TOTAL

UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR

Analysis of the performance of the Interior Penality Discontinuous Galerkin method

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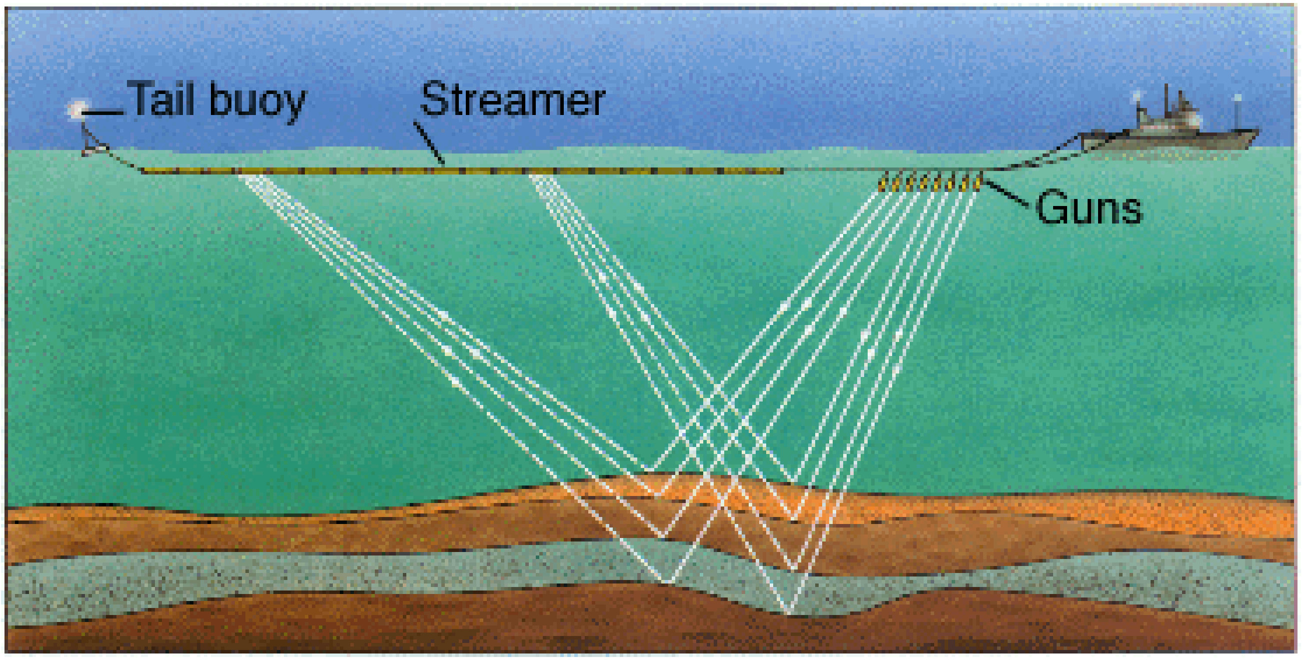
The seismic exploration



The Reverse Time Migration technique (RTM):

- Propagation of the wave field
- Retro-propagation of the data
- Application of an imaging condition

Motivation of the work

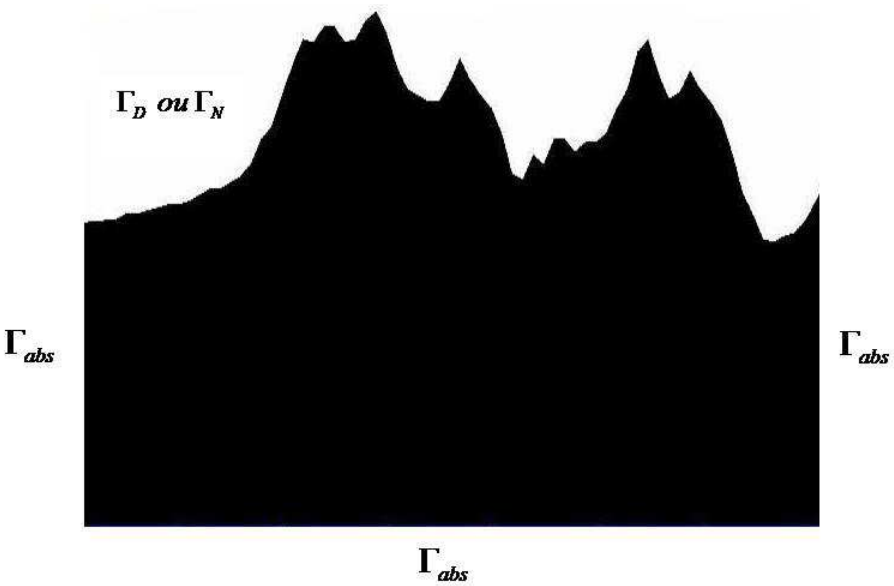
- Goal of the thesis: Migration including topography effects with wave equation in 3D (RTM)
- Both the accuracy and the computational cost of the numerical method to solve the direct problem are crucial
- Our choice: a finite element method which uses meshes adapted to the topography of the domain: the Interior Penality Discontinuous Galerkin method (IPDG)

Outline

- Presentation of the IPDG method
- Comparison 1D with a spectral finite element method (SEM)
- Comparison 2D with SEM and analytic solution
- Results of the propagation in an irregular top domain with IPDG
- Conclusion and ongoing works

