a very interesting structure because

$$\mathcal{A}^2 = \begin{bmatrix} I_{m-n} + \tilde{\mathbf{A}}\tilde{\mathbf{A}}^T & 0 \\ 0 & I_n + \tilde{\mathbf{A}}^T\tilde{\mathbf{A}} \end{bmatrix} = \mathcal{D}$$

and $\mathcal{D}\mathcal{A} = \mathcal{A}\mathcal{D}$

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The Krylov space

$$\mathcal{K}_k(\mathcal{A}, h_0) = Range\{\mathcal{A}^j h_0, \ldots, \mathcal{A}h_0, h_0\}$$

is

$$\mathcal{K}_k(\mathcal{A}, h_0) = \mathcal{K}_{\lfloor k/2 \rfloor}(\mathcal{D}, h_0) + \mathcal{K}_{\lfloor (k+1)/2 \rfloor}(\mathcal{D}, \mathcal{A}h_0)$$

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$\mathcal{A}-1$

$$\mathcal{A}^{-1} = \mathcal{D}^{-1}\mathcal{A} = \mathcal{A}\mathcal{D}^{-1}$$

Linear operators

Let $\mathbf{M} \in \mathrm{R}^{m\times m}$ and $\mathbf{N} \in \mathrm{R}^{n\times n}$ be symmetric positive definite matrices, and let $\mathbf{A} \in \mathrm{R}^{m\times n}$ be a full rank matrix.

$$\mathcal{M} = \{\mathbf{v} \in \mathrm{R}^m; \|\mathbf{u}\|^2_{\mathbf{M}} = \mathbf{v}^T\mathbf{M}\mathbf{v}\},\ \mathcal{N} = \{\mathbf{q} \in \mathrm{R}^n; \|\mathbf{q}\|^2_{\mathbf{N}} = \mathbf{q}^T\mathbf{N}\mathbf{q}\}$$

$$\mathcal{M}' = \{\mathbf{w} \in \mathrm{R}^m; \|\mathbf{w}\|^2_{\mathbf{M}^{-1}} = \mathbf{w}^T\mathbf{M}^{-1}\mathbf{w}\},$$
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$$\langle \mathbf{v}, \mathbf{A}\mathbf{q}\rangle_{\mathcal{M},\mathcal{M}'} = \mathbf{v}^T\mathbf{A}\mathbf{q}, \quad \mathbf{A}\mathbf{q} \in \mathcal{M}'\ \forall \mathbf{q} \in \mathcal{N}.$$

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The adjoint operator $\mathbf{A}^\star$ of $\mathbf{A}$ can be defined as

$$\langle \mathbf{A}^\star\mathbf{g}, \mathbf{f}\rangle_{\mathcal{N}',\mathcal{N}} = \mathbf{f}^T\mathbf{A}^T\mathbf{g}, \quad \mathbf{A}^T\mathbf{g} \in \mathcal{N}'\ \forall \mathbf{g} \in \mathcal{M}.$$

Generalized SVD

Given $\mathbf{q} \in \mathcal{M}$ and $\mathbf{v} \in \mathcal{N}$, the critical points for the functional

$$\frac{\mathbf{v}^T \mathbf{A} \mathbf{q}}{\|\mathbf{q}\|_{\mathbf{N}} \; \|\mathbf{v}\|_{\mathbf{M}}}$$

are the "*elliptic singular values and singular vectors*" of $\mathbf{A}$.

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are the "*elliptic singular values and singular vectors*" of **A**.
The saddle-point conditions are

$$\begin{cases} \mathbf{A}\mathbf{q}_i &= \sigma_i \mathbf{M}\mathbf{v}_i \qquad \mathbf{v}_i^T \mathbf{M}\mathbf{v}_j = \delta_{ij} \\ \mathbf{A}^T\mathbf{v}_i &= \sigma_i \mathbf{N}\mathbf{q}_i \qquad \mathbf{q}_i^T \mathbf{N}\mathbf{q}_j = \delta_{ij} \end{cases}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$$

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$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$$

The elliptic singular values are the standard singular values of $\tilde{\mathbf{A}} = \mathbf{M}^{-1/2}\mathbf{A}\mathbf{N}^{-1/2}$. The elliptic singular vectors $\mathbf{q}_i$ and $\mathbf{v}_i$, $i = 1, \ldots, n$ are the transformation by $\mathbf{M}^{-1/2}$ and $\mathbf{N}^{-1/2}$ respectively of the left and right standard singular vector of $\tilde{\mathbf{A}}$.

