Combining interval and affine arithmetic with linear reformulation in deterministic global optimization

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Introduction

We consider global optimization of constrained non-convex problems in a deterministic way

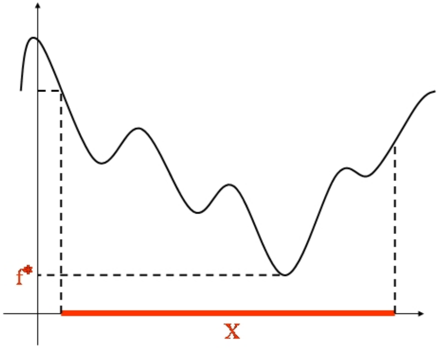
Problem

$$\begin{cases} \min\limits_{x \in X \subset \mathbb{R}^n} & f(x) \\ s.\ t. & \\ & g_i(x) \leq 0 \ ,\ \forall i \in \{1, \dots, n_g\} \\ & h_j(x) = 0 \ ,\ \forall j \in \{1, \dots, n_h\} \end{cases}$$

- Find better safe bound
- Eliminate intervals which do not contain global optimum

$\Longrightarrow$ Accelerate resolution of Interval Branch&Bound Algorithm (IBBA)

Interval Branch&Bound Algorithm (IBBA)



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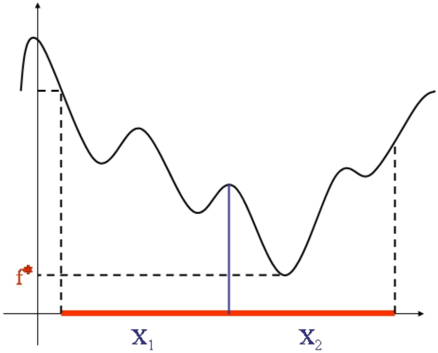

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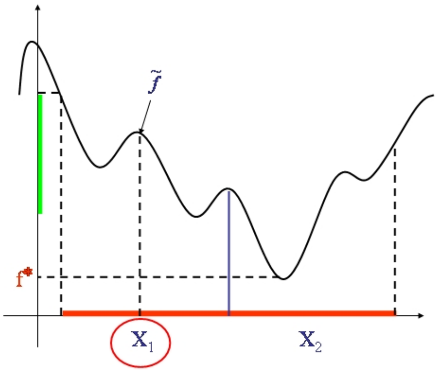

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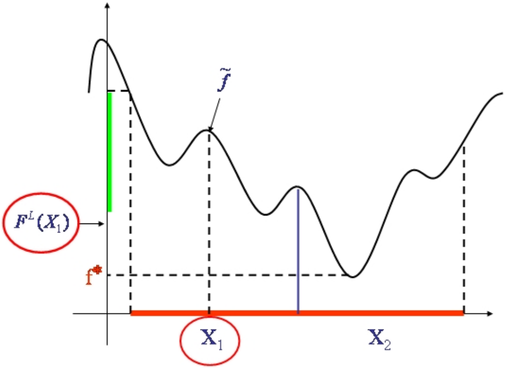

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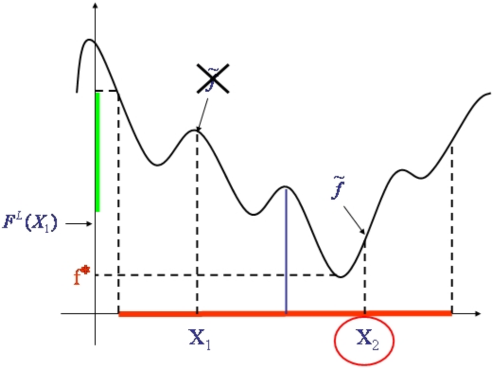

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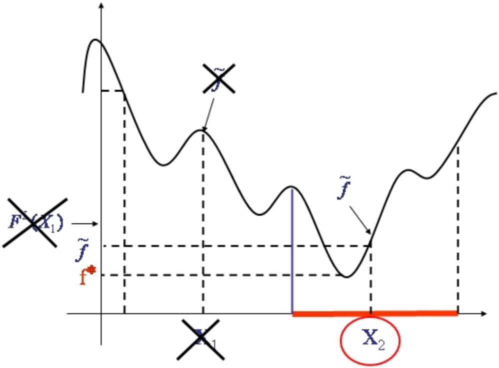

