A Stabilized Discontinuous Galerkin Formulation for Helmholtz Problems

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-PhD Thesis TOTAL-INRIA-UPPA-

29 février 2008

Helmholtz Equation

$$\Delta u + k^2 u = f \text{ in } \Omega, \text{ with } \Omega \subset \mathbb{R}^n \text{ bounded domain and } f \in L^2(\Omega)$$

- particular case of the waves equation: the solution: monochromatic wave;

Notations:

- $k = \frac{\omega}{c}$: wave number;
- ω: circular frequency of the wave;
- $\frac{\omega}{2\pi}$: wave frequency;
- c: velocity;
- $\frac{2\pi}{k}$: wavelength;

The solution u is here a stationary wave.

Goals of the thesis

- to develop a discontinuous Galerkin method that uses a special basis of functions constructed using the solutions of the homogeneous Helmholtz equation in 3D;

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- to develop a discontinuous Galerkin method that uses a special basis of functions constructed using the solutions of the homogeneous Helmholtz equation in 3D;

- to consider elements built on convex polyhedra/polygons; enforce a weak continuity of the solution at the inter-element boundaries using Lagrange multipliers;

Standard Galerkin

Problem: pollution effect due to the fact that the ellipticity is lost when frequency increases $\rightarrow$ decrease of the performance.

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Problem: solving the linear system is too expensive.
Solution: ?

Some existing methods

- PUM (Partition of unity method)[1] approximates the exact solution using a partition of unity method (continuous method).
- LSM (Least-squares method)[2] uses plane waves or Bessel functions in each element of the mesh; the continuity is enforced by minimizing a simple quadratic functional (discontinuous method).
- RFB (Residual Free Bubbles)[3] consists in approching exact solution in a discrete space spanned by polynomials + “bubble functions” (that verify the equation strongly in each element and are 0 on the boundary of each element) (continuous method).

[1] I. Babuska, M. Melenk, The partition of unity method,

[2] P. Monk, D. Wang, A least-squares method for the Helmholtz equation

[3] L. P. Franca, C. Farhat, A. P. Macedo, M. Lesoinne, Residual-free bubbles for the Helmholtz equation

A Discontinuous Galerkin Method (DGM)

- Discontinuous Enrichment Method (DEM): the standard finite element polynomial field is enriched within each element by free-space solutions of the homogeneous PDE to be solved (plane waves for Helmholtz equation)[1];
- for a large class of Helmholtz problems, the polynomial field is not necessary for capturing efficiently the solution [2]$\rightarrow$the polynomial field is dropped $\rightarrow$ DEM is a DGM wih plane waves basis functions;
- the weak continuity is enforced at the inter-element boundaries using Lagrange multipliers[2,3].

[1] C. Farhat, I. Harari, L. P. Franca, The discontinuous enrichment method
[2] C. Farhat, I. Harari, U. Hetmaniuk, A discontinuous Galerkin method with Lagrange multipliers for the solution of Helmholtz problems in the mid-frequency regime
[3] C. Farhat, P. Wiedemann-Goiran, R. Tezaur, A discontinuous Galerkin method with plane waves and Lagrange multipliers for the solution of short wave exterior Helmholtz problems on unstructured meshes

The DGM

Let us consider $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial\Omega)$ and the following problem:

$$(1) \begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that :} \\ \qquad \Delta u + k^2 u = f \quad \text{in } \Omega \\ \qquad \partial_n u = g \quad \text{on } \Gamma \\ \qquad \partial_n u = g + \mathrm{i}ku \quad \text{on } \Sigma \end{cases},$$

where:

- $\Omega \subset \mathbb{R}^2$: a bounded, polygonal-shaped domain;
- Γ: the interior boundary;
- Σ: the exterior boundary;
- $\mathbf{n}$: the unitary outward normal vector;
- ∂_n: the normal derivative.