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Generalized Golub-Kahan bidiagonalization

In Golub Kahan (1965), Paige Saunders (1982), several algorithms for the bidiagonalization of a $m \times n$ matrix are presented. All of them can be theoretically applied to $\tilde{\mathbf{A}}$ and their generalization to $\mathbf{A}$ is straightforward as shown by Bembow (1999). Here, we want specifically to analyse one of the variants known as the "Craig"-variant (see Paige Saunders (1982), Saunders (1995,1997)).

Generalized Golub-Kahan bidiagonalization

$$\begin{cases} \mathbf{A}\tilde{\mathbf{Q}} = \mathbf{M}\tilde{\mathbf{V}} \begin{bmatrix} \tilde{\mathbf{B}} \\ 0 \end{bmatrix} & \tilde{\mathbf{V}}^T\mathbf{M}\tilde{\mathbf{V}} = \mathbf{I}_m \\ \mathbf{A}^T\tilde{\mathbf{V}} = \mathbf{N}\tilde{\mathbf{Q}} \left[\tilde{\mathbf{B}}^T; 0\right] & \tilde{\mathbf{Q}}^T\mathbf{N}\tilde{\mathbf{Q}} = \mathbf{I}_n \end{cases}$$

where

$$\tilde{\mathbf{B}} = \begin{bmatrix} \tilde{\alpha}_1 & 0 & 0 & \cdots & 0 \\ \tilde{\beta}_2 & \tilde{\alpha}_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \tilde{\beta}_{n-1} & \tilde{\alpha}_{n-1} & 0 \\ 0 & \cdots & 0 & \tilde{\beta}_n & \tilde{\alpha}_n \\ 0 & \cdots & 0 & 0 & \tilde{\beta}_{n+1} \end{bmatrix}.$$

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Generalized Golub-Kahan bidiagonalization

$$\begin{cases} \mathbf{AQ} &= \mathbf{MV}\begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \\ \mathbf{A}^T\mathbf{V} &= \mathbf{NQ}\left[\mathbf{B}^T; 0\right] \end{cases} \qquad \begin{matrix} \mathbf{V}^T\mathbf{MV} = \mathbf{I}_m \\ \mathbf{Q}^T\mathbf{NQ} = \mathbf{I}_n \end{matrix}$$

where

$$\mathbf{B} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \beta_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & \alpha_{n-1} & \beta_{n-1} \\ 0 & \cdots & 0 & 0 & \alpha_n \end{bmatrix}.$$

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Algorithm

Thus, we can compute the first column of **B** and of **V**:
$\alpha_1 \mathbf{M}\mathbf{v}_1 = \mathbf{A}\mathbf{q}_1$, such as

$$\mathbf{w} = \mathbf{M}^{-1}\mathbf{A}\mathbf{q}_1$$
$$\alpha_1 = \sqrt{\mathbf{w}^T\mathbf{M}\mathbf{w}} = \sqrt{\mathbf{w}\mathbf{A}\mathbf{q}_1}$$
$$\mathbf{v}_1 = \mathbf{w}/\alpha_1.$$

