Balancing to prescribed row and column sums

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Balancing

- $A \in \mathbb{R}^{m \times n}$, $p \in \mathbb{R}^m$, $q, \in \mathbb{R}^n$, $A, p, q \geq 0$.
- Find diagonal matrices D_1 and D_2 so that $P = D_1 A D_2$ and $Pe_n = p$, $P^T e_m = q$.
- If $m = n$ and $p = q = e$, P is doubly stochastic.
- Existence and uniqueness: Sinkhorn, Knopp, Brualdi (1966–1972).

Existence

$$P_1 A P_2^T = \begin{bmatrix} F & 0 \\ G & H \end{bmatrix}, \quad P_1 p = \begin{bmatrix} f \\ g \end{bmatrix}, \quad P_2 q = \begin{bmatrix} c \\ d \end{bmatrix}.$$

- $\|p\|_1 = \|q\|_1$.
- $\|c\|_1 \geq \|f\|_1$.
- Equality only allowed if $\iff G = 0$.
- Uniqueness (up to scalar factor) $\iff G \neq 0$.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Application of Balancing

- m parties and n districts, A, p, q.
- a_{ij}: votes for party i in district j.
- p_i seats for party i, q_j seats in district j.
- $p_i = \left[\frac{\sum_j a_{ij}}{\sum_i \sum_j a_{ij}} \right]$ (round by, e.g., largest remainder rule).
- Find diagonal matrices D_1 and D_2 so that $P = [D_1 A D_2]$ satisfies $Pe_n = p$ and $P^T e_m = q$.
- Allocate p_{ij} seats to party i in district j.

Algorithm

- $r = e_m$, $c = e_n$.
- $c = q./(A^T r)$, $r = p./(Ac)$
- $D_1 = \mathcal{D}(r)$, $D_2 = \mathcal{D}(c)$.

Convergence of (p,q)-SK

- $P = \mathcal{D}(r)A\mathcal{D}(c)$. Let $Q = \mathcal{D}(p)^{-1/2} P \mathcal{D}(q)^{-1/2}$.
- Rate of convergence: $\sigma_2(Q)^2$.
- Lemma: $\sigma_1(Q) = 1$.
- Proof: Consider $\widehat{Q} = \mathcal{D}(q)^{-1/2} Q^T Q \mathcal{D}(q)^{1/2}$.
- $Q\mathcal{D}(q)^{1/2} = \mathcal{D}(p)^{-1/2} P$
- $\widehat{Q} = \mathcal{D}(q)^{-1} P^T \mathcal{D}(p)^{-1} P$
- $\widehat{Q} e = \mathcal{D}(q)^{-1} P^T e = \mathcal{D}(q)^{-1} q = e$.
- $Q^T Q$ is similar to a stochastic matrix. Result follows from Perron-Frobenius theorem.
- If P is unique, $\sigma_2(Q) < 1$.

A Fast Algorithm

$$S = \begin{bmatrix} \mathbf{0} & A \\ A^T & \mathbf{0} \end{bmatrix}, \quad s = \begin{bmatrix} p \\ q \end{bmatrix}.$$

- (p, q)-SK equivalent to $x_{k+1} = s./(Sx_k)$.
- $f(x) = \mathcal{D}(x)Sx - s$.
- Use Newton's method on $f(x) = \mathbf{0}$.
- CG for inner iteration.
- Quadratic convergence possible.
- Iteration matrices are singular but consistent.
- Convergence can stagnate.

Newton step

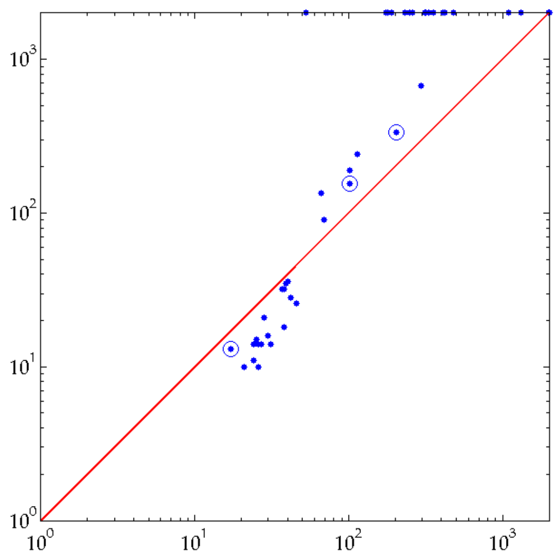
- Let $\mathcal{S}_k = S + \mathcal{D}(x_k)^{-1}\mathcal{D}(Sx_k)$.
- Newton step is $\mathcal{S}_k x_{k+1} = Sx_k + \mathcal{D}(x_k)^{-1}s$.
- If balancing solution exists, $\mathcal{S}_k$ is SSPD.
- Proof: $\mathcal{S}_k$ similar to weakly diagonally dominant matrix.