A mixed hybrid variational formulation

- $(\tau_h)_h$ a regular triangulation of the computational domain $\overline{\Omega}$ into rectangular-elements K;
- space of the primal variable:

$$X = \left\{ v \in L^2(\Omega);\ \forall K \in \tau_h,\ v_K = v|_K \in H^1(K) \right\}$$

$$\|v\|_X = \left(\sum_{K\in\tau_h} \|v_K\|^2_{x(K)} \right)^{1/2},\ \forall v \in X, \text{ where}$$

$$\|v_K\|_{X(K)} = \left(|v_K|^2_{1,K} + \frac{1}{|K|}\|v_K\|^2_{0,K} \right)^{1/2}$$

- we note $|v|_{1,\tau_h} = \left(\sum_{K\in\tau_h} |v_K|^2_{1,K} \right)^{1/2}, \forall v \in X$;
- space of the dual variable (that is the normal derivative of the solution on the interior edges):

$$M = \left\{ \mu \in \prod_{K\in\tau_h} M(K);\ \mu_K + \mu_{K'} = 0 \text{ on } \partial K \bigcap \partial K' \right\}, \text{ where}$$

$$M(K) = \left\{ \mu \in H^{-1/2}(\partial K);\ \mu = 0 \text{ on } \partial K \bigcap \partial\Omega \right\}$$

$$\|\mu\|_M = \left(\sum_{K\in\tau_h} \|\mu_K\|^2_{-\frac{1}{2},\partial K} \right)^{1/2},\ \forall \mu \in M.$$

A mixed hybrid variational formulation

$$(2)\left\{\begin{array}{l}\text{Find } (u,\lambda)\in X\times M \text{ such that :}\\ \begin{array}{rcll} a(u,v)+b(v,\lambda) & = & F(v) & \forall v\in X\\ b(u,\mu) & = & 0 & \forall \mu\in M\end{array}\end{array}\right. ,$$

with:

$\forall u, v \in X,\ \forall \mu \in M$:

$a(u,v) = \sum_{K\in\tau_h}\left(\int_K \nabla u.\nabla \overline{v} dx - k^2\int_K u\overline{v}dx - \mathrm{i}k\int_{\partial K\bigcap\Sigma} u\overline{v}ds\right),$

$b(v,\mu) = \sum_{K\in\tau_h} < \mu^K, \overline{v} >_{-\frac{1}{2},\frac{1}{2},\partial K}$

$F(v) = \sum_{K\in\tau_h}\left(\int_K f\overline{v}dx + \int_{\partial K\bigcap\partial\Omega} g\overline{v}ds\right)$

Main result*:

- exists a unique solution: $u \in H^1(\Omega)$.

*M. Amara, R. Djellouli, C. Farhat, Convergence analysis of a discontinuous Galerkin method with plane waves and Lagrange multipliers for the solution oh Helmholtz problems

Discrete formulation R-4-1

- choice of the basis functions: $\forall K \in \tau_h, \Phi_j^K = e^{ik\mathbf{n}_j.(\mathbf{x}-\mathbf{s}_j)}$, $1 \le j \le 4$, where:
  - T_j^K is the j^{th} edge of K, with summits $\left(\mathbf{s}_j^K, \mathbf{s}_{j+1}^K\right)$;
  - $\mathbf{n}_j$ the outward unitary normal vector of T_j;
- discrete space for the primal variable:

$X_h = \{v_h \in X;\ \forall K \in \tau_h, v_h|_K \in X_h(K)\}$, where

$$X_h(K) = \left\{v_h^K \in H^1(K); v_h^K = \sum_{j=1}^4 \alpha_j^K \Phi_j^K,\ \alpha_j^K \in \mathbb{C}\right\}$$

- discrete space of the dual variable :

$$M_h = \left\{\mu_h \in \mathcal{M}; \forall K \in \tau_h, \forall T_j^K \subset \partial K : \mu_j^K = \mu_j|_K \in \mathbb{C}, 1 \le j \le 4\right\},$$

where $\mathcal{M} = M \bigcap \prod_{K \in \tau_h} L^2(\partial K)$.

