Sparse Days at CERFACS

Regularization by Mollification

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MATHEMATICAL INSTITUTE OF TOULOUSE

Outline

- Introduction
- Fourier synthesis
- Asymptotic analysis (with N. Alibaud and Y. Saesor)
- Extension (with X. Bonnefond)
- A note on proximal inversion (with A. Rondepierre)

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Fourier Synthesis

Recover a function from a partial and approximate knowledge of its Fourier transform.

Example 1: Fourier series

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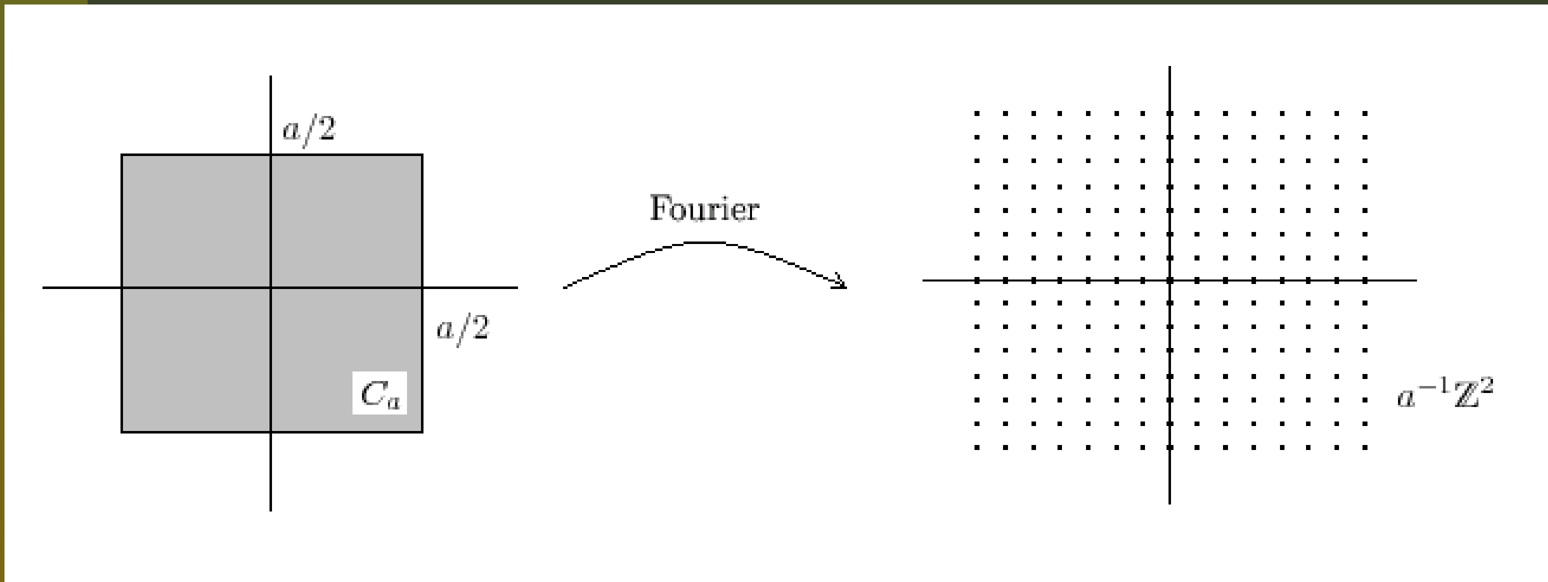

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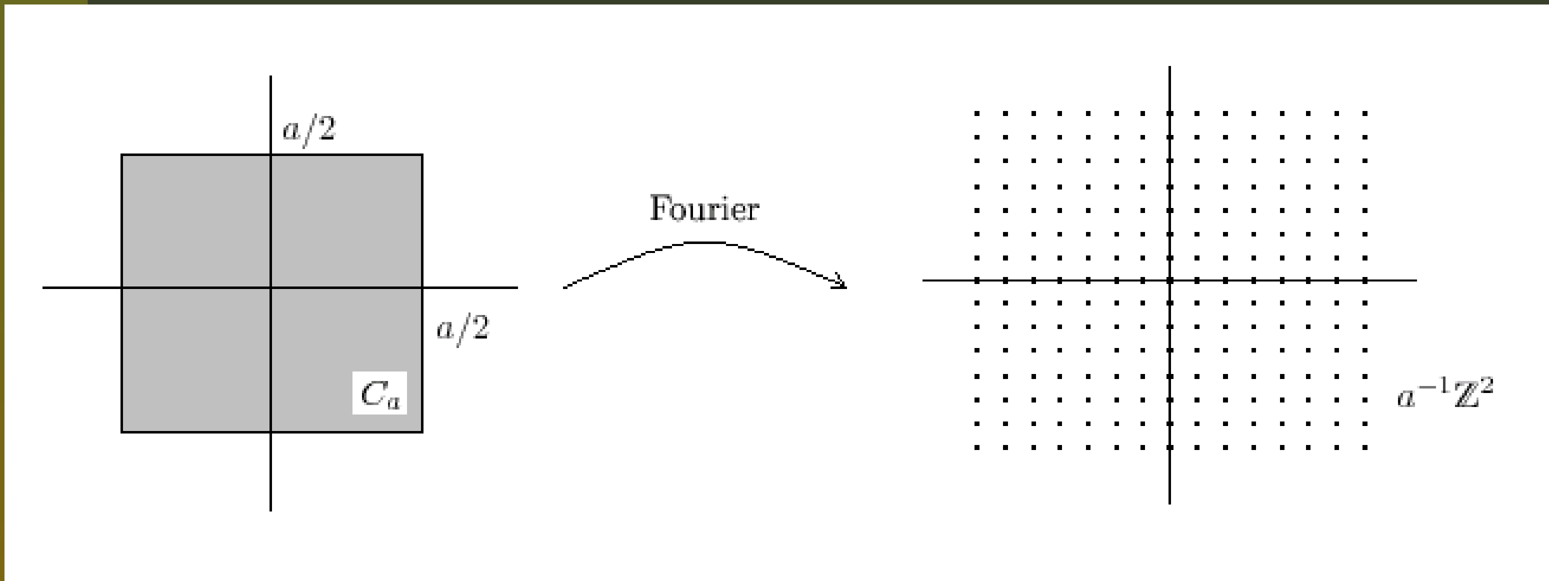