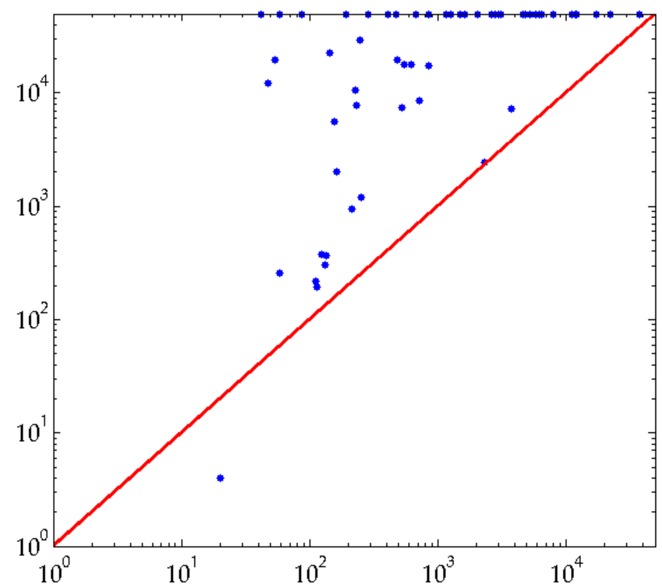
$$\begin{aligned} \mathcal{D}(x_k)^{-1}\mathcal{S}_k\mathcal{D}(x_k) &= \mathcal{D}(x_k)^{-1}S\mathcal{D}(x_k) + \mathcal{D}(x_k)^{-1}\mathcal{D}(Sx_k) \\ &= B + \mathcal{D}(Be). \end{aligned}$$

Implementation of CG

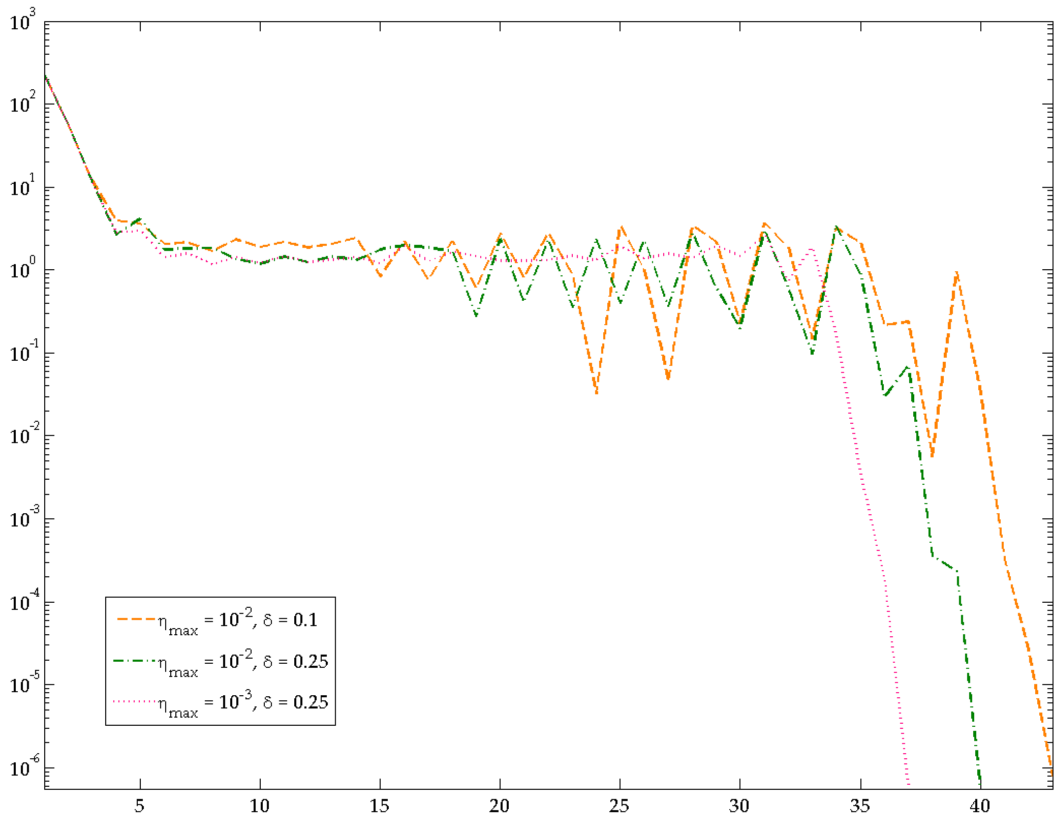
- Proof from previous slide leads to natural diagonal preconditioning.
- Systems are consistent if unique solution exists.
- $S_k = \mathcal{D}(x_k) S \mathcal{D}(x_k)$.
- Solve $(S_k + \mathcal{D}(S_k e))(x_{k+1}/x_k) = A_k e + s$.
- Need appropriate stopping criterion.