Discrete formulation R-4-1

$$
\text{(DVP)} \left\{ \begin{array}{rcll} \multicolumn{4}{l}{\text{Find } (u_h, \lambda_h) \in X_h \times M_h \text{ such that :}} \\ a(u_h, v_h) + b(v_h, \lambda_h) & = & F(v_h) & \forall v_h \in X_h \\ b(u_h, \mu_h) & = & 0 & \forall \mu_h \in M_h \end{array} \right.
$$

Main results*:

- unique solution $(u_h, \lambda_h) \in X_h \times M_h$;
- for $k\,(kh)^{2/3}$ “small enough”:
  - the error $\|u - u_h\|_{0,\Omega}$ is of order $k\,(kh)^{4/3}$;
  - the error $|u - u_h|_{1,\tau_h} + \|\lambda - \lambda_h\|_M$ is of order $k\,(kh)^{2/3}$.

*M. Amara, R. Djellouli, C. Farhat, Convergence analysis of a discontinuous Galerkin method with plane waves and Lagrange multipliers for the solution oh Helmholtz problems

Discrete formulation R-4-1

Main tools:

1. the normal derivative $\partial_n \Phi_j^K$ is constant along T_l^K $(1 \leq l, j \leq 4)$;
2. properties of the elementary matrix B^K, defined by:

$$B_{lj}^K = b(\mu_l, \Phi_j) = \frac{1}{h_l^K} \int_{T_l^K} \Phi_j^K ds, 1 \leq l, j \leq 4$$

- invertible if $kh_K \leq \pi$
- $\exists \hat{C} > 0$ such that:

$$\left\| B_K^{-1} \right\|_2 \leq \frac{\hat{C}}{k^2 h_K^2}.$$

Discrete formulation R-8-2

- choice of the basis functions: $\forall K \in \tau_h,\ \Phi_j^K = e^{\mathrm{i}k\overrightarrow{\theta_j}\cdot\mathbf{x}}$, with $\overrightarrow{\theta_j} = (\cos\theta_j, \sin\theta_j),\ \theta_j = j\frac{\pi}{4},\ j \in \{1, 2, ..., 8\}$
- discrete space of the primal variable:

$X_h = \{v_h \in X;\ \forall K \in \tau_h, v_h|_K \in X_h(K)\}$, where
$X_h(K) = \left\{v_h^K \in H^1(K); v_h^K = \sum_{j=1}^{8} \alpha_j^K \Phi_j^K,\ \alpha_j^K \in \mathbb{C}\right\};$

- discrete space of the dual variable:

$$\begin{aligned} M_h \quad = \quad & \Big\{\mu_h \in \mathcal{M}; \mu_h|_{\Gamma^{K,K'}}(x) = \lambda_1 e^{\mathrm{i}k\frac{\sqrt{2}}{4}x} + \lambda_2 e^{-\mathrm{i}k\frac{\sqrt{2}}{4}x},\ \Gamma^{K,K''} \parallel \overrightarrow{Ox} \\ & \mu_h|_{\Gamma^{K,K'}}(y) = \lambda_1 e^{\mathrm{i}k\frac{\sqrt{2}}{4}y} + \lambda_2 e^{-\mathrm{i}k\frac{\sqrt{2}}{4}y},\ \Gamma^{K,K'} \parallel \overrightarrow{Oy}, \lambda_1, \lambda_2 \in \mathbb{C}\Big\} \end{aligned}$$

where $\Gamma^{K,K'} = \partial K \bigcap \partial K'$.

Discrete formulation R-8-2

Differences between R-4-1 and R-8-2:

- the normal derivative is not constant along the interior edges;
- the elementary matrix B^K, defined by: $B^K_{lj} = b(\mu^K_l, \Phi^K_j)$ is not always invertible.