Initial problem

$$
\begin{cases}
\dfrac{1}{\mu}\dfrac{\partial^2 u}{\partial t^2} - \nabla.\left(\dfrac{1}{\rho}\nabla u\right) = f \quad in \quad I \times \Omega \\
u(0,x) = 0 \quad in \quad \Omega \\
\dfrac{\partial u}{\partial t}(0,x) = 0 \quad in \quad \Omega \\
u = 0 \quad on \quad \Gamma_D \\
\nabla u \cdot n = 0 \quad on \quad \Gamma_N \\
\dfrac{1}{\sqrt{\mu}}\dfrac{\partial u}{\partial t} + \dfrac{1}{\sqrt{\rho}}\nabla u \cdot n = 0 \quad on \quad \Gamma_{abs}
\end{cases}
$$



Notations

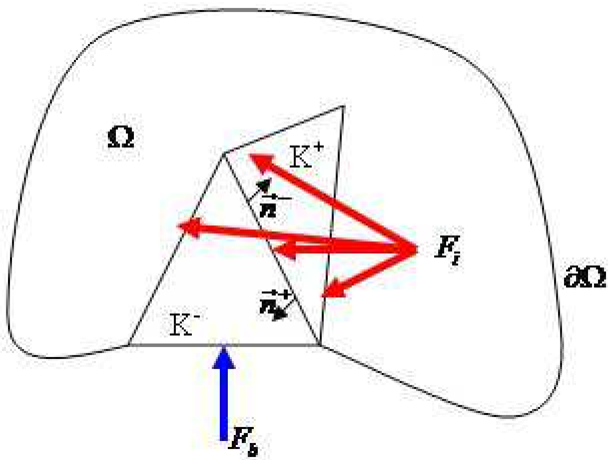
- $I =]0,T[$: a finite time interval
- Ω : a bounded domain in IR^d, $d = 2,3$
- $\Gamma_D \cup \Gamma_N \cup \Gamma_{abs} = \partial\Omega$
- ρ : the density of the medium satisfying $0 < \rho_* < \rho(x) < \rho^* < \infty$
- μ : the compressibility satisfying $0 < \mu_* < \mu(x) < \mu^* < \infty$
- n : the unit outward normal of Ω

The IPDG method

- Method proposed by Douglas and Dupont in the 70's
- Applied to the wave equation by Grote, Schneebeli and Schötzau in 2005

- Continuity is weakly enforced across interfaces by adding bilinear forms, so-called fluxes
- Method based on meshes made of triangles in 2D or tetrahedra in 3D

Notations -1-



T_h : a shape-regular mesh of Ω composed by elements (triangles) K; $\overline{\Omega} = \bigcup_{K \in T_h} \overline{K}$

F_i an interior face of T_h define by two elements ; $F_i = \partial K^+ \bigcap \partial K^-$

$\mathrm{F_i}$ the set of all F_i

F_b a boundary face of T_h ; $F_b = \partial K \bigcap \partial \Omega$

$\mathrm{F_b}$ the set of all F_b

$\mathrm{F_h} = \mathrm{F_i} \bigcup \mathrm{F_b}$

$n^{\pm}$ the unit outward vectors on the boundaries $\partial K^{\pm}$

$v^{\pm}$ the traces of a fonction v on $K^{\pm}$

Notations -2-

$[[v]] = v^+ n^+ + v^- n^-$: the jump of v at $x \in$ F ; F $\in$ $\mathrm{F_i}$

$\{\{v\}\} = \left(v^+ + v^-\right)/2$: the average of v at $x \in$ F ; F $\in$ $\mathrm{F_i}$

$[[v]] = vn$: the jump of v at $x \in$ F ; F $\in$ $\mathrm{F_b}$

$\{\{v\}\} = v$: the average of v at $x \in$ F ; F $\in$ $\mathrm{F_b}$

For a smooth vector-valued function, we analogously define

the jump and the average

$V_l^h = \left\{ v \in L^2(\Omega) : v_{|K} \in P_l(K)\, \forall K \in T_h \right\}$: the finite element space

Space discretization -1-

After space discretization, we obtain the scheme:

$$M\frac{\partial^2 U_h}{\partial t^2} + B\frac{\partial U_h}{\partial t} + KU_h = F_h$$

where

U_h is the vector defined by the composants of $u_h \in V_l^h$

The mass matrix M is block-diagonal and its coefficients are:

$$M_{ij} = \sum_{K\in T_h} \int_K \frac{1}{\mu} v_i v_j dx, \quad \forall v_i, v_j \in V_l^h$$

The matrix B is equal to zero except for Γ_{abs}'s elements:

$$B_{ij} = \sum_{F_b \in \mathrm{F_b} \cap \Gamma_{abs}} \int_{F_b} \frac{1}{\sqrt{\mu\rho}} v_i v_j dF, \quad \forall v_i, v_j \in V_l^h$$

