Domain decomposition methods applied to flow computation in heterogeneous porous media

IFSIC master internship

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Sub-surface water

- Difficulty to have direct observation
- Environmental protection : Prediction and anticipation
- Modelisation

Porous flow

Discretization

Domain decomposition

Results

Equations

- Darcy's law :

$$K\nabla(h) = v$$

- Mass conservation

$$\nabla.(v) = 0$$

- Then :

$$\nabla.(K\nabla(h)) = 0$$

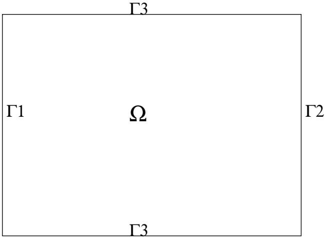
Boundary conditions

- Dirichlet condition :

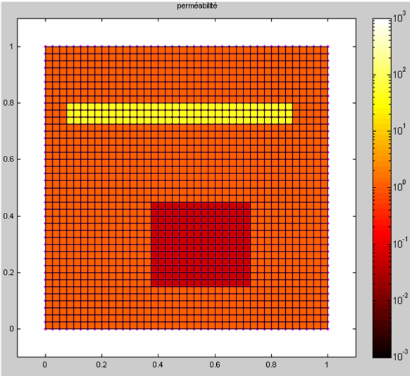
$$h = constant$$

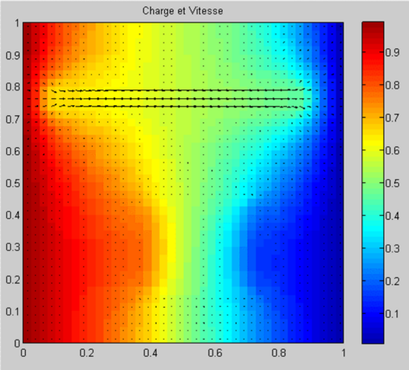
- Homogenous Neumann condition :

$$\frac{\partial h}{\partial n} = 0$$



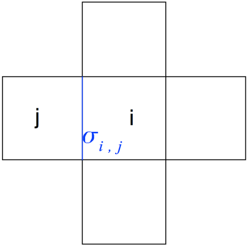
Permeability and charge





Finite volume

- $\int_{M_i} \nabla.(K \nabla h(x,y)) dxdy = 0$
- $\int_{\partial M_i} K \nabla h(x,y).n(x,y) d_\gamma (x,y) = 0$
- $\sum_j \int_{\sigma_{i,j}} K \nabla h(x,y).n(x,y) d_\gamma (x,y) = 0$
- $\sum_j F_{M_i,\sigma_{i,j}} = 0$
- $F_{M_i,\sigma_{i,j}} = \frac{2K_i K_j}{K_i + K_j}(h_j - h_i)$



System constructions

- $\sum_{j=neighbor} A_{i,j}h_j + A_{i,i}h_i = b_i$
- $A_{i,j} = -mean(K_j, K_i)$
- $A_{i,i} = \sum_{j=neighbor} mean(K_j, K_i)$
- $b_i = 0$

Boundary conditions

- $A_{i,i} = 1$
- $b_i = constant$
- $b_j = A_{i,j}$
- $A_{i,j} = A_{j,i} = 0$

Domain decomposition

- $Ax = b$, with A Symmetric Positive Definite
- Divide and conquer
- Speedup and parallelism
- With or without overlap