Comparison of algorithms



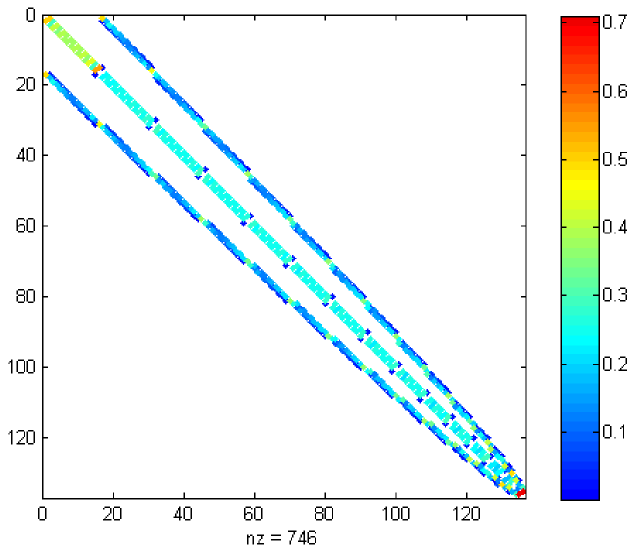


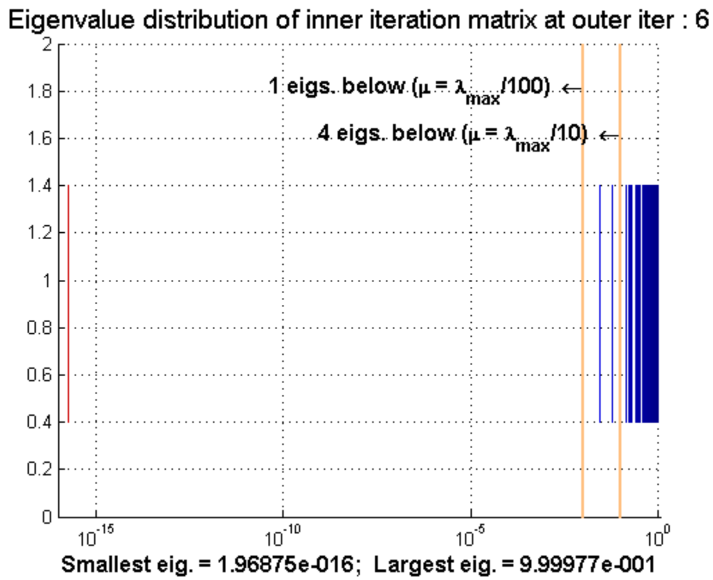
Erratic Convergence



When things work well...

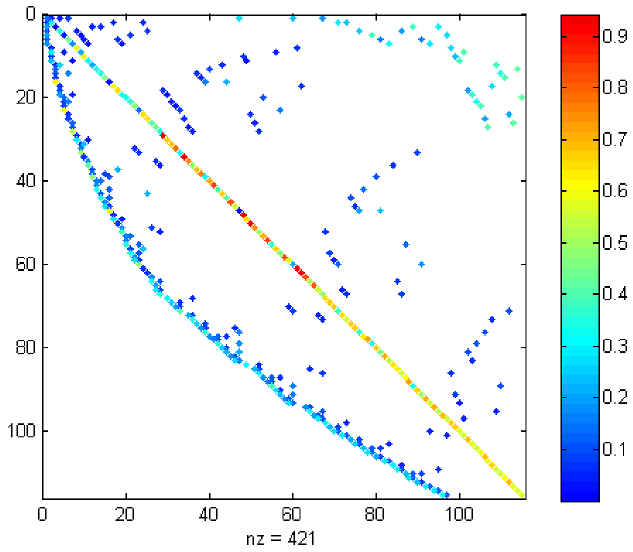
- When A is symmetric...