Space discretization -2-

The stiffness matrix K is symmetric and has for entries the terms:

$$
\begin{aligned}
K_{ij} = & \sum_{K \in T_h} \int_K \frac{1}{\rho} \nabla v_i \cdot \nabla v_j \, dx - \sum_{F_i \in \mathrm{F_i}} \int_{F_i} \left[\left[v_i \right]\right] \cdot \left\{\left\{ \frac{1}{\rho} \nabla v_j \right\}\right\} dF \\
& - \sum_{F_i \in \mathrm{F_i}} \int_{F_i} \left[\left[v_j \right]\right] \cdot \left\{\left\{ \frac{1}{\rho} \nabla v_i \right\}\right\} dF + \sum_{F_i \in \mathrm{F_i}} \int_{F_i} \gamma \left[\left[v_i \right]\right] \cdot \left[\left[v_j \right]\right] dF \\
& - \sum_{F_b \in \mathrm{F_b}} \int_{F_b} \left[\left[v_i \right]\right] \cdot \left\{\left\{ \frac{1}{\rho} \nabla v_j \right\}\right\} dF - \sum_{F_b \in \mathrm{F_b}} \int_{F_b} \left[\left[v_j \right]\right] \cdot \left\{\left\{ \frac{1}{\rho} \nabla v_i \right\}\right\} dF \\
& + \sum_{F_b \in \mathrm{F_b}} \int_{F_b} \gamma \left[\left[v_i \right]\right] \cdot \left[\left[v_j \right]\right] dF, \quad \forall v_i, v_j \in V_l^h
\end{aligned}
$$

Space discretization -3-

The function γ penalizes the jump of v_i and v_j on T_h's faces.

It is defined by: $\forall \mathrm{F} \in \mathrm{F}_\mathrm{h} = \mathrm{F}_\mathrm{i} \cup \mathrm{F}_\mathrm{b}$, $\gamma_{|\mathrm{F}} = \alpha c_{\mathrm{max}} h_{\mathrm{min}}^{-1}$

The source vector F_h has for components the terms:

$$F_{h,i} = \sum_{K \in T_h} \int_K f v_i dx$$

IPDG’s advantages

- Meshes made of triangles in 2D or tetrahedra in 3D. Thus the topography of the computational domain is easily discretized.

- The representation of the solution is quasi-explicit because the mass matrix is block-diagonal.

  To compute easily its coefficients, we use an exact quadrature formula which does not hamper the order of convergence.

Comparison 1D IPDG versus SEM

ORDER 2	Uniform mesh		Random mesh	
	IPDG	SEM	IPDG	SEM
nb_ddl = 90	dx = 0.200 dt = 0.0808 err = 24.9565	dx = 0.100 dt = 0.0990 err = 0.2700	dx = 0.200 dt = 0.0761 err = 24.9720	dx = 0.100 dt = 0.0913 err = 1.7403
nb_ddl = 180	dx = 0.100 dt = 0.0404 err = 11.0781	dx = 0.0500 dt = 0.0495 err = 0.0676	dx = 0.100 dt = 0.0380 err = 10.9472	dx = 0.0500 dt = 0.0448 err = 0.5149
nb_ddl = 360	dx = 0.0500 dt = 0.0202 err = 2.9120	dx = 0.0250 dt = 0.0247 err = 0.0169	dx = 0.0500 dt = 0.0190 err = 2.8747	dx = 0.0250 dt = 0.0224 err = 0.1261
nb_ddl = 540	dx = 0.0333 dt = 0.0135 err = 1.3004	dx = 0.0167 dt = 0.0165 err = 0.0075	dx = 0.0333 dt = 0.0127 err = 1.2836	dx = 0.0167 dt = 0.0149 err = 0.0565
nb_ddl = 720	dx = 0.0250 dt = 0.0101 err = 0.7325	dx = 0.0125 dt = 0.0124 err = 0.0042	dx = 0.0250 dt = 0.0095 err = 0.7219	dx = 0.0125 dt = 0.0112 err = 0.0318