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Finally, knowing $\mathbf{q}_1$ and $\mathbf{v}_1$ we can start the recursive relations

$$\mathbf{g}_{i+1} = \mathbf{N}^{-1}\left(\mathbf{A}^T\mathbf{v}_i - \alpha_i\mathbf{N}\mathbf{q}_i\right)$$
$$\beta_{i+1} = \sqrt{\mathbf{g}^T\mathbf{N}\mathbf{g}}$$
$$\mathbf{q}_{i+1} = \mathbf{g}\ \beta_{i+1}$$
$$\mathbf{w} = \mathbf{M}^{-1}\left(\mathbf{A}\mathbf{q}_{i+1} - \beta_{i+1}\mathbf{M}\mathbf{v}_i\right)$$
$$\alpha_{i+1} = \sqrt{\mathbf{w}^T\mathbf{M}\mathbf{w}}$$
$$\mathbf{v}_{i+1} = \mathbf{w}/\alpha_{i+1}.$$

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Generalized Least Squares

Normal equations: $(\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A} + \mathbf{N})\mathbf{y} = \mathbf{A}^T\mathbf{M}^{-1}\mathbf{b}$.

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At k-th iteration, seek $y \approx \mathbf{y}_k := \tilde{\mathbf{V}}_k\bar{\mathbf{y}}_k$:

$$(\tilde{\mathbf{B}}_k^T\tilde{\mathbf{B}}_k+\mathbf{I})\bar{\mathbf{y}}_k=\tilde{\mathbf{B}}_k^T\beta_1\mathbf{e}_1$$

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$$(\tilde{\mathbf{B}}_k^T\tilde{\mathbf{B}}_k+\mathbf{I})\bar{\mathbf{y}}_k=\tilde{\mathbf{B}}_k^T\beta_1\mathbf{e}_1$$

i.e.:

$$\min_{\bar{\mathbf{y}}\in\mathrm{R}^k} \ \tfrac{1}{2}\left\| \begin{bmatrix}\tilde{\mathbf{B}}_k\\ \mathbf{I}\end{bmatrix}\bar{\mathbf{y}}-\begin{bmatrix}\beta_1\mathbf{e}_1\\ 0\end{bmatrix}\right\|_2^2$$

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i.e.:

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or:

$$\begin{bmatrix}\mathbf{I} & \tilde{\mathbf{B}}_k\\ \tilde{\mathbf{B}}_k^T & -\mathbf{I}\end{bmatrix}\begin{bmatrix}\bar{\mathbf{x}}_k\\ \bar{\mathbf{y}}_k\end{bmatrix}=\begin{bmatrix}\beta_1\mathbf{e}_1\\ 0\end{bmatrix}.$$

Generalized LSQR

Solve

$$\min_{\bar{\mathbf{y}} \in \mathbb{R}^k} \ \frac{1}{2} \left\| \begin{bmatrix} \tilde{\mathbf{B}}_k \\ \mathbf{I} \end{bmatrix} \bar{\mathbf{y}} - \begin{bmatrix} \beta_1 \mathbf{e}_1 \\ 0 \end{bmatrix} \right\|_2^2$$

by specialized Givens Rotations (Eliminate $\mathbf{I}$ first and $\tilde{\mathbf{R}}_k$ will be upper bidiagonal)

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$$\min_{\bar{\mathbf{y}} \in \mathrm{R}^k} \ \frac{1}{2} \left\| \begin{bmatrix} \tilde{\mathbf{B}}_k \\ \mathbf{I} \end{bmatrix} \bar{\mathbf{y}} - \begin{bmatrix} \beta_1 \mathbf{e}_1 \\ 0 \end{bmatrix} \right\|_2^2$$

by specialized Givens Rotations (Eliminate $\mathbf{I}$ first and $\tilde{\mathbf{R}}_k$ will be upper bidiagonal)

$$\min_{\bar{\mathbf{y}} \in \mathrm{R}^k} \ \frac{1}{2} \left\| \begin{bmatrix} \tilde{\mathbf{R}}_k \\ 0 \end{bmatrix} \bar{\mathbf{y}} - \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} \right\|_2^2.$$

Generalized LSQR

Solve

$$\min_{\bar{\mathbf{y}}\in \mathbb{R}^k} \frac{1}{2} \left\| \begin{bmatrix} \tilde{\mathbf{B}}_k \\ \mathbf{I} \end{bmatrix} \bar{\mathbf{y}} - \begin{bmatrix} \beta_1 \mathbf{e}_1 \\ 0 \end{bmatrix} \right\|_2^2$$