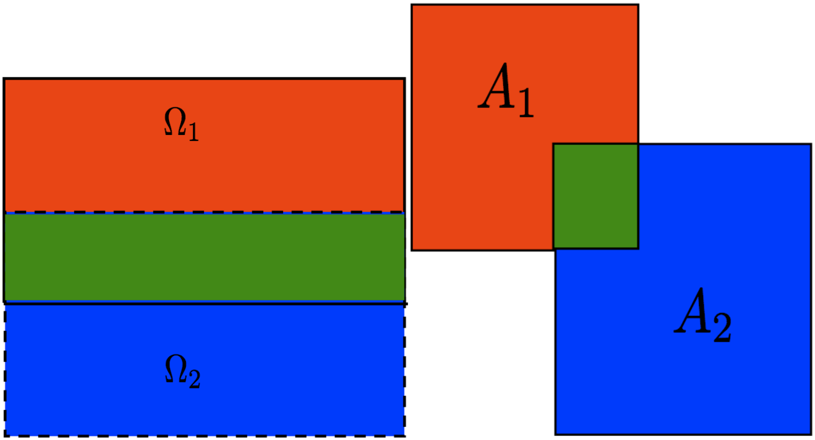
Standard iterative method

$$Ax^c = b - Ax^k = r^k$$

$$x^{k+1} = x^k + x^c$$

x^k : Approximation after k steps
r^k : Residual after k steps
x^c : Correction

Overlapping decomposition



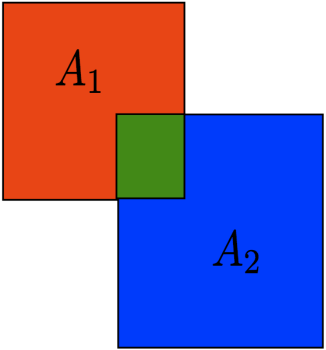
Multiplicative Schwarz

- $B_i = R_i^T (R_i A R_i^T)^{-1} R_i$

  $B_1 = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$

- $x^{2k} = x^{2k-1} + B_1(b - Ax^{2k-1})$

  $x^{2k+1} = x^{2k} + B_2(b - Ax^{2k})$

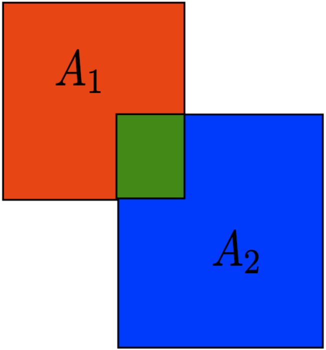


Additive Schwarz

- $B_i = R_i^T (R_i A R_i^T)^{-1} R_i$

  $B_1 = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$

- $x^{k+1} =$
  $x^k + B_1(b - Ax^k) + B_2(b - Ax^k)$



Preconditioning

- Preconditioning for PCG
- Equivalent system resolution :

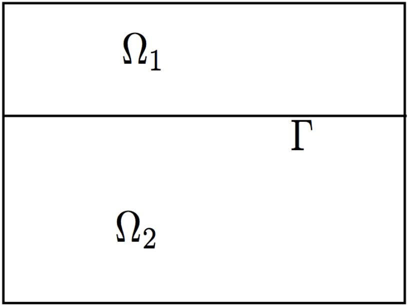
$$M^{-1}Ax = M^{-1}b$$

- Additive Schwarz :

$$M^{-1} = B_1 + B_2$$

Non-overlapping decomposition

$$\begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix}$$



Schur complement

- $$\begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$
- $$x_1 = A_{11}^{-1}(b_1 - A_{13}x_3)$$
  $$x_2 = A_{22}^{-1}(b_2 - A_{23}x_3)$$

Schur complement

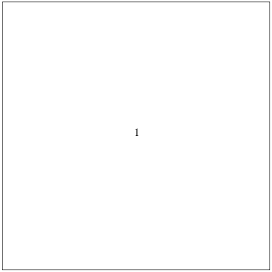
- With
$$S = A_{33} - A_{13}^T A_{11}^{-1} A_{13} - A_{23}^T A_{22}^{-1} A_{23}$$
- And
$$\bar{b}_3 = b_3 - A_{13}^T A_{11}^{-1} b_1 - A_{23}^T A_{22}^{-1} b_2$$
- We have :
$$S x_3 = \bar{b}_3$$
- Benefits :
  - Smaller system
  - PCG resolution : only matrix-vector product
    $S v = (A_{33} - A_{13}^T A_{11}^{-1} A_{13} - A_{23}^T A_{22}^{-1} A_{23}) v$

Neumann-Neumann

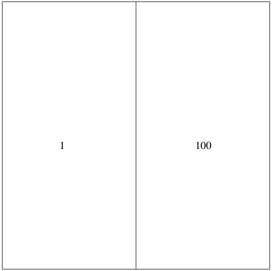
- Preconditioning using Schur complement
- S_i : contribution of A_{ii} to S