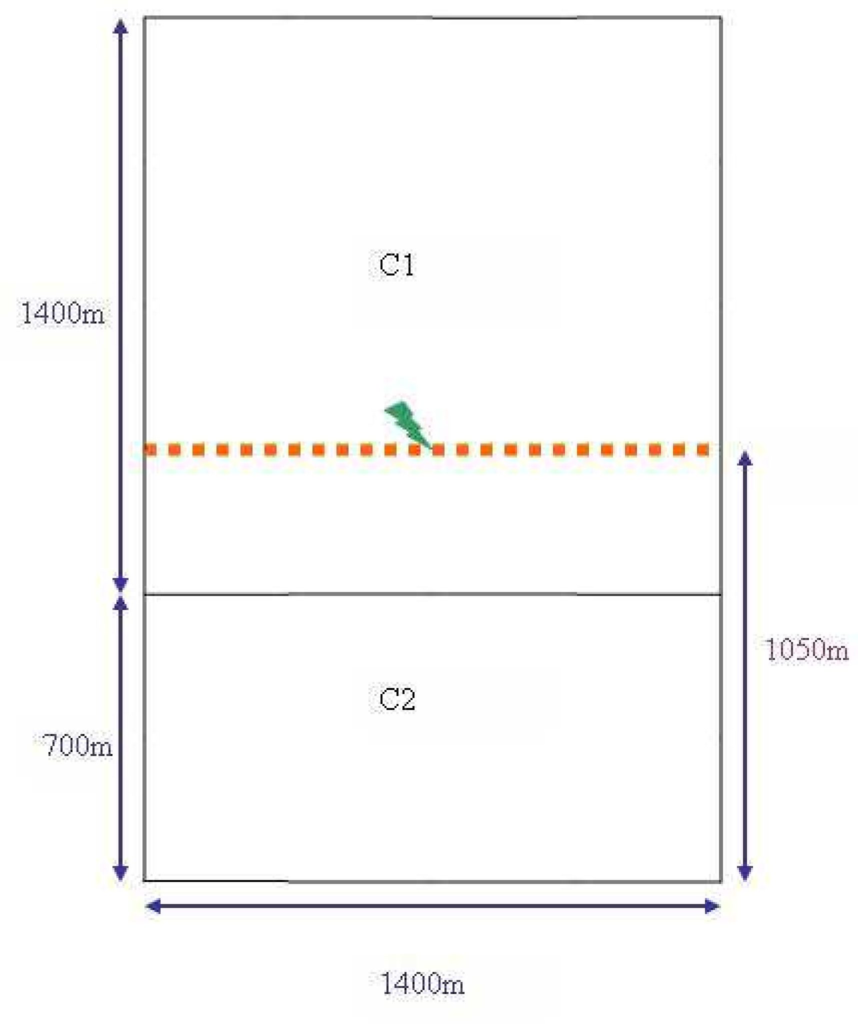
Comparison 1D IPDG versus SEM

ORDER 3	Uniform mesh		Random mesh	
	IPDG	SEM	IPDG	SEM
nb_ddl = 180	dx = 0.1500 dt = 0.0347 err = 1.2187	dx = 0.100 dt = 0.0400 err = 1.5961	dx = 0.1500 dt = 0.0320 err = 1.0594	dx = 0.100 dt = 0.0363 err = 1.3152
nb_ddl = 360	dx = 0.0750 dt = 0.0174 err = 0.3040	dx = 0.0500 dt = 0.0200 err = 0.4018	dx = 0.0750 dt = 0.0159 err = 0.2570	dx = 0.0500 dt = 0.0182 err = 0.3327
nb_ddl = 720	dx = 0.0375 dt = 0.0087 err = 0.0760	dx = 0.0250 dt = 0.0100 err = 0.1007	dx = 0.0375 dt = 0.0079 err = 0.0632	dx = 0.0250 dt = 0.0091 err = 0.0827
nb_ddl = 1080	dx = 0.0250 dt = 0.0058 err = 0.0338	dx = 0.0167 dt = 0.0067 err = 0.0448	dx = 0.0250 dt = 0.0053 err = 0.0281	dx = 0.0167 dt = 0.0060 err = 0.0368
nb_ddl = 1440	dx = 0.0187 dt = 0.0043 err = 0.0190	dx = 0.0125 dt = 0.0050 err = 0.0252	dx = 0.0187 dt = 0.0040 err = 0.0159	dx = 0.0125 dt = 0.0125 err = 0.0207

Comparison 1D IPDG versus SEM

ORDER 4	Uniform mesh		Random mesh	
	IPDG	SEM	IPDG	SEM
nb_ddl = 180	dx = 0.200 dt = 0.0307 err = 0.4865	dx = 0.1500 dt = 0.0348 err = 0.6263	dx = 0.200 dt = 0.0305 err = 0.4795	dx = 0.1500 dt = 0.0335 err = 0.5916
nb_ddl = 360	dx = 0.100 dt = 0.0153 err = 0.1215	dx = 0.0750 dt = 0.0174 err = 0.1565	dx = 0.100 dt = 0.0152 err = 0.1195	dx = 0.0750 dt = 0.0168 err = 0.1460
nb_ddl = 720	dx = 0.0500 dt = 0.0077 err = 0.0304	dx = 0.0375 dt = 0.0087 err = 0.0391	dx = 0.0500 dt = 0.0076 err = 0.0299	dx = 0.0375 dt = 0.0084 err = 0.0362
nb_ddl = 1080	dx = 0.0333 dt = 0.0051 err = 0.0135	dx = 0.0250 dt = 0.0058 err = 0.0174	dx = 0.0333 dt = 0.0051 err = 0.0132	dx = 0.0250 dt = 0.0056 err = 0.0163
nb_ddl = 1440	dx = 0.0250 dt = 0.0038 err = 0.0076	dx = 0.0187 dt = 0.0044 err = 0.0098	dx = 0.0250 dt = 0.0038 err =0.0074	dx = 0.0187 dt = 0.0042 err = 0.0091