$$f(\mathbf{x}) = \frac{1}{a^n} \sum_{\mathbf{k} \in \mathbb{Z}^n} \hat{f}\left(\frac{\mathbf{k}}{a}\right) \exp\left[2i\pi \left\langle \frac{\mathbf{k}}{a}, \mathbf{x} \right\rangle\right] \mathbb{1}_{C_a}(\mathbf{x})$$

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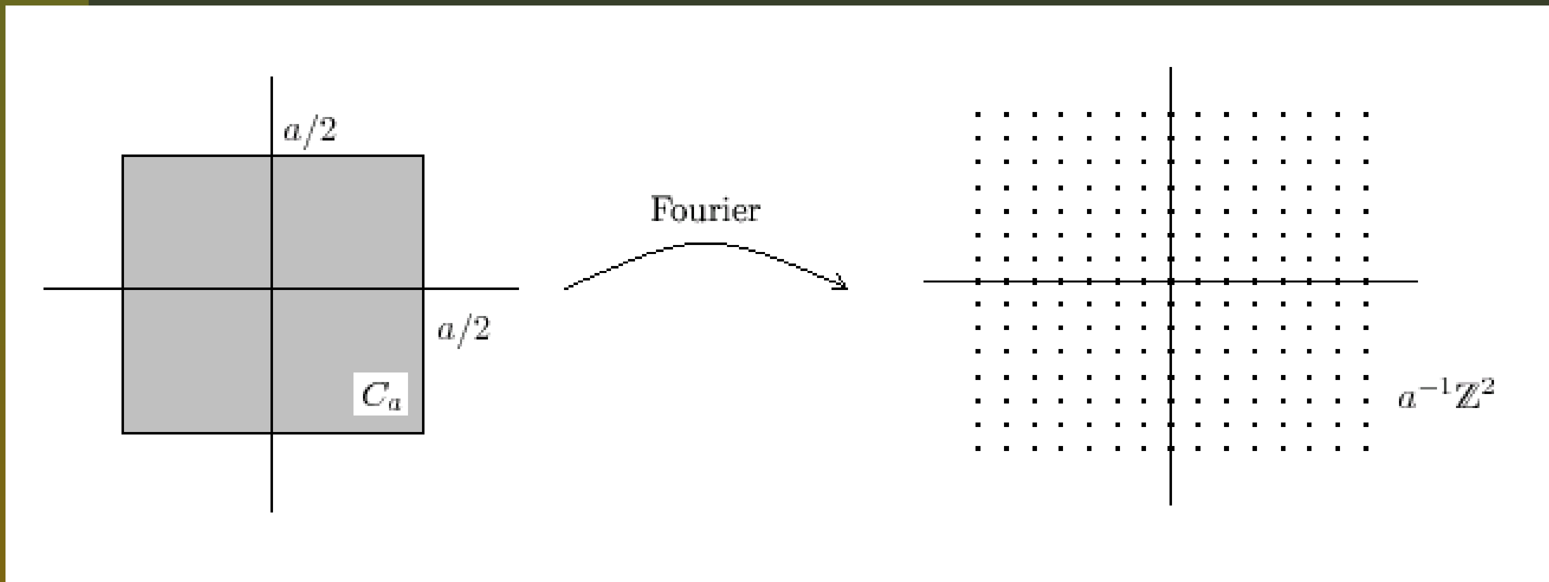
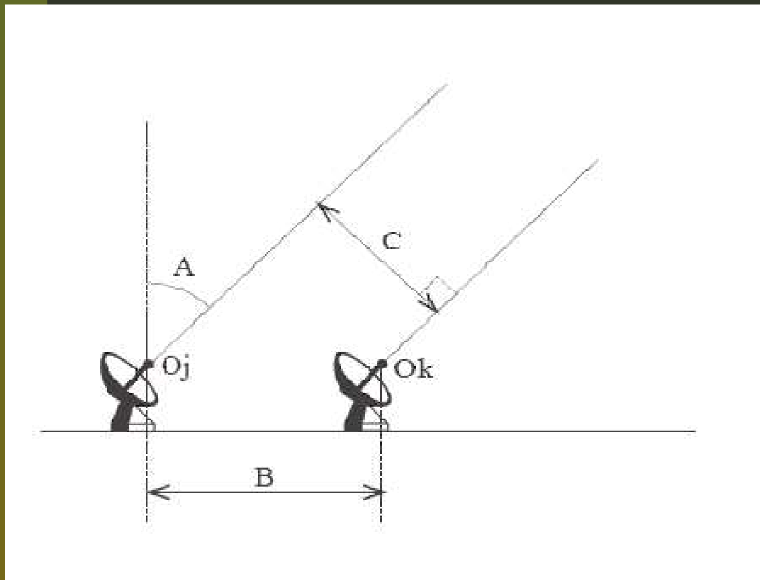
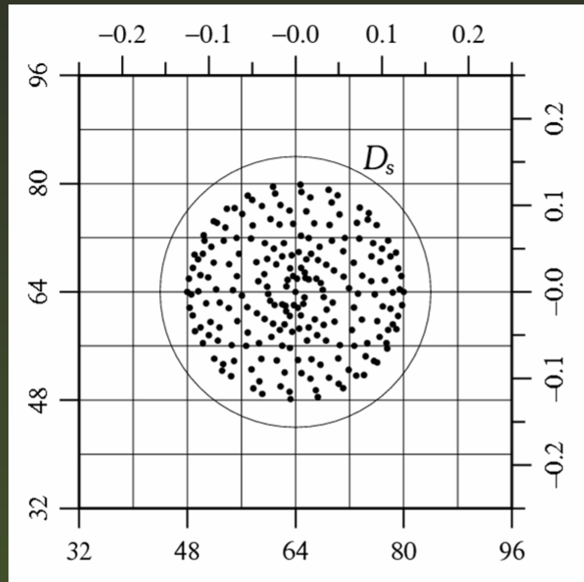