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$$\min_{\bar{\mathbf{y}}\in \mathbb{R}^k} \frac{1}{2} \left\| \begin{bmatrix} \tilde{\mathbf{R}}_k \\ 0 \end{bmatrix} \bar{\mathbf{y}} - \begin{bmatrix} \phi_k \\ 0 \end{bmatrix} \right\|_2^2 .$$

As in Paige-Saunders '82 we can build recursive expressions of $\mathbf{y}_k$

$$\mathbf{y}_{k+1} = \mathbf{y}_k + \mathbf{d}_k \phi_k \quad \left(\mathbf{D}_k = \tilde{\mathbf{V}}_k \tilde{\mathbf{R}}_k^{-1} \right)$$

and we have that

$$||\bar{\mathbf{y}}||^2_{\mathbf{N}+\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A}} = \sum_{j=1}^{m} \phi_j^2 \quad \text{and} \quad ||\bar{\mathbf{y}} - \mathbf{y}_k||^2_{\mathbf{N}+\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A}} = \sum_{j=k+1}^{m} \phi_j^2$$

Error bound

Lower bound We can estimate $||\bar{\mathbf{y}} - \mathbf{y}_k||^2_{\mathbf{N}+\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A}}$ by the lower bound

$$\xi^2_{k,d} = \sum_{j=k+1}^{k+d+1} \phi_j^2 < ||\bar{\mathbf{y}} - \mathbf{y}_k||^2_{\mathbf{N}+\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A}}.$$

and $||\bar{\mathbf{y}}||^2_{\mathbf{N}+\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A}}$ by the lower bound $\sum_{j=1}^{k} \phi_j^2$. Given a threshold $\tau < 1$ and an integer d, we can stop the iterations when

$$\xi^2_{k,d} \leq \tau \sum_{j=1}^{k+d+1} \phi_j^2 < \tau \sum_{j=1}^{k} \phi_j^2 < \tau ||\bar{\mathbf{y}}||^2_{\mathbf{N}+\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A}}.$$

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and $||\bar{\mathbf{y}}||^2_{\mathbf{N}+\mathbf{A}^T\mathbf{M}^{-1}\mathbf{A}}$ by the lower bound $\sum_{j=1}^{k} \phi_j^2$.
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Upper bound Despite being very inexpensive, the previous estimator is still a lower bound of the error. We can use an approach inspired by the Gauss-Radau quadrature algorithm and similar to the one described in Golub-Meurant (2010).

Generalized CRAIG

$$\min_{\mathbf{y},\mathbf{x}} \ \frac{1}{2}(\|\mathbf{y}\|_{\mathbf{N}}^2 + \|\mathbf{x}\|_{\mathbf{M}}^2) \quad \text{s.t. } \mathbf{A}\mathbf{y} + \mathbf{M}\mathbf{x} = \mathbf{b}.$$

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Generalized CRAIG

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At step k of GK bidiagonalization, we seek

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or:

$$\min_{\bar{\mathbf{y}} \in \mathrm{R}^k} \ \tfrac{1}{2} \left\| \begin{bmatrix} \mathbf{B}_k \\ \mathbf{I} \end{bmatrix} \bar{\mathbf{y}} - \begin{bmatrix} \beta_1 \mathbf{e}_1 \\ 0 \end{bmatrix} \right\|_2^2 .$$

Generalized CRAIG

By contrast with generalized LSQR, we solve the SQD subsystem

$$\begin{bmatrix} \mathbf{I}_k & \mathbf{B}_k \\ \mathbf{B}_k^T & -I_k \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_k \\ \bar{\mathbf{y}}_k \end{bmatrix} = \begin{bmatrix} \beta_1 \mathbf{e}_1 \\ 0 \end{bmatrix}$$

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Following Saunders (1995) and Paige (1974), we compute an LQ factorization to the k-by-$2k$ matrix $\begin{bmatrix} \mathbf{B}_k & \mathbf{I}_k \end{bmatrix}$ by applying $2k-1$ Givens rotations that zero out the identity block.