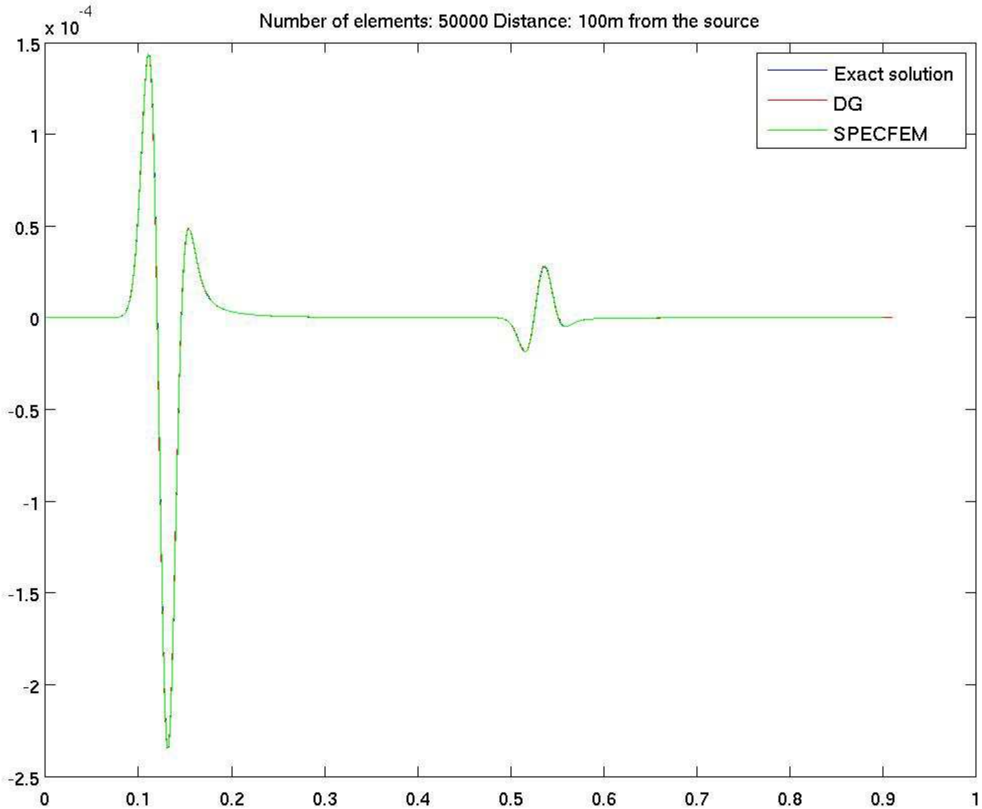
Comparison 2D IPDG/SEM/Exact solution



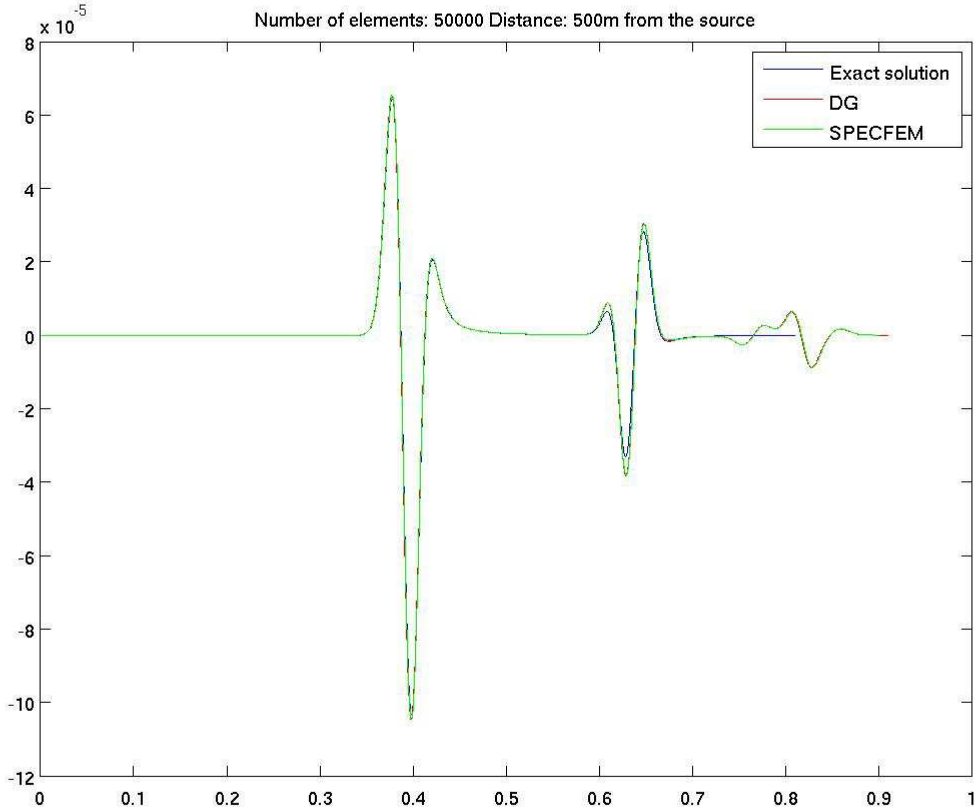
- [0 1400] x [0 2100]
- C1=1500m/s
- C2=3000m/s
- Source position: (700,1050)
- First derivative of a Gaussian
- *f* =20Hz
- Dirichlet condition on the top, absorbing condition elsewhere
- Time propagation: 0.9s
- Position of the receivers:

  (5*i,1050) i=1,…,280

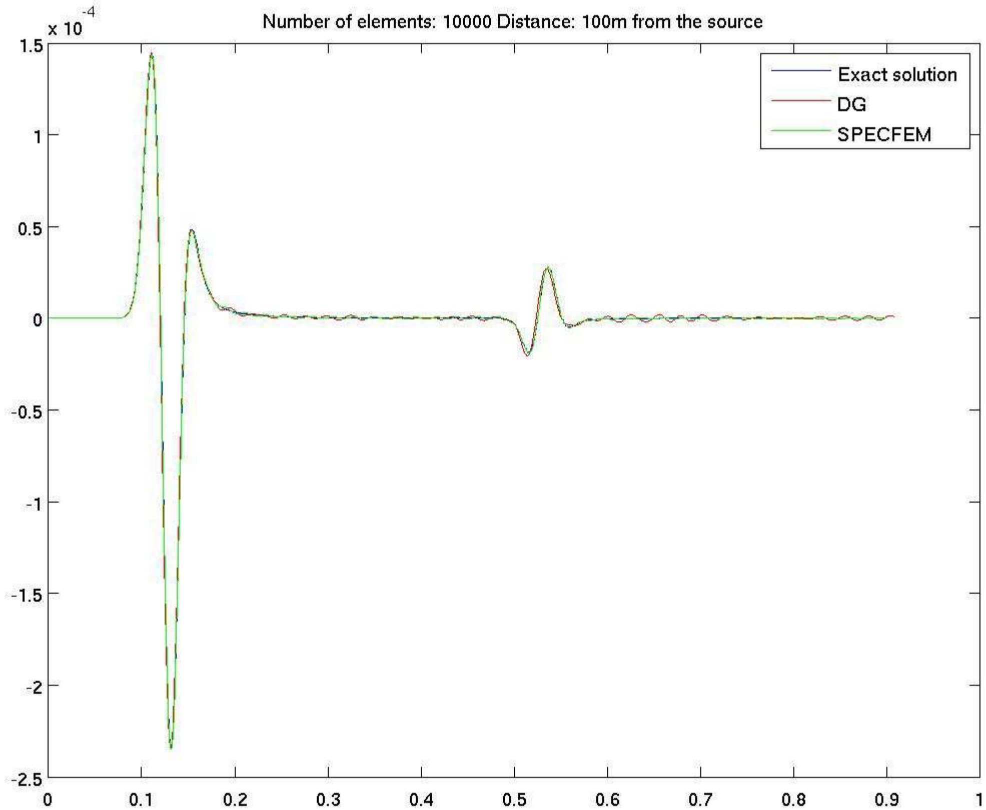
Results for a fine mesh (52 pts/λ) -1-



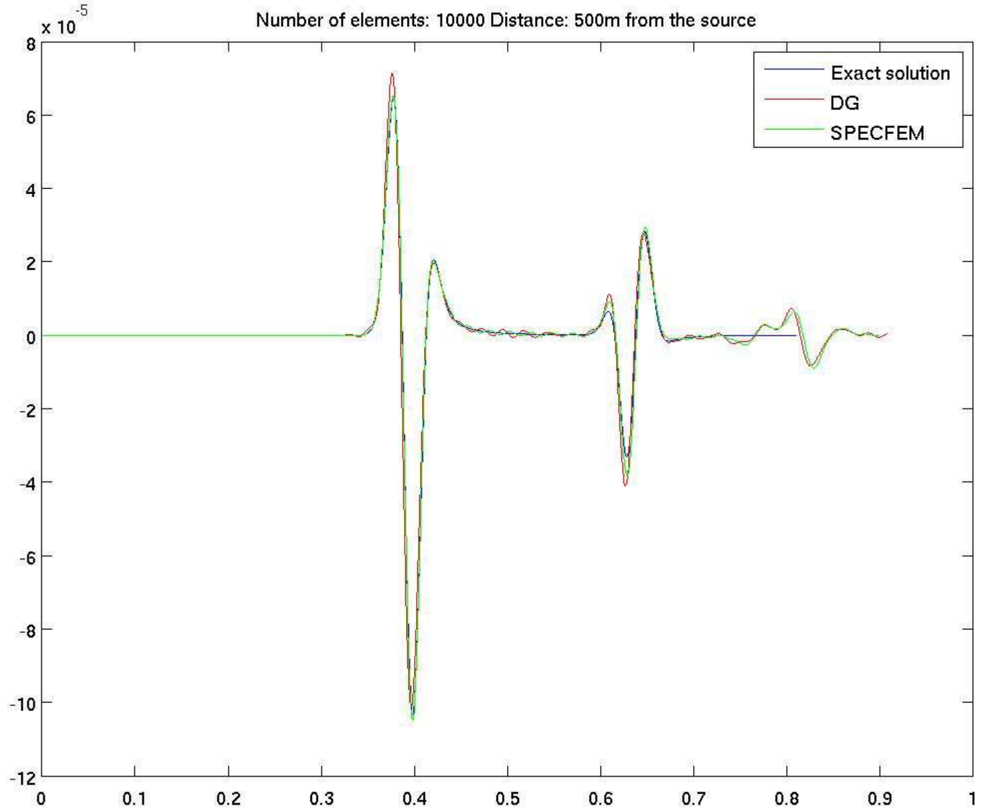
Results for a fine mesh (52 pts/λ) -2-



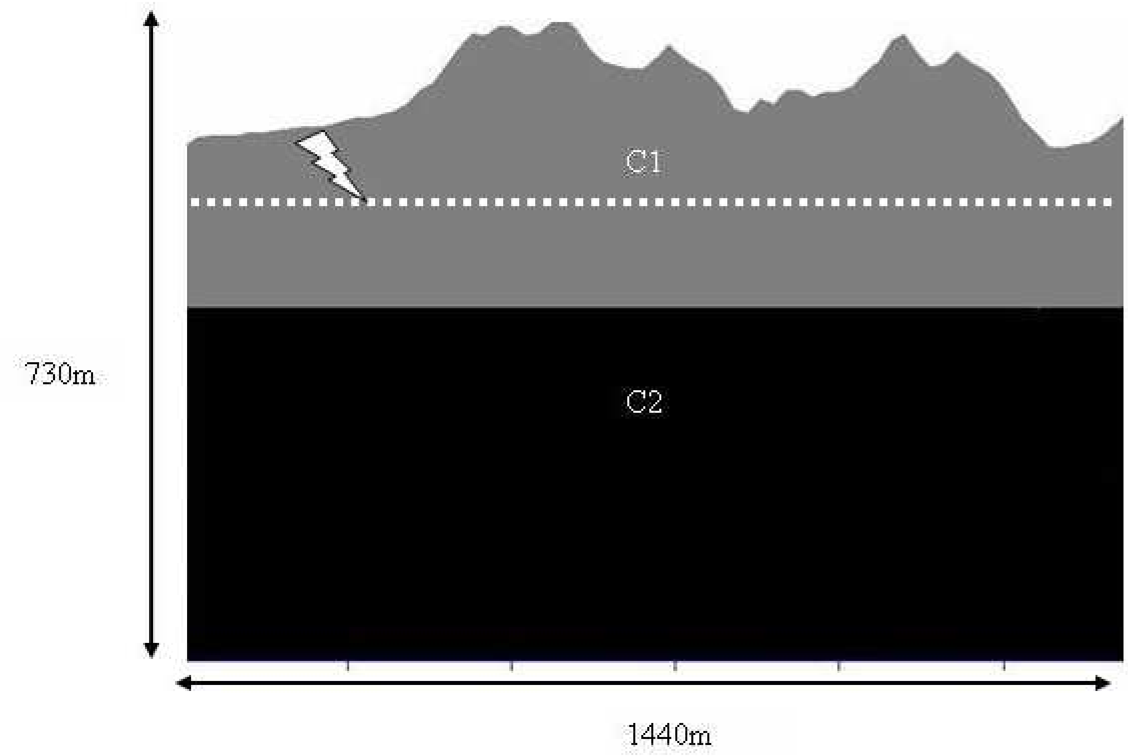
Results for a coarse mesh (20 pts/λ) -1-