Discrete formulation R-8-2

Differences between R-4-1 and R-8-2:

- the normal derivative is not constant along the interior edges;
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Conclusion:

- we cannot adapt the technics used for R-4-1;
- stability problem;

Remark: numerical results improved on the case R-4-1.

Our method

Basic problem: let $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$;

$$\left\{ \begin{array}{l} \text{Find } u \in H^1(\Omega) \text{ such that :} \\ \begin{array}{rcll} \Delta u + k^2 u & = & f & \text{in} \quad \Omega \\ \partial_n u & = & \mathrm{i} k u + g & \text{on} \quad \Sigma \\ \partial_n u & = & g & \text{on} \quad \Gamma \end{array} \end{array} \right.$$

Main result*: existence and uniqueness of the solution $u \in H^1(\Omega)$.

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Our method

We keep the previous notations:

$$X = \left\{v \in L^2(\Omega); \forall K \in \tau_h, v_K = v\mid_K \in H^1(K)\right\},$$

$$M = \left\{\mu \in \prod_{K \in \tau_h} M(K); \mu^K + \mu^{K'} = 0 \text{ on } \partial K \cap \partial K'\right\},$$

$$M(K) = \left\{\mu \in H^{-1/2}(\partial K); \mu = 0 \text{ on } \partial K \cap \partial \Omega\right\},$$

$$\mathcal{M}(K) = L^2(\partial K) \cap M(K),$$

$$\mathcal{M} = \left\{\mu \in \prod_{K \in \tau_h} \mathcal{M}(K); \mu^K + \mu^{K'} = 0 \text{ on } \partial K \cap \partial K'\right\}.$$

Our method

We set:

$$a_K(u,v) = \int_K \left(\nabla u.\nabla \overline{v} - k^2 u\overline{v}\right) dx - ik \int_{\partial K\cap\Sigma} u\overline{v} ds$$

$$F_K(v) = \int_K f\overline{v} dx + \int_{\partial K\cap\partial\Omega} g\overline{v} ds$$

Previous variational formulation:

$$\left\{\begin{array}{l} \forall K \in \tau_h,\ \forall v \in H^1(K), \\ \\ a_K(u,v) - < \lambda^K, \overline{v} >_{-\frac{1}{2},\frac{1}{2},\partial K} = F_K(v) \\ \\ \forall \mu \in M,\ \ \sum_{K\in\tau_h} < \mu, \overline{u} >_{-\frac{1}{2},\frac{1}{2},\partial K} = 0 \\ \\ u \in X,\ \lambda \in M \end{array}\right. .$$

- stability problem.

Our method

- "replace" the second equation of the system:

$$\begin{cases} \forall K \in \tau_h,\ \forall v \in H^1(K), \\ a_K(u,v) - < \lambda^K, \overline{v} >_{-\frac{1}{2},\frac{1}{2},\partial K} = F_K(v) \\ \forall v \in X,\ \sum_{e-internal} \int_e [u][\overline{v}]ds = 0 \\ u \in X,\ \lambda \in M \end{cases},$$

where $[u]|_e = u|_K - u|_{K'}$, with $e = \partial K \cap \partial K'$;

- the new equation implies $[u] = 0$ along all interior edges, hence $u \in H^1(\Omega)$;
- equivalence with the basic problem.