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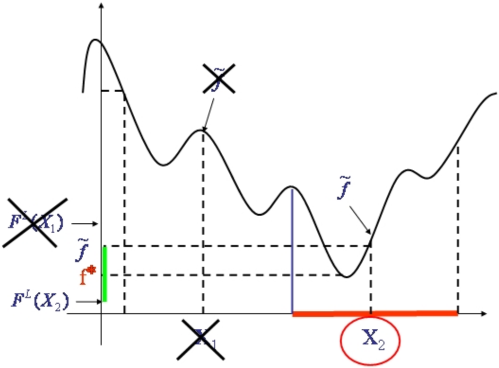

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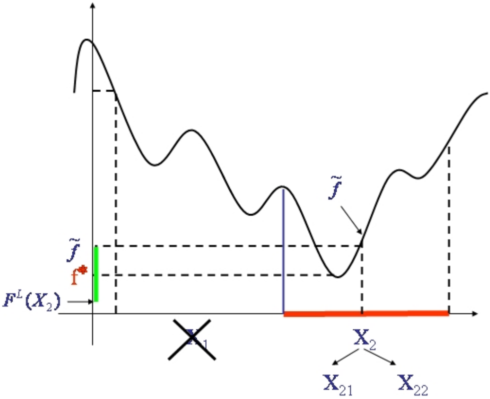

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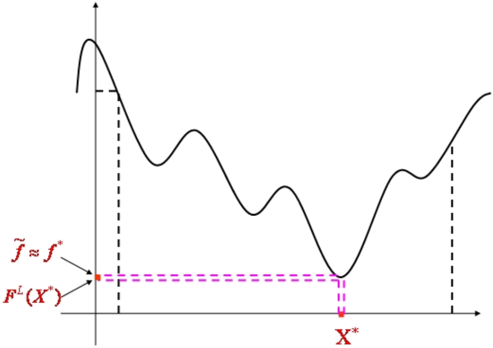

Contents

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Interval Arithmetic : RE Moore (1966)

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Example :$1/3 \rightarrow [0.33333333; 0.33333334]$

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$$f = 33.75y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 5.5y^8 + \frac{x}{2y}$$

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$x = 77617$ and $y = 33096$, with FORTRAN double precision,

we obtain : $f = 1.1726039400531...$

But the real value is : $f = -\frac{54767}{66192} = -0.8273960599$

Operator

$$\begin{cases} [a,b]+[c,d]=[a+c,b+d] \\ [a,b]-[c,d]=[a-d,b-c] \\ [a,b]\times[c,d]=[\min\{a\times c,a\times d,b\times c,b\times d\}, \\ \qquad\qquad\qquad \max\{a\times c,a\times d,b\times c,b\times d\}] \\ [a,b]\div[c,d]=[a,b]\times[\frac{1}{d},\frac{1}{c}] \ si\ 0\notin[c,d] \end{cases}$$

- Example :

$$\begin{aligned} [1;2]+[3;4] &= [4;6] \\ [1;2]-[3;4] &= [-3;-1] \\ [1;2]\times[3;4] &= [3;8] \\ [1;2]\div[3;4] &= [1/4;2/3] \\ [-1;1]\times([1;2]+[3;4]) &= [-6;4] \end{aligned}$$

Natural Extension

Theorem

The natural extension into intervals of an expression of a function provides lower and upper bounds of this function over the studied interval vector.

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In Interval Arithmetic, $\quad \forall x \in [1;2] \times [3;4] \ , \ f(x) \in [2;17]$

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In Interval Arithmetic, $\forall x \in [1;2] \times [3;4]\ ,\ f(x) \in [2;17]$

In real, $\forall x \in [1;2] \times [3;4]\ ,\ f(x) \in [6.75;13]$

Affine Arithmetic : MVA Andrade, J Comba, J Stolfi (1994)

Definition

All real numbers are represented by an affine form $\hat{x}$

$$\hat{x} = x_0 + \sum_{i=1}^{n} x_i \epsilon_i$$
$$\text{with } \forall i \in [1; n], x_i \in \mathbb{R} \text{ and } \epsilon_i = [-1; 1]$$

- Example : with $n = 1$

$$\begin{aligned} 1/3 &\rightarrow 0.333 + 0.001 * [-1; 1] \\ log(2) &\rightarrow 0.693 + 0.001 * [-1; 1] \end{aligned}$$