Results for a coarse mesh (20 pts/λ) -2-



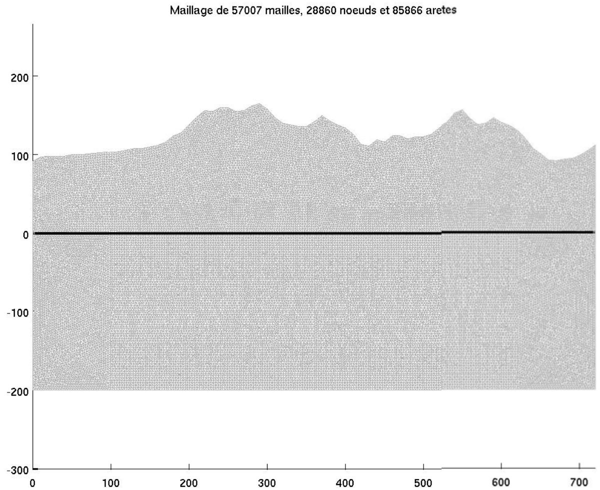
The foothill case



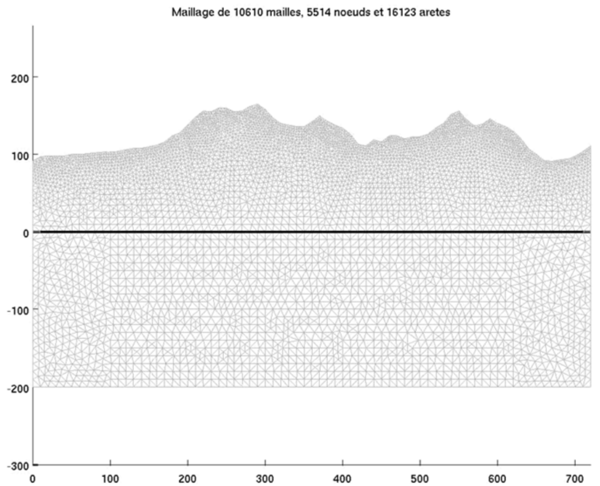
- [0 1440] x [0 730]
- C1=1500m/s
- C2=3000m/s
- Source position: (300,530)
- First derivative of a Gaussian
- *f* =20Hz
- Dirichlet condition on the top, absorbing condition elsewhere
- Time propagation: 1s
- Position of the receivers:

  (5*i,530) i=1,…,288

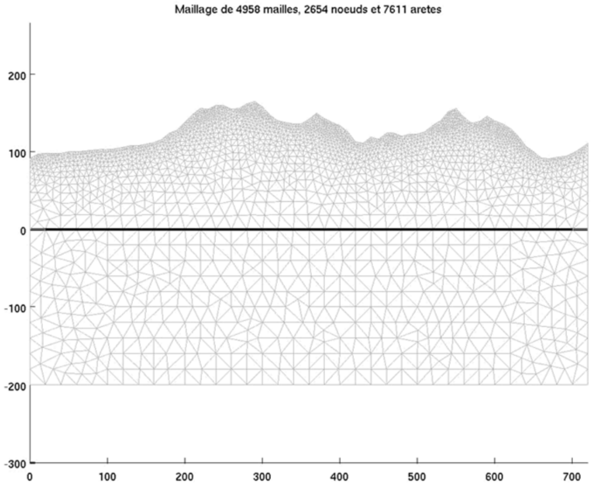
Influence of the size of the mesh for IPDG