$$\hat{f}(\boldsymbol{\xi}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \hat{f}\left(\frac{\mathbf{k}}{a}\right) \operatorname{sinc} \pi a \left(\boldsymbol{\xi} - \frac{\mathbf{k}}{a}\right)$$

Example 2: Aperture synthesis





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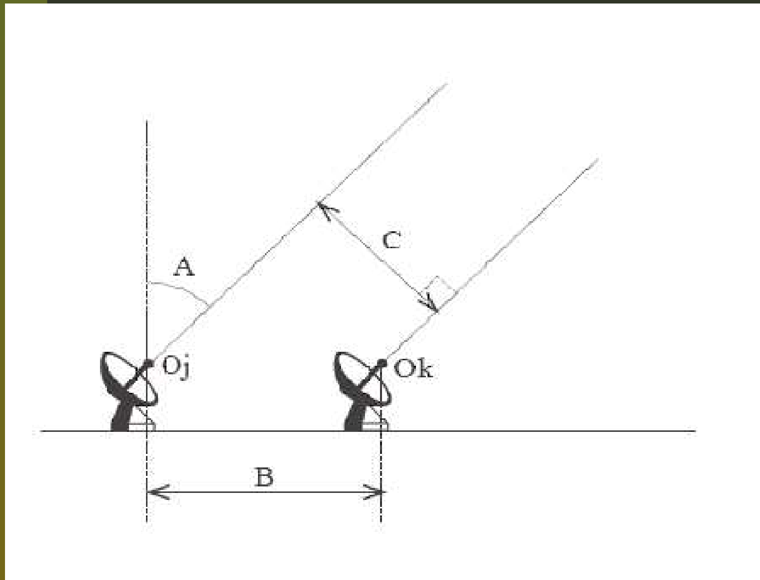
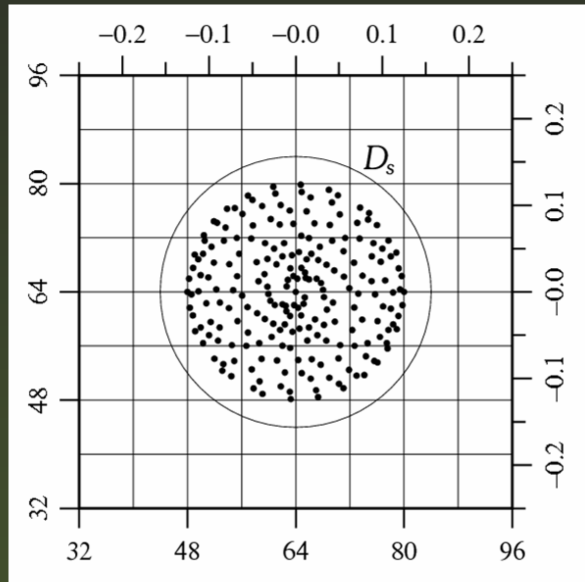
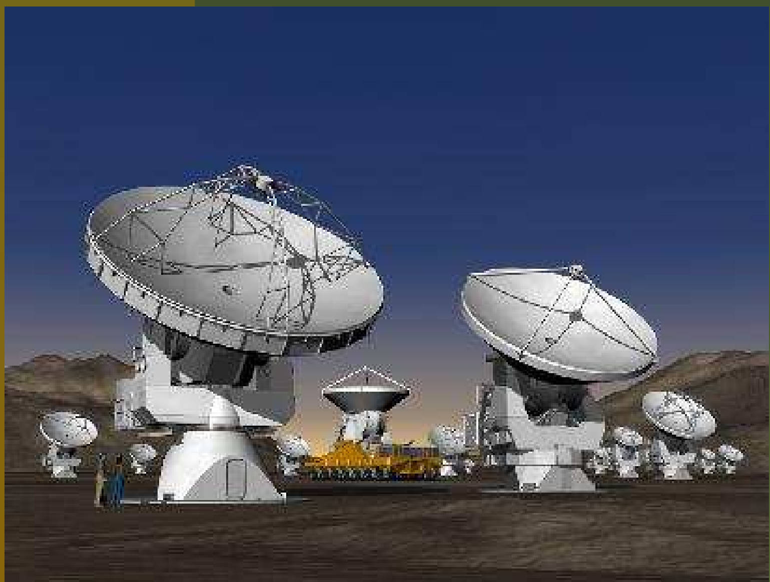
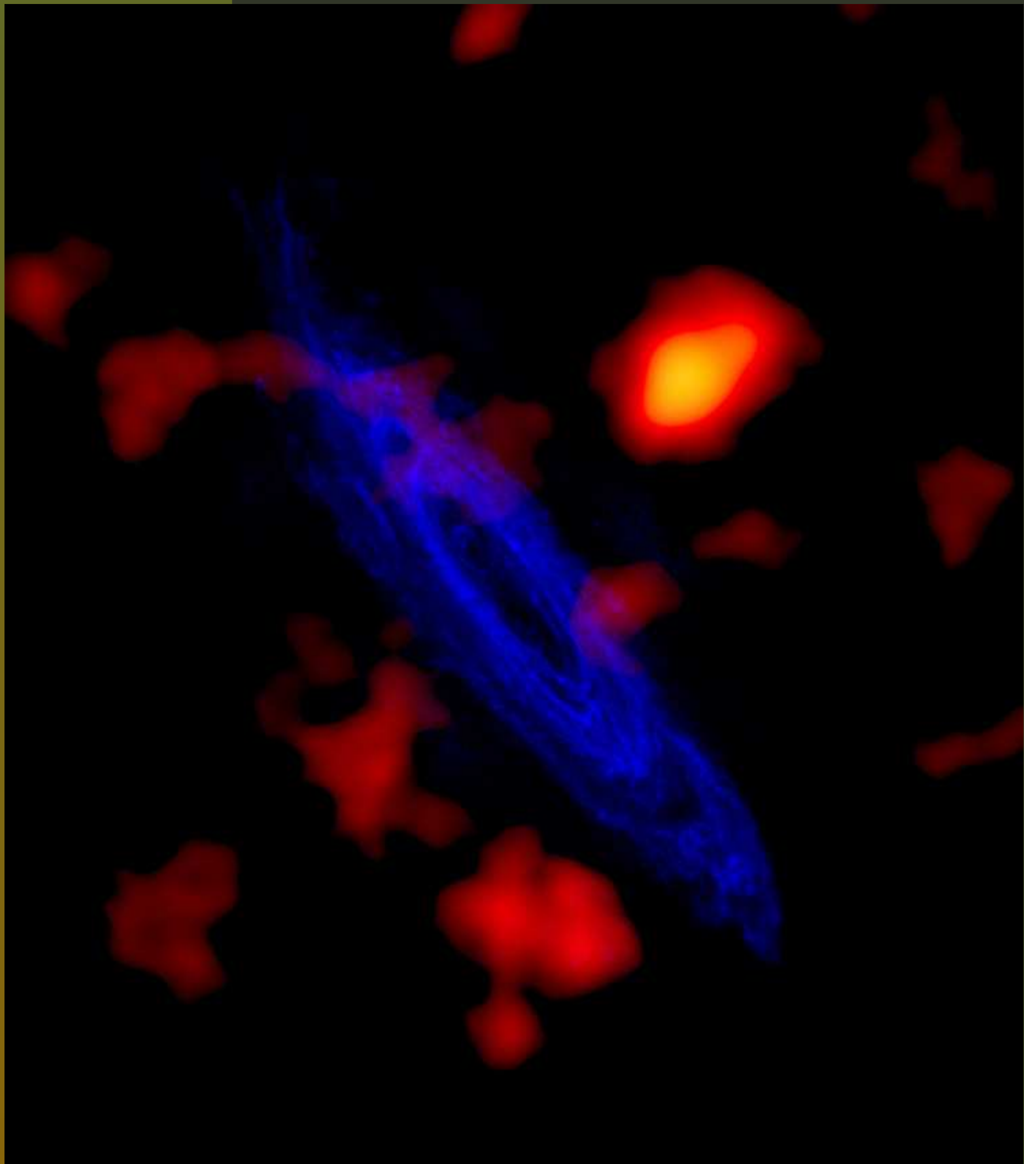