Affine Operator

$$\hat{x} \pm \hat{y} = (x_0 \pm y_0) + \sum_{i=1}^{n} (x_i \pm y_i)\epsilon_i$$

$$a \pm \hat{x} = (a \pm x_0) + \sum_{i=1}^{n} x_i \epsilon_i$$

$$a \times \hat{x} = a x_0 + \sum_{i=1}^{n} a x_i \epsilon_i$$

- Example : $A = [1; 3]$ and $B = [-2; 0]$

$$\begin{aligned} \hat{A} &\rightarrow 2 + \epsilon_1 \\ \hat{B} &\rightarrow -1 + \epsilon_2 \\ \hat{A} + \hat{B} &= 1 + \epsilon_1 + \epsilon_2 \end{aligned}$$

Non-Affine Operator

$$\begin{aligned} x \times y &= (x_0 + \sum_{i=1}^{n} x_i \epsilon_i) \times (y_0 + \sum_{i=1}^{n} y_i \epsilon_i) \\ &= x_0 y_0 + \sum_{i=1}^{n} (x_0 y_i + x_i y_0) \epsilon_i + \left(\sum_{i=1}^{n} \mid x_i \mid \times \sum_{i=1}^{n} \mid y_i \mid \right) \epsilon_{n+1} \end{aligned}$$

$$\hat{f}(\hat{x}) = \zeta + \alpha \hat{x} + \delta \epsilon_{n+1}$$

$$\textit{with } \alpha, \delta, \zeta \in \mathbb{R} \textit{ and } \hat{x} = x_0 + \sum_{i=1}^{n} x_i \epsilon_i$$

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$\Longrightarrow$ All non-affine operations add a new variable

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$$\forall x \in [1;2] \times [3;4]\ ,\ f(x) = x_1^2 + x_2^2 - x_1 x_2$$

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New Affine Forms : F. Messine (2002)

Difficulties with AF

- Non-affine operators $\Rightarrow$ add a new variable
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- AF2 : $\hat{x} = x_0 + \sum_{i=1}^{n} x_i \epsilon_i + e_{\pm} \epsilon_{n+1} + e_{+} \epsilon_{n+2} + e_{-} \epsilon_{n+3}$

  with $\epsilon_{n+1} = [-1; 1]$, $\epsilon_{n+2} = [0; 1]$ and $\epsilon_{n+3} = [-1; 0]$

Non-Affine Operator

Multiplication with AF1

$$
\begin{aligned}
\hat{x} \times \hat{y} &= (x_0 + \sum_{i=1}^{n+1} x_i \epsilon_i) \times (y_0 + \sum_{i=1}^{n+1} y_i \epsilon_i) \\
&= x_0 y_0 + \sum_{i=1}^{n} (x_0 y_i + x_i y_0) \epsilon_i + \\
&\left(x_0 y_{n+1} + x_{n+1} y_0 + \left(\sum_{i=1}^{n+1} \mid x_i \mid \times \sum_{i=1}^{n+1} \mid y_i \mid \right) \right) \epsilon_{n+1}
\end{aligned}
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With AF1, $f(x) \in [5.5; 13]$

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With AF2, $f(x) \in [6; 13]$

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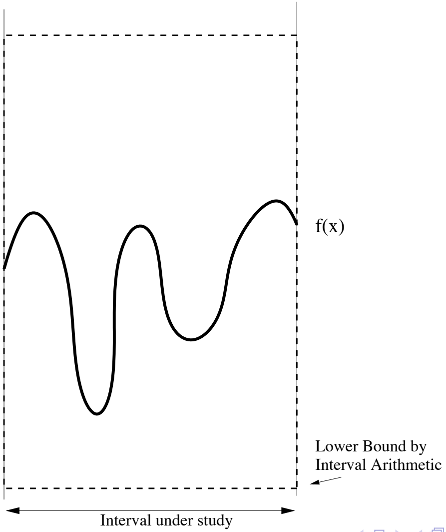
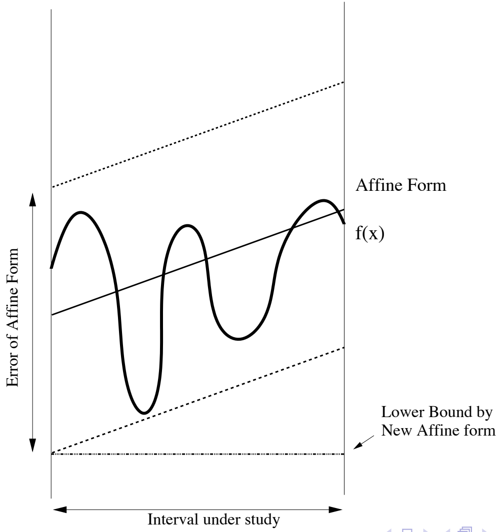
In real, $f(x) \in [6.75; 13]$