Our method

Steps:

- writing the problem in the following way:

$$(\mathcal{P})\begin{cases} \forall K \in \tau_h,\ \forall \mu \in M(K),\ \text{find } U(\mu) \in H^1(K) \text{ such that :} \\ \\ a_K(U(\mu), v) = F_K(v) + < \mu, \overline{v} >_{-\frac{1}{2},\frac{1}{2},\partial K} \end{cases}$$

- "splitting" of $\mathcal{P}$:

$$(\mathcal{P}_1)\begin{cases} \text{Find } \varphi \in H^1(K) \text{ such that :} \\ a_K(\varphi, v) = F_K(v), \forall v \in H^1(K) \end{cases}$$

$$(\mathcal{P}_2)\begin{cases} \text{For all } \mu, \text{ find } \Phi(\mu) \in H^1(K) \text{ such that :} \\ a_K(\Phi(\mu), v) =< \mu, v >_{-\frac{1}{2},\frac{1}{2},\partial K}, \forall v \in H^1(K) \end{cases}$$

- purpose: solve $\mathcal{P}_1$ and $\mathcal{P}_2$ and write $U(\mu) = \varphi + \Phi(\mu)$;
- find $\lambda \in M$ such that $U(\lambda) \in H^1(\Omega) \Rightarrow U(\lambda) = u$.

Discretization of the problem

- $M_h \subset M$; existence and uniqueness of $\lambda_h \in M_h$ such that: $\sum_{e-interne} \int_e [U(\lambda_h)][\overline{\Phi}(\mu_h)] = 0, \forall \mu_h \in M_h$

$$\begin{aligned}
\sum_{e-internal} \int_e [U(\lambda_h)][\overline{\Phi}(\mu_h)] &= 0 \Longleftrightarrow \\
\sum_{e-internal} \int_e [\Phi(\lambda_h)][\overline{\Phi}(\mu_h)] &= -\sum_{e-internal} \int_e [\varphi][\overline{\Phi}(\mu_h)]
\end{aligned}$$

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- positive definite Hermitian matrix:

$$A_{lm} = \sum_{e-internal} \int_e [\Phi(\mu_l)][\overline{\Phi}(\mu_m)]$$

Interest of the approach

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Interest of the approach

- we avoid the inf-sup condition;
- the discretization leads us to a positive definite Hermitian matrix $\longrightarrow$ conjugate gradient method;
- the method is allowed for irregular mesh and may be applied to problems in 2D and 3D.

Interest of the approach

Conjugate gradient method:

- solve in 8 elements:

$$\left\{\begin{array}{l} \text{Find } \xi_h^K(v) \in X_h(K): \\ \begin{array}{rcll} -\Delta\xi_h^K(v) - k^2\xi_h^K(v) & = & 0 & \text{in } K \\ \partial_n\xi_h^K(v) & = & -\mathrm{i}k\xi_h^K(v) & \text{on } \partial K \cap \Sigma \\ \partial_n\xi_h^K(v) & = & 0 & \text{on } \partial K \cap \Gamma \\ \partial_n\xi_h^K(v) & = & [v] & \text{on } e-\text{internal} \end{array} \end{array}\right. \quad \text{with}$$

$v = \sum_{l=1}^{d} x_l \Phi_h(\mu_l)$

Interest of the approach

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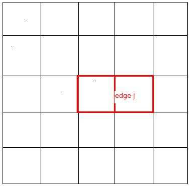
$v = \sum_{l=1}^{d} x_l \Phi_h(\mu_l)$

- $(Ax)_j = \int_{e_j} [\xi_h(v)]$: no need to compute and store the matrix A!!!!!

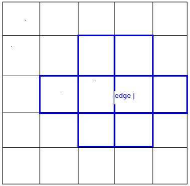
Numerical results

- the matrix of the system is sparse as for R-4-1 and R-8-2:

Figure:



(a) R-4-1 and R-8-2



(b) our method

Numerical results

- computational domain: $\Omega = [0,1] \times [0,1]$;
- $f = 0$ in Ω;
- $g = \mathrm{i}k\,(\mathbf{d}^*.\mathbf{n} - 1)\,e^{\mathrm{i}k\mathbf{x}.\mathbf{d}^*}$ on Σ, with $\mathbf{d}^* = (\cos\theta^*, \sin\theta^*)$;
- exact solution: $u = e^{\mathrm{i}k\mathbf{x}.\mathbf{d}^*}$;
- discrete spaces:

$$X_h = \{v_h \in X;\ \forall K \in \tau_h, v_h|_K \in X_h(K)\},\ \text{where}$$

$$X_h(K) = \left\{v_h^K \in H^1(K); v_h^K = \sum_{j=1}^4 \alpha_j^K \Phi_j^K,\ \alpha_j^K \in \mathbb{C}\right\}\ \text{and}$$

$$M_h = \left\{\mu_h \in \mathcal{M}; \forall K \in \tau_h, \forall T_j^K \subset \partial K : \mu_j^K = \mu_j|_K \in \mathbb{C},\ 1 \le j \le 4\right\}$$

Numerical results

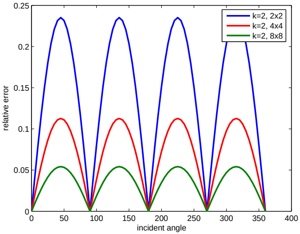


Figure:

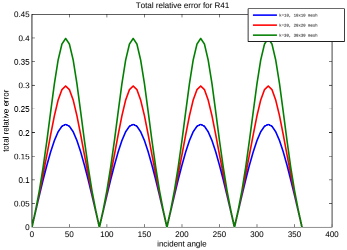
(a) Behavior of the error: k=2, 2x2, 4x4, 8x8

Numerical results

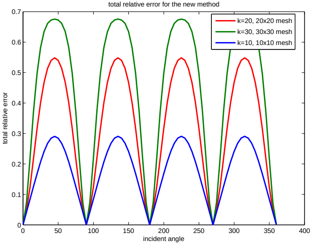
- relative error: modified H^1 norm:

$$\|u-u_h\| = \sqrt{\sum_{K\in\tau_h} \|u-u_h\|^2_{H^1(K)} + \sum_{e-interne} \|[u_h]\|^2_{L^2(e)}}$$

Figure:



(c) Performance of R41 for kh=1

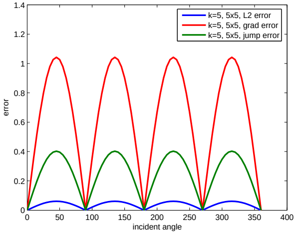


(d) Performance of our method for kh=1

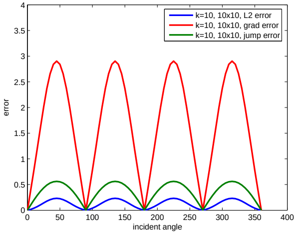
- win in stability...

Numerical results

Figure:



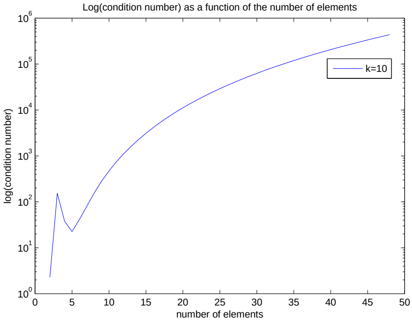
(a) Contribution of different errors for k=5, 5x5



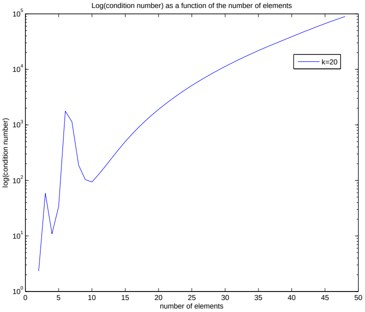
(b) Contribution of different errors for k=10, 10x10

Numerical results

Figure:



(c) Condition number for k=10



(d) Condition number for k=20

- same order as for R41.

Goals

- precondition the matrix; analyze the results;
- develop the code, using the spaces defined for R-8-2;
- mathematical analysis of the method;

Further works:

- development of the 3D code.

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