Andromeda Galaxy

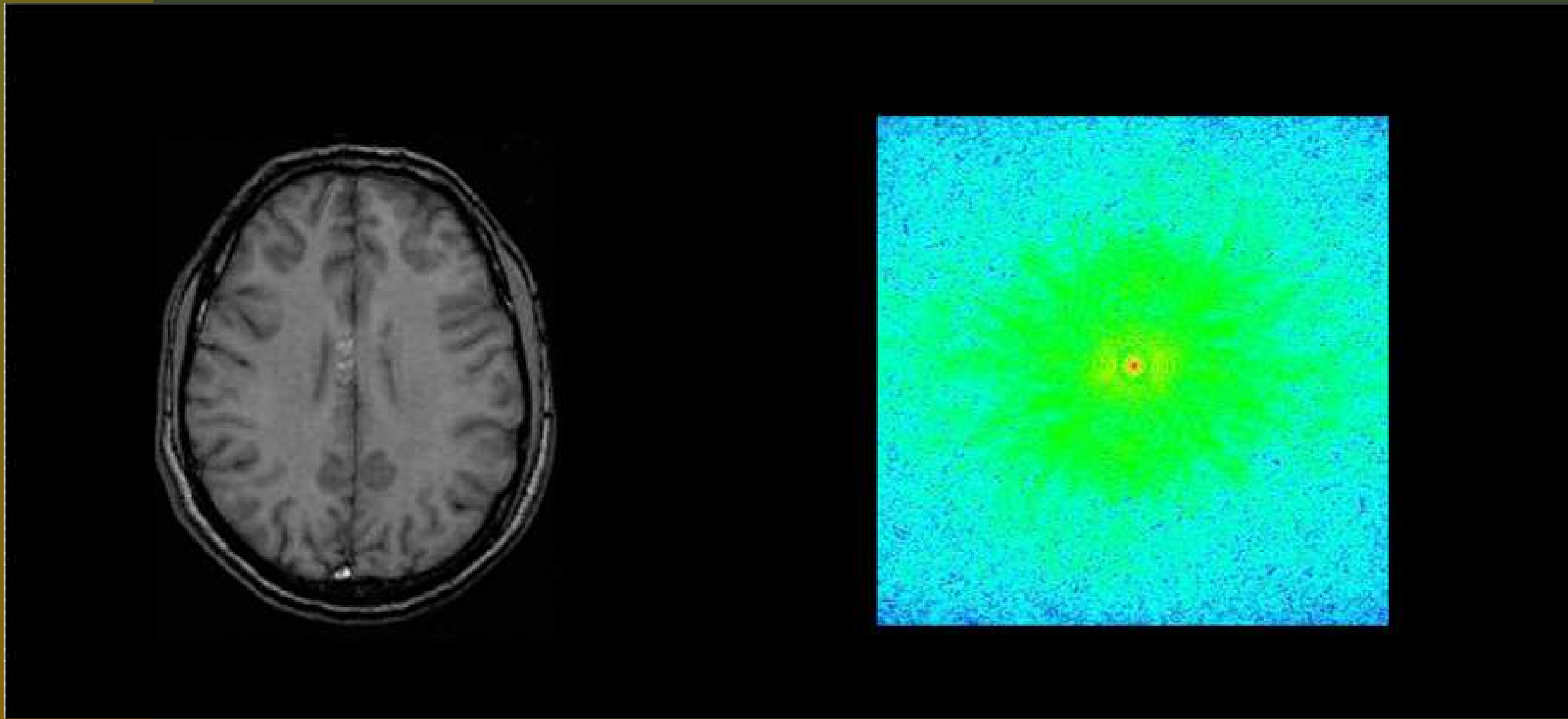


WSRT

Courtesy of *National Radio Astronomy Observatory / Associated Universities, Inc. / National Science Foundation*

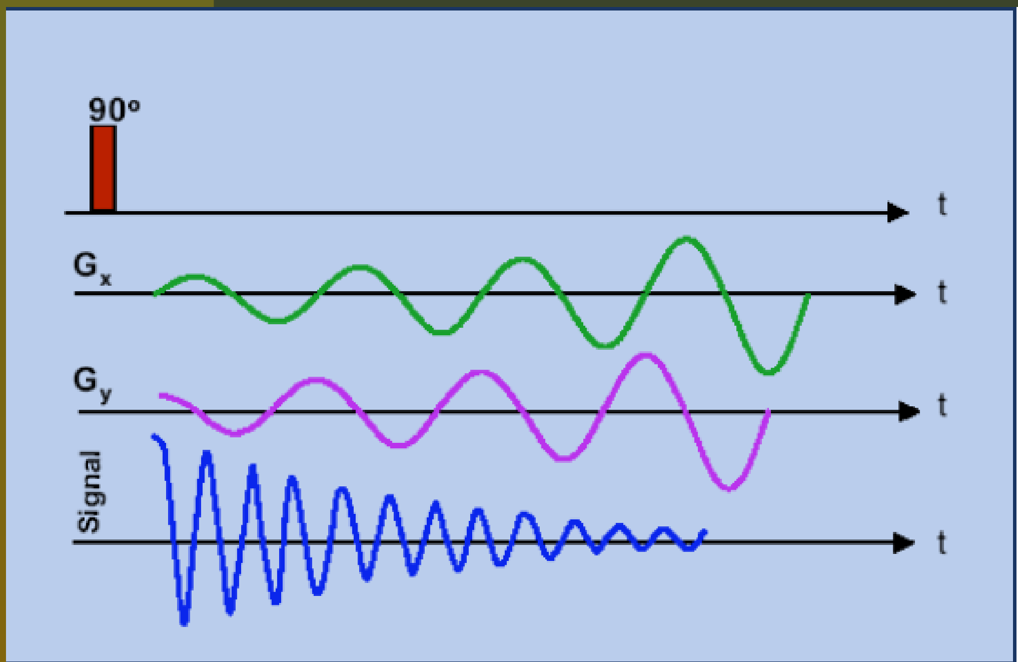
Example 3: MRI

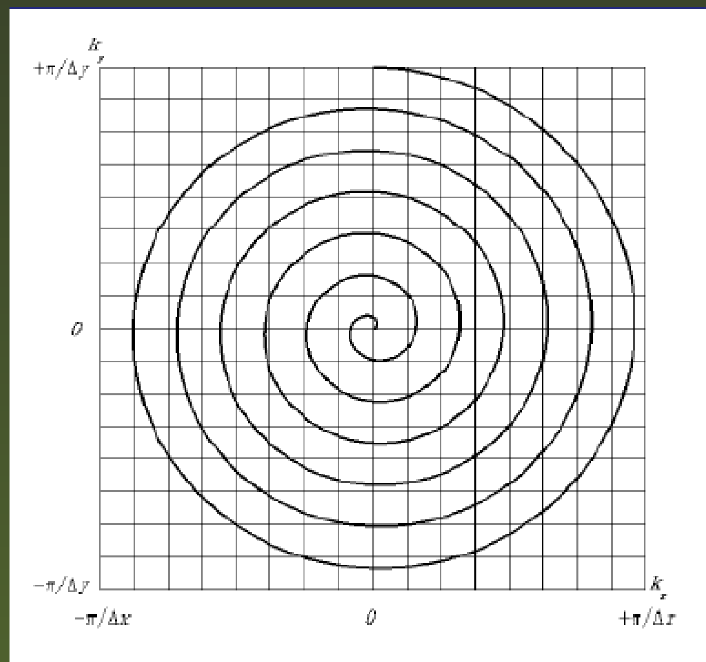
Standard acquisitions:



Example 3: MRI

Non-Cartesian and sparse acquisitions:





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Let V and W be subsets of $\mathbb{R}^d$. Assume that V is bounded and that W has a non-empty interior. Recover $f_0 \in L^2(V)$ from the knowledge of its Fourier transform on W.