Interval Form



New Affine Form



Contents

Notation

$$\forall x \in X \subset \mathbb{R}^n, f(x) - L_f(T(x)) \in E_f$$

where X is the domain under study,
T is the affine transformation of X to $[-1;1]^n$,
L_f is a linear function of$[-1;1]^n$ to $\mathbb{R}^n$,
$E_f = [\underline{E}_f; \overline{E}_f]$ is the interval corresponding to the error generated.

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With AF1, $L_f(\epsilon) = \sum_{i=1}^{n} x_i \epsilon_i$ and $E_f = x_0 + x_{n+1}\epsilon_{n+1}$

With AF2, $L_f(\epsilon) = \sum_{i=1}^{n} x_i \epsilon_i$ and $E_f = x_0 + e_{\pm}\epsilon_{n+1} + e_{+}\epsilon_{n+2} + e_{-}\epsilon_{n+3}$

Reformulation

$$\begin{cases} \min\limits_{x \in X^n} & f(x) \\ s.\ t. & \\ & g_i(x) \leq 0\ ,\ \forall i \in \{1, ..., n_g\} \\ & h_j(x) = 0\ ,\ \forall j \in \{1, ..., n_h\} \end{cases}$$

Reformulation

$$\begin{cases} \min_{x \in X^n} & f(x) \\ s.\ t. & \\ & g_i(x) \leq 0\ ,\ \forall i \in \{1, ..., n_g\} \\ & h_j(x) = 0\ ,\ \forall j \in \{1, ..., n_h\} \end{cases}$$

$$\begin{cases} \min_{\epsilon \in [-1;1]^n} & L_f(\epsilon) \\ s.\ t. & \\ & L_{g\,i}(\epsilon) \leq -\underline{E}_{g_i}\ , \quad \forall i \in \{1, ..., n_g\} \\ & L_{hj}(\epsilon) \leq -\underline{E}_{h_j}\ , \quad \forall j \in \{1, ..., n_h\} \\ & -L_{hj}(\epsilon) \leq \overline{E}_{h_j}\ , \quad \forall j \in \{1, ..., n_h\} \end{cases}$$

Property

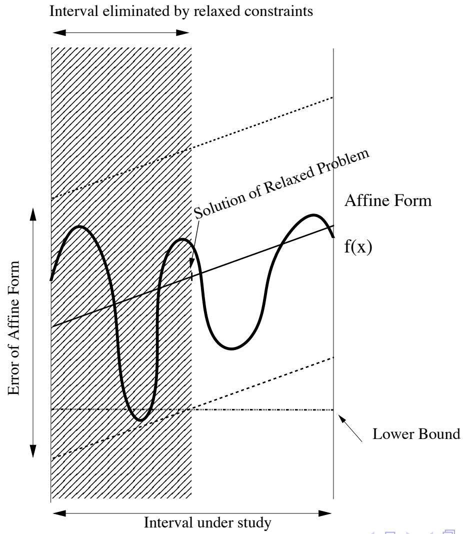
Let ϵ_R the solution of relaxed problem,

$$\forall x \in X \ , \ L_f(\epsilon_R) + E_f^L \leq f(x)$$

Property

If the relaxed problem has no feasible solution,
the general problem has no feasible solution.

with New Affine Form



$$\left\{ \begin{array}{ll} \min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3 + (x_1 + x_2 + x_3)x_1 x_4 \\ & x_1 x_2 x_3 x_4 \geq 25 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40 \end{array} \right.$$

$$\left\{\begin{array}{ll}\min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3+(x_1+x_2+x_3)x_1x_4\\ & x_1x_2x_3x_4\geq 25\\ & x_1^2+x_2^2+x_3^2+x_4^2=40\end{array}\right.$$

$$x_3+(x_1+x_2+x_3)x_1x_4 \Leftrightarrow \quad 18.98+3.44\epsilon_1+0.39\epsilon_2+0.64\epsilon_3+3.05\epsilon_4+1.13\epsilon_5$$

$$\left\{\begin{array}{ll} \min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3+(x_1+x_2+x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2+x_2^2+x_3^2+x_4^2=40 \end{array}\right.$$