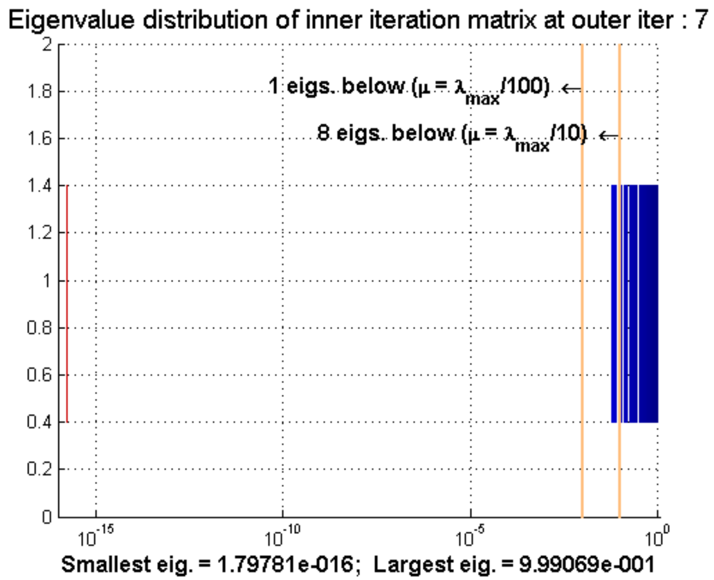




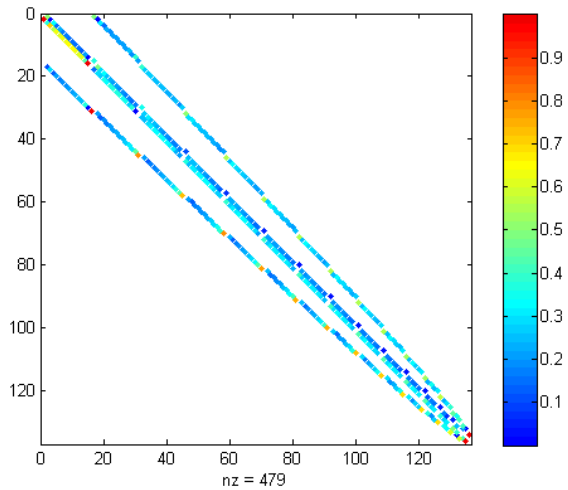
When things work well...

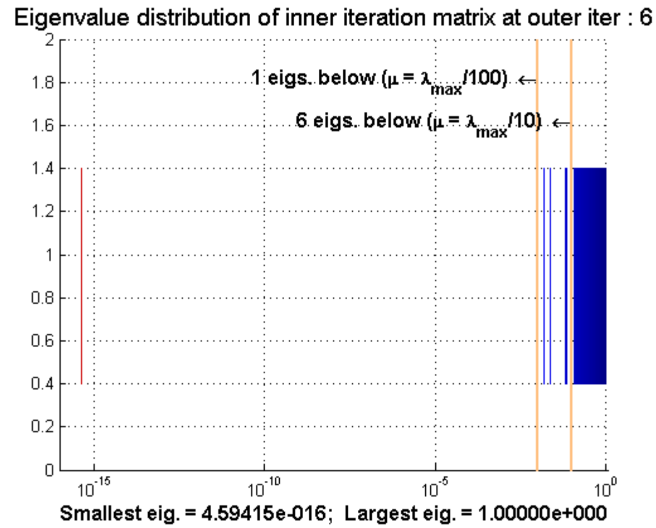
- When A is not too sparse...