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Truncated Fourier operator:

$$\begin{array}{rrcl} T_W : & L^2(V) & \longrightarrow & L^2(W) \\ & f & \longmapsto & T_W f := \mathbb{1}_W \hat{f} = \mathbb{1}_W \mathbb{U} f. \end{array}$$

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$$(T_W f)(\xi) = \int_{\mathbb{R}^d} \underbrace{e^{-2i\pi\langle x,\xi\rangle}\mathbb{1}_V(x)\mathbb{1}_W(\xi)}_{\alpha(x,\xi) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)} f(x)\,\mathrm{d}x.$$

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The operator equation $T_W f = g$ is ill-posed

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Ill-posed equation : $T f_0 = g$ with : $T : F \longrightarrow G$

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Main issues

Well-posedness

Asymptotic behavior ($\alpha \downarrow 0$)

Alternative approach

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Minimiser $\;\frac{1}{2}\,\|\,\Phi_\beta g - Tf\,\|_G^2 + \frac{\alpha}{2}\,\|\,(I - C_\beta) f\,\|_F^2$

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$T = K = U^{-1}\hat{k}U$, convolution by k

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$$\Phi_\beta = C_\beta$$

Examples

$$TC_\beta = \Phi_\beta T \quad \text{with} \quad C_\beta := U^{-1}\hat{\phi}_\beta U$$

$T = R$, the Radon operator

$$(Rf)(\boldsymbol{\theta}, s) = \int f(\mathbf{x})\delta(s - \langle \boldsymbol{\theta}, \mathbf{x} \rangle)\, \mathrm{d}\mathbf{x}$$

$$R(f_1 * f_2) = Rf_1 \circledast Rf_2$$

$\circledast$ convolution w.r.t. s

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Proposition Let $\alpha, \beta > 0$ be fixed. Then $(\mathcal{P}_{\alpha,\beta})$ has a unique solution $f_{\alpha,\beta}$, which depends continuously on $g \in L^2(W)$.

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Theorem Let $\beta > 0$ be fixed and let $g \in \mathcal{D}(T_W^+)$.

(i) If $\hat{\phi}_\beta g \in \mathcal{D}(T_W^+)$, then $f_{\alpha,\beta} \longrightarrow T_W^+(\hat{\phi}_\beta g)$ as $\alpha \downarrow 0$.

(ii) If $\hat{\phi}_\beta g \notin \mathcal{D}(T_W^+)$, then $\| f_{\alpha,\beta} \|_{L^2(V)} \longrightarrow \infty$ as $\alpha \downarrow 0$.

Approximation of T_W^+: $\alpha \downarrow 0$

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Proposition Assume that $\phi \in L^1(\mathbb{R}^d)$ is such that $\hat{\phi}$ is analytical, and let $\beta > 0$ be fixed and $g \in \mathcal{D}(T_W^+)$. Then, the following are equivalent:

(i) $\hat{\phi}_\beta g \in \mathcal{D}(T_W^+)$;

(ii) $\operatorname{supp}(\phi_\beta * T_W^+ g) \subseteq V$.

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If $g \in T_W(L^2(V) \cap H^s(\mathbb{R}^d))$, then $f_{\alpha,\beta} \to T_W^+ g$ strongly as $\beta \downarrow 0$.

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$$\left.\begin{array}{r} (f_n) \text{ bounded} \\ \lim\limits_{R\to\infty} \sup\limits_n \displaystyle\int_{\|x\|>R} |f_n(x)|^2 \,\mathrm{d}x = 0 \\ \sup\limits_n \|\mathcal{T}_h f_n - f_n\| \to 0 \text{ as } \|h\| \to 0 \end{array}\right\} \Rightarrow (f_n) \text{ precompact}$$

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- V bounded $\hookrightarrow \lim\limits_{R\to\infty} \sup\limits_{n} \int_{\|x\|>R} |f_n(x)|^2 \, \mathrm{d}x = 0$

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Step 3: the convergence is in fact strong

- (f_n) is bounded (Step 1)
- V bounded $\hookrightarrow$ $\lim\limits_{R\to\infty} \sup\limits_n \displaystyle\int_{\|x\|>R} |f_n(x)|^2\,\mathrm{d}x = 0$
- $\sup_n \|\mathcal{T}_h f_n - f_n\| \to 0$ as $\|h\| \to 0$

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$\hookrightarrow$ ϕ is positive, isotropic, radially decreasing, C^∞