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$$\begin{bmatrix} \mathbf{B}_k & \mathbf{I}_k \end{bmatrix} \mathbf{Q}_k^T = \begin{bmatrix} \hat{\mathbf{B}}_k & 0 \end{bmatrix} \qquad \mathbf{Q}_k^T \mathbf{Q}_k = \mathbf{I}$$

where

$$\hat{\mathbf{B}}_k := \begin{bmatrix} \hat{\alpha}_1 & & & \\ \hat{\beta}_2 & \hat{\alpha}_2 & & \\ & \ddots & \ddots & \\ & & \hat{\beta}_k & \hat{\alpha}_k \end{bmatrix}.$$

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Generalized CRAIG

$$\beta_1 \mathbf{e}_1 = B_k \bar{\mathbf{y}}_k + \bar{\mathbf{x}}_k = \begin{bmatrix} B_k & I_k \end{bmatrix} \begin{bmatrix} \bar{\mathbf{y}}_k \\ \bar{\mathbf{x}}_k \end{bmatrix} =$$

$$\begin{bmatrix} \hat{B}_k & 0 \end{bmatrix} Q_k \begin{bmatrix} \bar{\mathbf{y}}_k \\ \bar{\mathbf{x}}_k \end{bmatrix} = \begin{bmatrix} \hat{B}_k & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{z}}_k \\ 0 \end{bmatrix} = \hat{B}_k \bar{\mathbf{z}}_k,$$

for some $\bar{\mathbf{z}}_k \in \mathrm{R}^k$: $\bar{\mathbf{z}}_k = (\zeta_1, \ldots, \zeta_k)$

Generalized CRAIG

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for some $\bar{\mathbf{z}}_k \in \mathrm{R}^k$: $\bar{\mathbf{z}}_k = (\zeta_1, \ldots, \zeta_k)$

$$\zeta_1 = \beta_1/\hat{\alpha}_1, \quad \zeta_{i+1} = -\hat{\beta}_{i+1}\zeta_i/\hat{\alpha}_{i+1}, \quad (i = 1, \ldots, k-1).$$

Generalized CRAIG: errors bound

Moreover, for $k = 1, \ldots, p := \min(m, n)$,

$$\|\mathbf{x}_k\|_{\mathbf{M}}^2 + \|\mathbf{y}_k\|_{\mathbf{N}}^2 = \sum_{i=1}^{k} \zeta_i^2,$$

$$\|\mathbf{x}^* - \mathbf{x}_k\|_{\mathbf{M}}^2 + \|\mathbf{y}^* - \mathbf{y}_k\|_{\mathbf{N}}^2 = \sum_{i=k+1}^{p} \zeta_i^2.$$

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As for generalized LSQR, we can estimate the error using the windowing technique and we can give a lower bound of the error by

$$\xi^2_{k,d} = \sum_{j=k+1}^{k+d+1} \zeta_i^2 \leq \|\mathbf{x}^* - \mathbf{x}_k\|^2_{\mathbf{M}} + \|\mathbf{y}^* - \mathbf{y}_k\|^2_{\mathbf{N}}$$

and $\|\mathbf{x}_k\|^2_{\mathbf{M}} + \|\mathbf{y}_k\|^2_{\mathbf{N}}$ by the lower bound $\sum_{j=1}^{k} \zeta_j^2$.

Generalized CRAIG: errors bound

Moreover, for $k = 1, \ldots, p := \min(m, n)$,

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As for GLSQR. If we know a lower bound of singular values we can use an approach inspired by the Gauss-Radau quadrature algorithm and similar to the one described in Golub-Meurant (2010).