$$M^{-1} = \frac{1}{2}(S_1^{-1} + S_2^{-1})$$

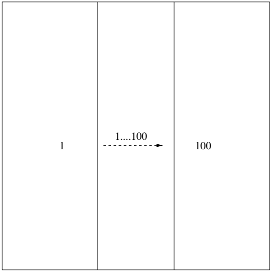
Simple models



Homogenous



Without transition

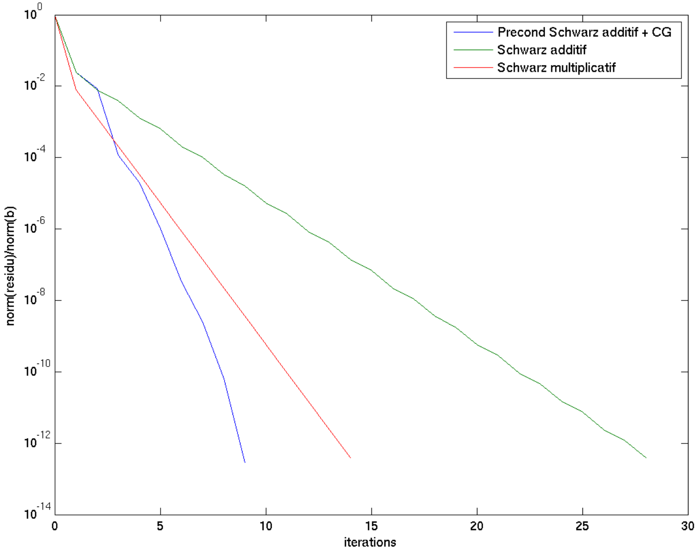


Progressive transition

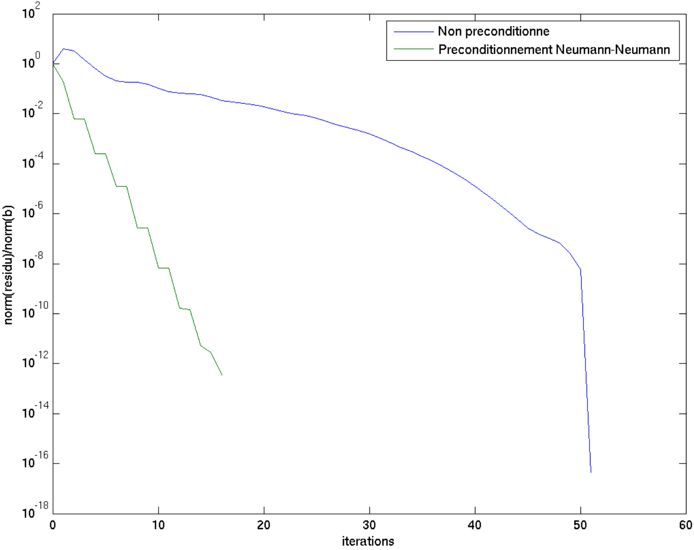
First tests

- With overlap :
  - Multiplicative Schwarz
  - Additive Schwarz
  - PCG with additive Schwarz preconditioning
- Without overlap:
  - Schur complement resolution
  - With and without Neumann-Neumann preconditioning

Results with overlap



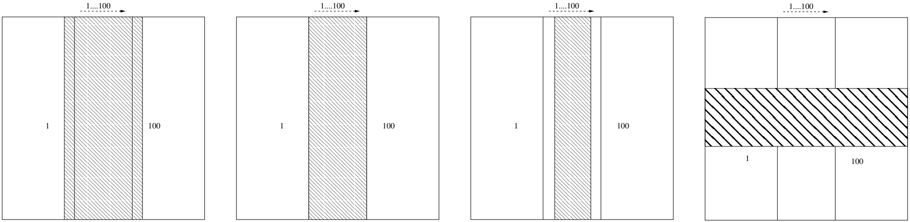
Results without overlap



Selected methods

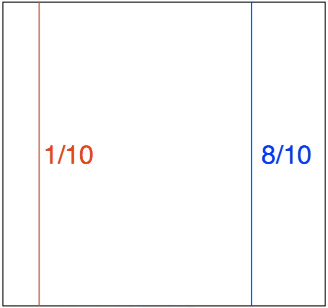
- With overlap :
  - Multiplicative Schwarz
  - PCG with additive Schwarz preconditioning
- Without overlap :
  - Schur complement resolution with Neumann-Neumann preconditioning

Overlaps



Sizes and models

- 3 sizes
  - 50x50, 100x100, 150x150
- Two heterogeneous zones :
  - with or without transition zone
  - K=1 and K=100, K=1 and K=2



heterogeneity positions

Multiplicative Schwarz

Hétérogénéité	Petite largeur		Moyenne largeur		Grande largeur		Horizontal	
1	98	10,2	53	6,13	36	4,60	22	3,33
2	98	10,2	55	6,36	37	4,73	38	2,91
3	98	10,2	55	6,34	37	4,71	32	2,46
4	98	10,2	23	2,75	8	1,17	26	2,02
5	9	1,08	7	0,95	6	0,92	21	1,64
6	16	1,81	6	0,85	6	0,92	16	1,27
7	27	2,93	14	1,75	9	1,29	13	1,05
8	35	3,75	18	2,20	12	1,65	14	1,12
9	41	4,38	22	2,64	15	2,03	16	1,28

PCG and additive Schwarz

Hétérogénéité	Petite largeur		Moyenne largeur		Grande largeur		Horizontal	
1	8	1,08	7	1,06	6	1,01	11	1,02
2	8	1,09	7	1,06	6	1,02	10	0,94
3	9	1,19	7	1,06	6	1,01	10	0,95
4	9	1,20	5	0,83	3	0,64	9	0,86
5	3	0,55	3	0,59	3	0,64	9	0,86
6	4	0,66	3	0,59	3	0,64	9	0,86
7	5	0,77	4	0,71	3	0,64	9	0,87
8	6	0,88	5	0,83	4	0,77	9	0,86
9	9	1,20	7	1,06	6	1,02	10	0,95

Schur complement

Hétérogénéité	Vertical		Horizontal	
1	4	1,50	14	2,06
2	4	1,28	14	2,06
3	4	1,29	15	2,17
4	8	2,02	15	2,17
5			13	1,95
6	7	1,85	15	2,18
7	6	1,65	15	2,20
8	5	1,49	15	2,17
9	4	1,29	15	2,17

Perspectives

- Only two sub-domains
  - Multiple sub-domains
- Very simple models
  - More zones
  - Random distribution
- Best choice for each method
  - Comparison between methods
  - Mixed resolution