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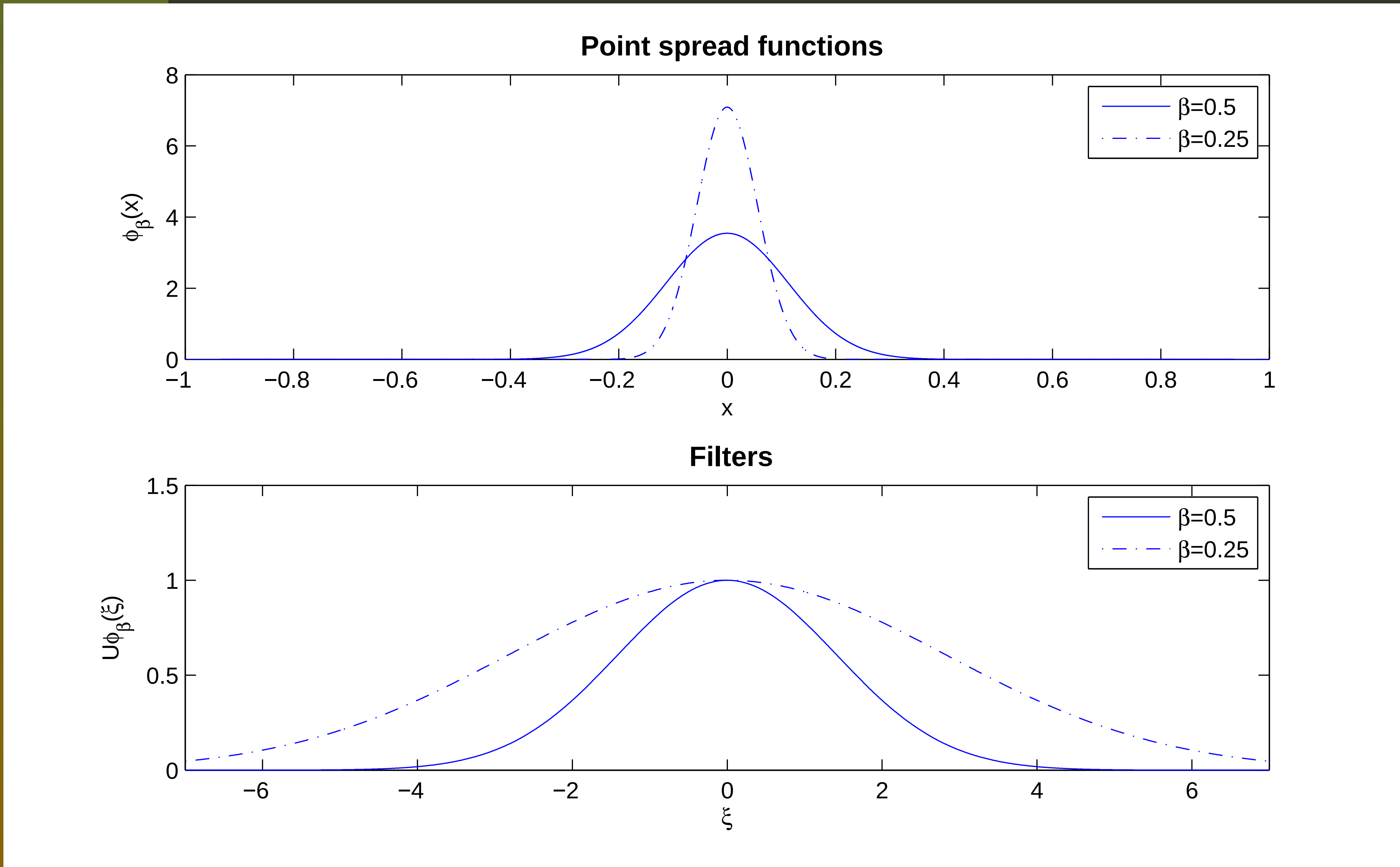

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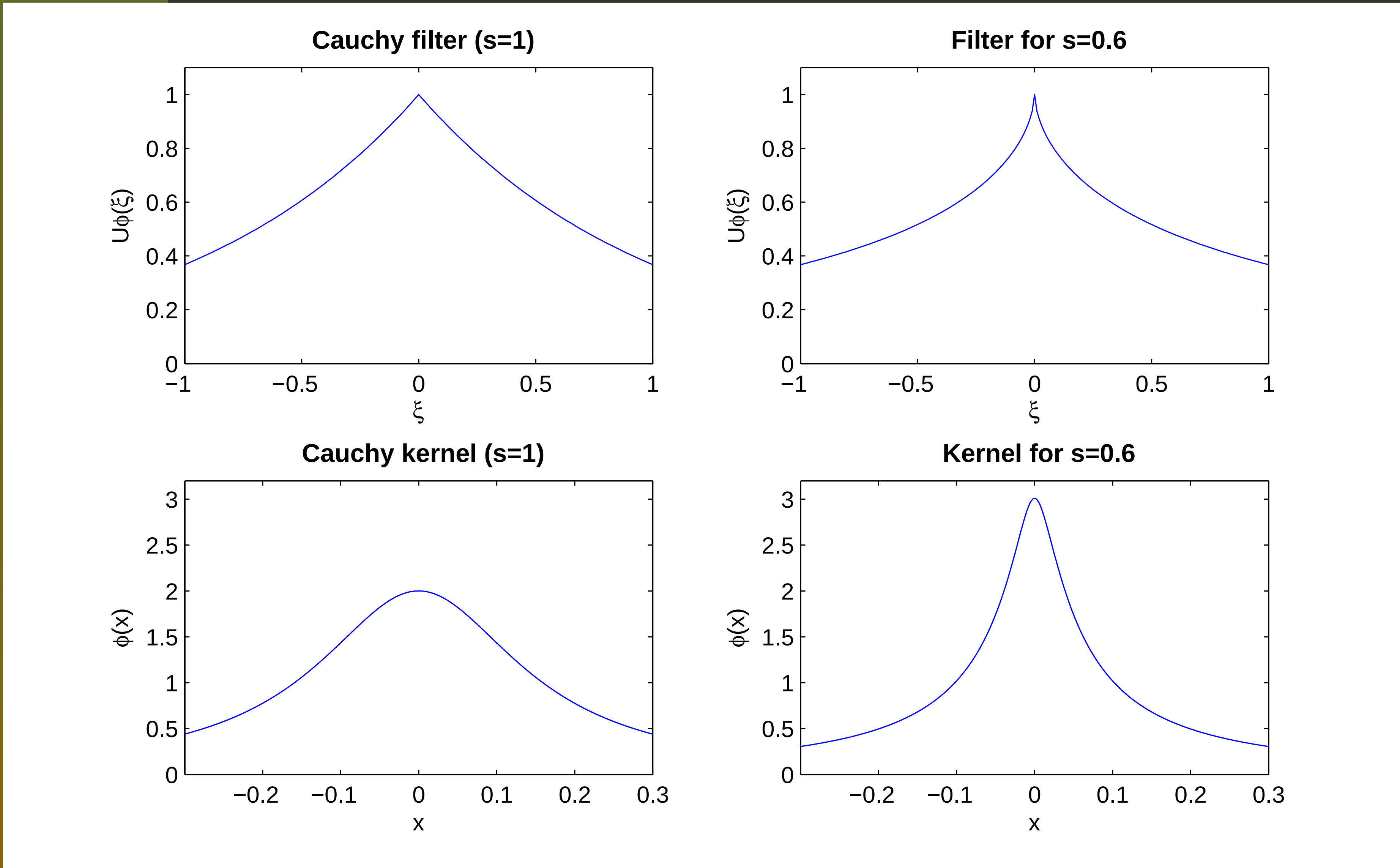