Mesh n°1

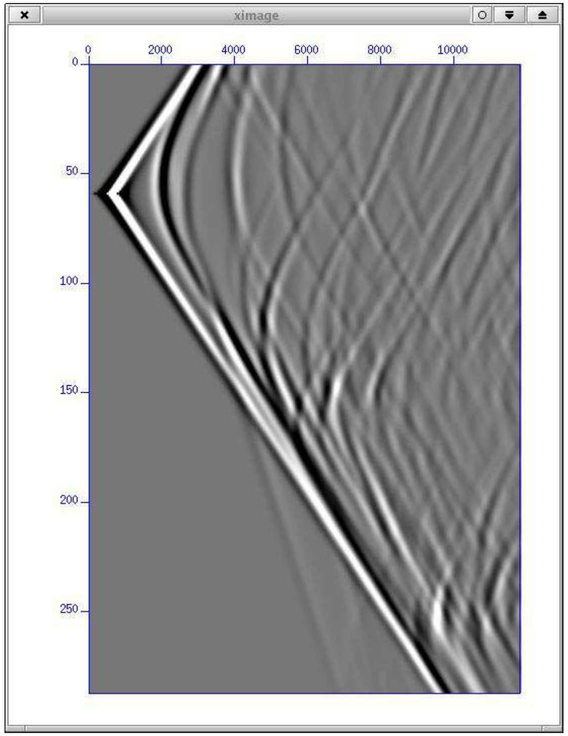


Mesh n°2

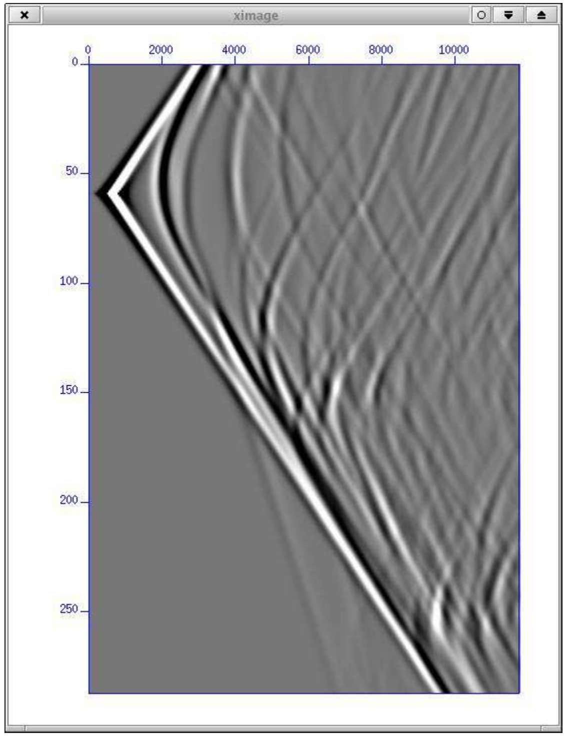


Mesh n°3

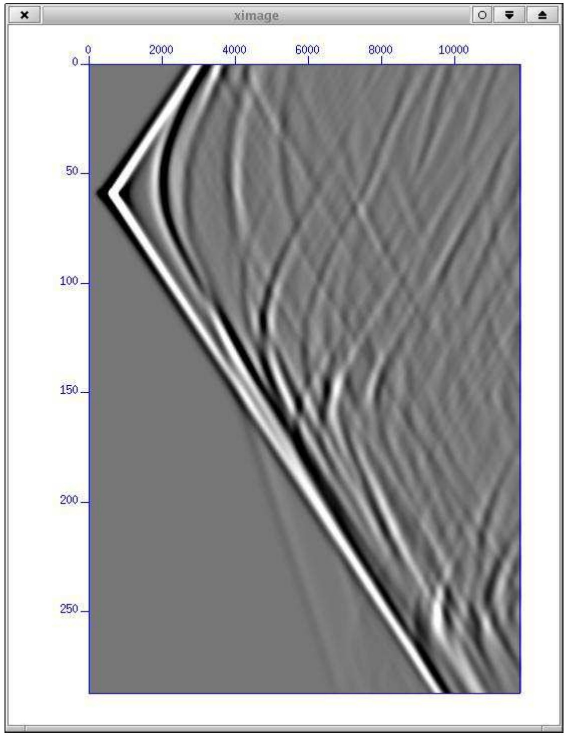
Seismograms



Mesh n°1
100 pts/ λ



Mesh n°2
30 pts/ λ



Mesh n°3
15 pts/ λ

Conclusions and ongoing works

- *Conclusions:*
  - Absorbing conditions must be improved
  - It's only necessary to have a fine mesh at the surface of the domain

- *Ongoing works:*
  - Analysis of the numerical dispersion
  - Improvement of the absorbing conditions
  - Implementation into the MigWE code
  - Local time stepping

Fonction test v_j telle que:

$$\left[\left[v_j\right]\right] \neq 0 \text{ et } \left[\left[\frac{1}{\rho}\nabla v_j\right]\right] = 0$$

$$K_{ij} = \sum_{K\in T_h}\int_K \frac{1}{\rho}\nabla v_i \cdot \nabla v_j dx - \sum_{F_i\in \mathrm{F_i}}\int_{F_i}\left[\left[v_i\right]\right]\cdot\left\{\left\{\frac{1}{\rho}\nabla v_j\right\}\right\}dF$$

$$- \sum_{F_i\in \mathrm{F_i}}\int_{F_i}\left[\left[v_j\right]\right]\cdot\left\{\left\{\frac{1}{\rho}\nabla v_i\right\}\right\}dF + \sum_{F_i\in \mathrm{F_i}}\int_{F_i}\gamma\left[\left[v_i\right]\right]\cdot\left[\left[v_j\right]\right]dF$$

$$- \sum_{F_b\in \mathrm{F_b}}\int_{F_b}\left[\left[v_i\right]\right]\cdot\left\{\left\{\frac{1}{\rho}\nabla v_j\right\}\right\}dF - \sum_{F_b\in \mathrm{F_b}}\int_{F_b}\left[\left[v_j\right]\right]\cdot\left\{\left\{\frac{1}{\rho}\nabla v_i\right\}\right\}dF$$

$$+ \sum_{F_b\in \mathrm{F_b}}\int_{F_b}\gamma\left[\left[v_i\right]\right]\cdot\left[\left[v_j\right]\right]dF$$