Numerical experiments

We will focus on optimization problems:

$$\min_{x \in \mathbb{R}^n} g^T x + \frac{1}{2} x^T Q x \quad \text{s. t.} \quad Dx = d, \quad x \geq 0,$$

where $g \in \mathbb{R}^n$ and $Q = Q^T \in \mathbb{R}^{n \times n}$ is positive semi-definite, and result in linear systems with coefficient matrix

$$\begin{bmatrix} Q + X^1 Z + \rho I & D^T \\ D & -\delta I \end{bmatrix}$$

where $\rho > 0$ and $\delta > 0$ are regularization parameters.

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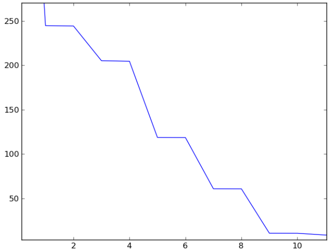
Numerical experiments MINRES

This is a blow-up of some iterations



Numerical experiments GLSQR

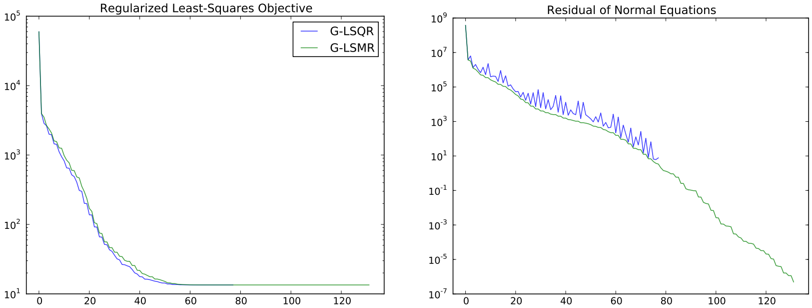


Figure: Problem DUAL1 $(255, 171)$.

Numerical experiments GLSQR

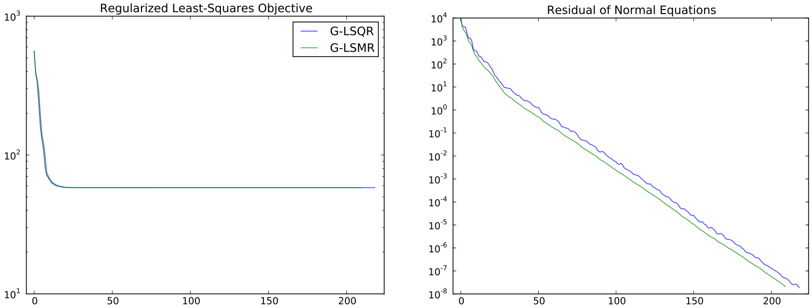
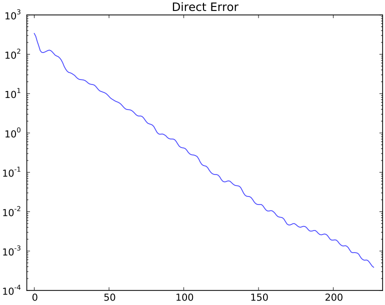


Figure: Problem MOSARQP1 $(5700, 3200)$.

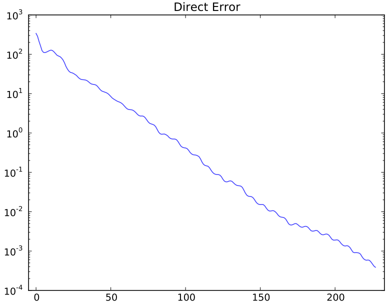
Numerical experiments GCraig

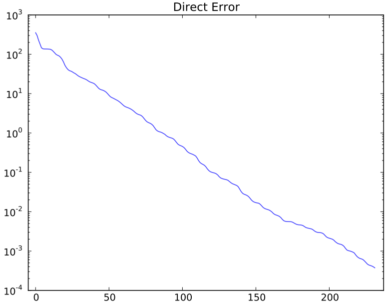
$d = 5, 10$



Numerical experiments GCraig

$d = 10, 15$





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- A. and Orban "Iterative methods for symmetric quasi definite systems" in preparation. WORK IN PROGRESS