$$\begin{aligned} x_3+(x_1+x_2+x_3)x_1x_4 \Leftrightarrow\ & 18.98+3.44\epsilon_1+0.39\epsilon_2+0.64\epsilon_3+3.05\epsilon_4+1.13\epsilon_5 \\ 25-x_1x_2x_3x_4 \Leftrightarrow\ & -2.83-5.57\epsilon_1-1.46\epsilon_2-1.86\epsilon_3-5.57\epsilon_4+2.71\epsilon_5 \end{aligned}$$

$$\left\{ \begin{array}{ll} \min_{x \in [1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3 + (x_1 + x_2 + x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40 \end{array} \right.$$

$$\begin{array}{rcl} x_3 + (x_1 + x_2 + x_3)x_1x_4 & \Leftrightarrow & 18.98 + 3.44\epsilon_1 + 0.39\epsilon_2 + 0.64\epsilon_3 + 3.05\epsilon_4 + 1.13\epsilon_5 \\ 25 - x_1x_2x_3x_4 & \Leftrightarrow & -2.83 - 5.57\epsilon_1 - 1.46\epsilon_2 - 1.86\epsilon_3 - 5.57\epsilon_4 + 2.71\epsilon_5 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 - 40 & \Leftrightarrow & -0.25 + 0.62\epsilon_1 + 2.38\epsilon_2 + 1.88\epsilon_3 + 0.63\epsilon_4 + 0.25\epsilon_5 \end{array}$$

$$\left\{ \begin{array}{ll} \displaystyle\min_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3 + (x_1 + x_2 + x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40 \end{array} \right.$$

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$$\left\{\begin{array}{ll} \min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3+(x_1+x_2+x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2+x_2^2+x_3^2+x_4^2 = 40 \end{array}\right.$$

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$$\left\{\begin{array}{lrl} \min\limits_{\epsilon\in[-1;1]^4} & 3.44\epsilon_1+0.39\epsilon_2+0.64\epsilon_3+3.05\epsilon_4 & \\ & -5.57\epsilon_1-1.46\epsilon_2-1.86\epsilon_3-5.57\epsilon_4 & \leq 5.54 \\ & 0.62\epsilon_1+2.38\epsilon_2+1.88\epsilon_3+0.63\epsilon_4 & \leq 0.5 \\ & -0.62\epsilon_1-2.38\epsilon_2-1.88\epsilon_3-0.63\epsilon_4 & \leq 0 \end{array}\right.$$

$$\left\{ \begin{array}{ll} \min\limits_{x \in [1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3 + (x_1+x_2+x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40 \end{array} \right.$$

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Solution of linear program :
$$\begin{array}{ll}\epsilon_R & =(-1;-0.24;1;-0.26) \\ L_f(\epsilon_R) & =-3.70\end{array}$$

$$\left\{\begin{array}{ll} \min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3+(x_1+x_2+x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2+x_2^2+x_3^2+x_4^2 = 40 \end{array}\right.$$

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$L_f(\epsilon_R) = -3.70$

Lower bound by IA = 12.5 ;

$$\left\{\begin{array}{ll} \min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3+(x_1+x_2+x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2+x_2^2+x_3^2+x_4^2 = 40 \end{array}\right.$$

$$\left\{\begin{array}{lll} \min\limits_{\epsilon\in[-1;1]^4} & 3.44\epsilon_1+0.39\epsilon_2+0.64\epsilon_3+3.05\epsilon_4 & \\ & -5.57\epsilon_1-1.46\epsilon_2-1.86\epsilon_3-5.57\epsilon_4 & \leq 5.54 \\ & 0.62\epsilon_1+2.38\epsilon_2+1.88\epsilon_3+0.63\epsilon_4 & \leq 0.5 \\ & -0.62\epsilon_1-2.38\epsilon_2-1.88\epsilon_3-0.63\epsilon_4 & \leq 0 \end{array}\right.$$

Solution of linear program : $\epsilon_R = (-1; -0.24; 1; -0.26)$
$L_f(\epsilon_R) = -3.70$

Lower bound by IA = 12.5 ; New lower bound = 14.15

Integration in IBBA

IBBA (Interval Branch&Bound Algorithm) is a deterministic global optimization algorithm based on Interval Arithmetic.