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- Fourier synthesis
- Asymptotic analysis
- Extension
- A note on proximal inversion

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Ill-posed equation : $Tf_0 = g$ with : $T : F \longrightarrow G$

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$$T\colon L^2(V) \to G, \quad G \text{ Hilbert space}$$

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Remark TCT^+ is bounded if and only if there exists a positive number K such that

$$\forall f \in (\ker T)^\perp, \quad \|TCf\|_F \leq K\|Tf\|_G.$$

Example

Proposition Let T be the integral operator of kernel α, that is: $Tf(x) = \int \alpha(x,y) f(y)\,\mathrm{d}y$, with $\alpha \colon \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$. Assume that

(i) $\int_{\mathbb{R}^d \times V} |\alpha(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y < \infty$

(ii) for all $x, y, z \in \mathbb{R}^d$, $\alpha(x, y+z) = \alpha(x,y) g(x,z)$;

(iii) there exists a positive constant M_ϕ, depending on ϕ only, such that

$$\forall x \in \mathbb{R}^d, \quad \left| \int_{\mathbb{R}^d} \phi(z) g(x,z)\,\mathrm{d}z \right| < M_\phi.$$

Then T is well-defined on $\operatorname{ran} C$ and TCT^+ is bounded on its domain.

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Introduction

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$$(Q_\beta) \quad \left| \begin{array}{rl} \text{Minimize} & X \mapsto \|TC_\beta - XT\| \\ \text{s.t.} & X \in M_{m\times n}(\mathbb{R}), \;\; X = 0 \;\text{ on }\; (\operatorname{ran} T)^\perp \end{array} \right.$$

Introduction

For simplicity, we consider here the problem of computing the pseudo-inverse of an ill-posed matrix M.

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Theorem The pseudo-inverse of a matrix $M \in M_{m\times n}(\mathbb{R})$ is the solution of minimum Frobenius norm of the optimization problem

$$(\mathcal{P}) \quad \left| \begin{array}{rl} \text{Minimize} & f(\Phi) := \frac{1}{2}\| M\Phi - I \|_F^2 \\ \text{s.t.} & \Phi \in M_{n\times m}(\mathbb{R}) \end{array} \right.$$

The proximal algorithm

The proximal algorithm

The proximal point algorithm is a general algorithm for computing zeros of maximal monotone operators.

The proximal algorithm

A well-known application is the minimization of a convex function f by finding a zero in its subdifferential.

The proximal algorithm

In our setting, it consists in the following steps:

1. Choose an initial matrix $\Phi_0 \in M_{m\times n}(\mathbb{R})$;
2. Generate a sequence $(\Phi_k)_{k\geq 0}$ according to the formula

$$\Phi_{k+1} = \operatorname*{argmin}_{\Phi\in M_{n\times m}(\mathbb{R})} \left\{ f(\Phi) + \frac{1}{2\mu_k} \|\Phi - \Phi_k\|^2 \right\},$$

in which $(\mu_k)_{k\geq 0}$ is a sequence of positive numbers, until some stopping criterion is satisfied.

Optimality conditions

$$\Phi_{k+1} = \underset{\Phi \in M_{n\times m}(\mathbb{R})}{\operatorname{argmin}} \left\{ \frac{1}{2} \| M\Phi - I \|_F^2 + \frac{1}{2\mu_k} \|\Phi - \Phi_k\|^2 \right\}$$

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Since $M^\top M$ is positive semi-definite and μ_k is positive for all k, the matrix $(I + \mu_k M^\top M)$ is nonsingular and the proximal iteration also reads

$$\Phi_{k+1} = (I + \mu_k M^\top M)^{-1}\left(\Phi_k + \mu_k M^\top\right).$$

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2) The proximal iteration may be performed by means of any efficient minimization algorithm.

Convergence

Convergence

Proposition Let α_1 denote the smallest nonzero eigenvalue of $M^\top M$ and let E_1 be the corresponding eigenspace. Assume that $\mu_k = \mu$ for all k and that Φ_0 is not orthogonal to the eigenspace E_1. Then,

$$\frac{\|M(\Phi_{k+1} - \Phi_k)\|}{\|\Phi_{k+1} - \Phi_k\|} \to \frac{1}{1 + \alpha_1 \mu} \quad \text{and} \quad \frac{\Phi_{k+1} - \Phi_k}{\|\Phi_{k+1} - \Phi_k\|} \to \Psi_1$$

in which Ψ_1 is a unit eigenvector in E_1. Moreover the sequence (Φ_k) generated by the proximal algorithm converges linearly to the orthogonal projection of Φ_0 onto the solution set $\operatorname{argmin} f = M^+ + \ker \mathcal{M}$.

Additional comments

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Tikhonov approximation. A standard approximation of the pseudo-inverse of an ill-conditioned matrix M is $(M^\top M + \varepsilon I)^{-1} M^\top$, where ε is a small positive number. This approximation is nothing but the Tikhonov regularization of M^+, with regularization parameter ε.

The choice $\Phi_0 = 0$ yields the latter approximation for $\varepsilon = 1/\mu$ after one proximal iteration.

Additional comments

Link with existing iterative methods. In the case where $\mu_k = \mu$ for all k, the proximal algorithm belongs to the class of fixed point methods, along with the algorithms of Jacobi, Gauss-Seidel, SOR and SSOR.

It is easy to check that M^+ satisfies the *fixed point equation*

$$\Phi = \varphi(\Phi) := B\Phi + C,$$

with

$$B := (I+\mu M^\top M)^{-1} \quad \text{and} \quad C := (M^\top M+\mu^{-1} I)^{-1} M^\top.$$

Clearly, φ is a contraction and, if M is invertible, then $M^\top M$ is positive definite and φ is a strict contraction.