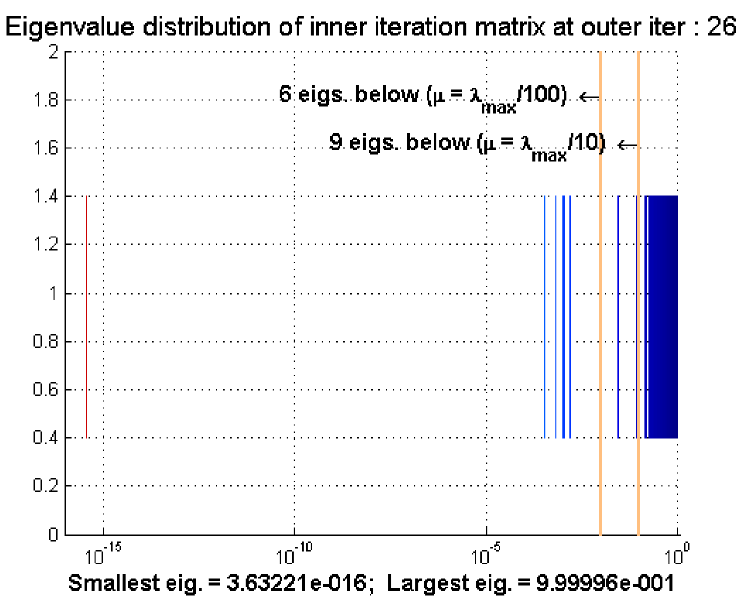


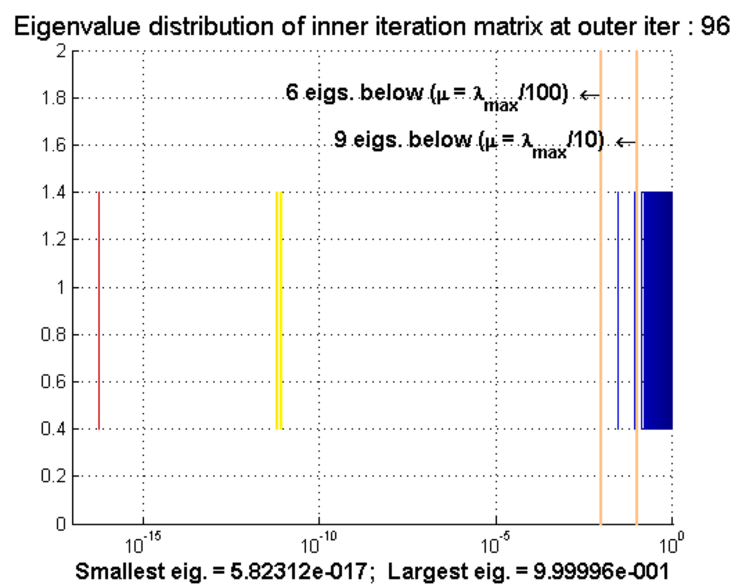


But if A is "nearly" decomposable. . .

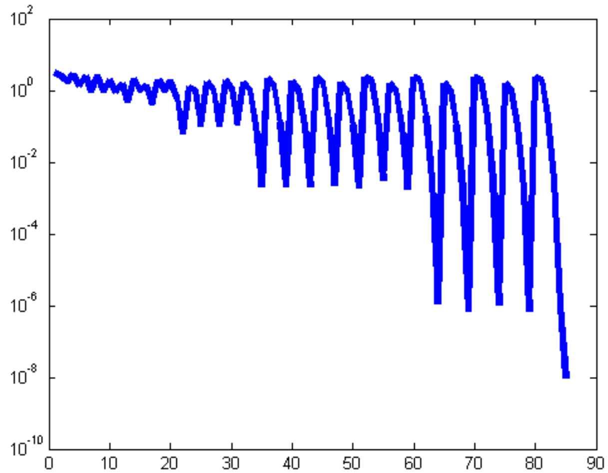


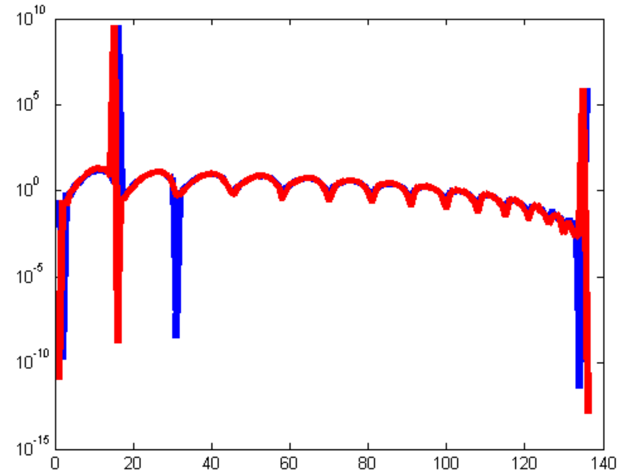






And the cause. . .





- Chain of connections necessary for scaling gets harder to satisfy.
- In symmetric case, modifications hit the right spot.
- But overlooking lack of symmetry causes stagnation.