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- eliminate intervals which do not contain global minimum

Examples

- 4 polynomial problems

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EX 1 (H. Tuy)

$$\begin{cases} \min\limits_{x\in[1;5]^4} & x_3 + (x_1 + x_2 + x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40 \end{cases}$$

EX 3

$$\begin{cases} \min\limits_{x\in[0;3]\times[0;4]} & -x_1 - x_2 \\ & x_2 \leq 2x_1^4 - 8x_1^3 + 8x_1^2 + 2 \\ & x_2 \leq 4x_1^4 - 32x_1^3 + 88x_1^2 - 96x_1 + 36 \end{cases}$$

Examples

- 4 polynomial problems
- 5 global optimization problems (COCONUT library)

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EX 7.2.3

$$\left\{\begin{array}{ll}
\min\limits_{x\in[10^2;10^4]\times[10^3;10^4]^2\times[10;10^3]^5} & x_1+x_2+x_3 \\
& \frac{833.33252x_4}{x_1x_6}+\frac{100}{x_6}-\frac{83333.333}{x_1x_6}\leq 1 \\
& \frac{1250x_5}{x_2x_7}+\frac{x_4}{x_7}-\frac{1250x_4}{x_2x_7}\leq 1 \\
& \frac{1250000}{x_3x_8}+\frac{x_5}{x_8}-\frac{2500x_5}{x_3x_8}\leq 1 \\
& 0.0025x_4+0.0025x_6\leq 1 \\
& -2.5x_4+2.5x_5+2.5x_7\leq 1000 \\
& -0.01x_5+0.01x_8\leq 1
\end{array}\right.$$

Examples

- 4 polynomial problems
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PB Elec (optimization of a electrical motor)

$$\left\{\begin{array}{ll} \min\limits_{x \in X} & \pi x_6 \frac{x_1}{x_2}(x_1 + x_4 - x_{10} - x_3)(2x_5 + x_4 + x_{10} + x_3) \\ & \Gamma_{em} = \frac{\pi}{2x_2}(1 - x_9)\sqrt{k_r x_6 E_{ch} x_4} {x_1}^2 (x_1 + x_4) x_7 \\ & E_{ch} = k_r x_4 {x_8}^2 \\ & x_9 = 1.5 x_{11} x_6 \frac{x_{10} + x_4}{x_1} \\ & x_7 = \frac{2 x_3 P}{x_1 \ln\left(\frac{x_1 + 2x_4}{x_1 - 2(x_3 + x_{10})}\right)} \\ & x_5 = \frac{\pi x_6 x_7}{4 x_{11} B_{iron}} x_1 \\ & x_{11} = \frac{\pi x_1}{\Delta_p} \end{array}\right.$$

only Reformulation Method

	only IBBA		IBBA with AF1		IBBA with AF2	
	Nb Its	T	Nb Its	T	Nb Its	T
Ex 1	-		1 702	0.2s	1 684	0.21s
Ex 2	715 234	17.02s	112 967	9.7s	220	0.05s
Ex 3	2 598 426	27.84s	691	0.05s	695	0.05s
Ex 4	701 594	17.14s	199	0.02s	137	0.02s
Ex 7.2.1	-		6 891	2.27s	4 565	1.42s
Ex 7.2.2	-		6 042	1.36s	6 036	1.42s
Ex 7.2.3	-		-		-	
Ex 7.2.5	150 761	1.5s	7 898	0.7s	7 898	0.87s
Ex 7.2.6	-		-		-	
PB Elec	-		-		-	

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with Constraints Propagation

	IBBA with CP		AF1+CP		AF2+CP	
	Nb Its	T	Nb Its	T	Nb Its	T
Ex 1	15631707	16min13s	1 202	0.14s	1 255	0.15s
Ex 2	6 358	0.16s	1 556	0.17s	1 510	0.19s
Ex 3	426 512	8.6s	271	0.03s	197	0.03s
Ex 4	322 737	14.28s	594	0.04s	550	0.04s
Ex 7.2.1	-		3 235	1.11s	2 039	0.85s
Ex 7.2.2	-		210	0.03s	210	0.03s
Ex 7.2.3	-		1776 730	5min48s	1754 806	6min15s
Ex 7.2.5	11 153	0.3s	960	0.12s	960	0.13s
Ex 7.2.6	-		1 633	0.14s	1 633	0.15s
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Conclusion

Main Contributions :

- Find better safe bound
- Eliminate intervals which do not contain global optimum
- Reduce computing times

Further research :

- Study other way to calculate reformulations
- Integrate a free linear solver

Thank you for your attention