Multigrid techniques applied in an optimization context

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joint work with

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Outline

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Does it have a sense ?

YES !

Solve a linear system $\Leftrightarrow$ Minimize a quadratic function

$$\min f(x) = \frac{x^T A x}{2} - x^T b + c \Leftrightarrow \nabla_x f(x) = Ax - b = 0 \Leftrightarrow Ax = b$$

Why multigrid

- Solution based on discretization :
  High accuracy $\Rightarrow$ computational cost
- Use of coarse grids :
  1. find a good starting point
  2. solve a subproblem (e.g. the TR subproblem)
- Well-known for solving SPD linear systems resulting of the discretization of a continuous problem
  [W. Briggs, V.E. Henson and S. McCormick, 2000]
- Nonlinear systems
  [W. Hackbuch and A. Reusken, 1989]

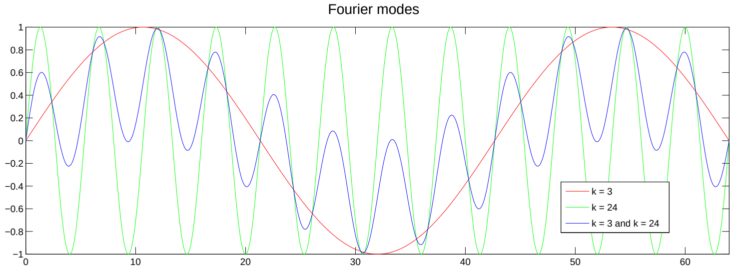
Linear systems

Solve $Ax = b$

$\longrightarrow$ Choice of using an iterative method

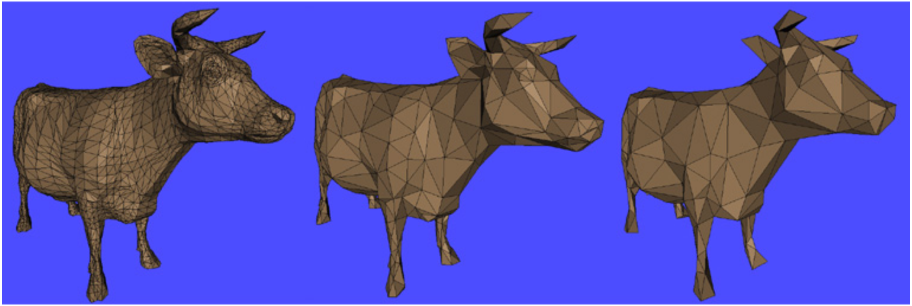
Smoothing/Relaxation methods (Gauss-Seidel)

- Cheap ($\mathcal{O}(n)$)
- Quick in reducing oscillatory components of the error
- Slow in reducing smooth components of the error



Solution :

Use coarser representations



Outline

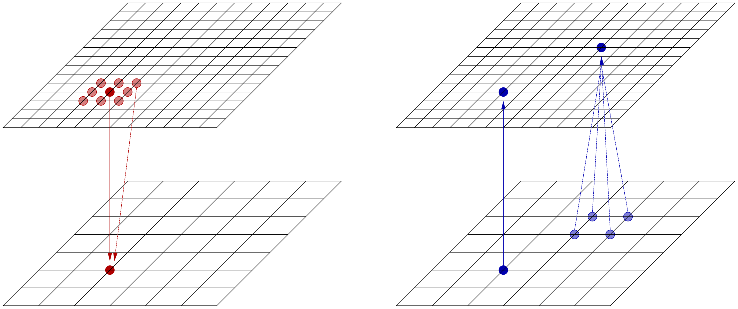
Transfert operators : Geometric case

Restriction

$R_i : \mathbb{R}^{n_i} \longrightarrow \mathbb{R}^{n_{i-1}}$

Prolongation

$P_i : \mathbb{R}^{n_{i-1}} \longrightarrow \mathbb{R}^{n_i}$



$$R_i = \sigma P_i^T$$

Coarse problem definition :

Use the transfert operators

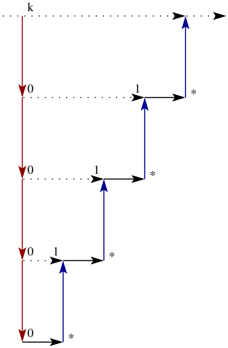
If $A_i x_i = b_i$ is the linear system at level i
Then we define at level $i-1$

$$\boxed{\begin{array}{rcl} A_{i-1} & = & R_i A_i P_i \\ b_{i-1} & = & R_i b_i \end{array}}$$

(Or you already have a coarse definition of A and b)

Mesh refinement : Find a good starting point

- Solve the problem on the coarsest level
  $\Rightarrow$ Good starting point for the next fine level
- Do the same on each level
  $\Rightarrow$ Good starting point for the finest level
- Finally solve the problem on the finest level



2-levels scheme

Residual equation at level i :

$$\boxed{A_i e_{i,k} = r_{i,k}}$$

where $e_{i,k}$ = error and $r_{i,k}$ = residual at iteration k

Solving a linear system$\Rightarrow$ too expensive

2-levels scheme

Approximate the error using coarse grids

In practice :

1. Compute A_{i-1} and $r_{i-1,k}$
2. Find $e_{i-1,*}$ the solution of the residual equation at level $i-1$
3. Prolongate $e_{i-1,*}$ to define an approximation of the error at level i :

$$\boxed{e_{i,k} = P_i e_{i-1,*}}$$

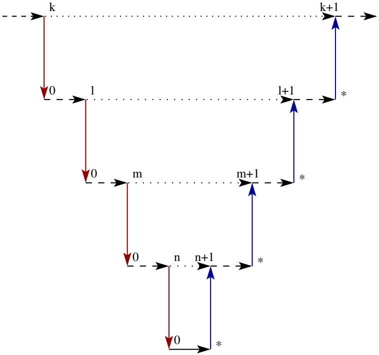
4. Correct your current iterate : $x_{i,k} = x_{i,k} - e_{i,k}$

Combining smoothing and 2-levels scheme

Error behaviour during the multigrid process :

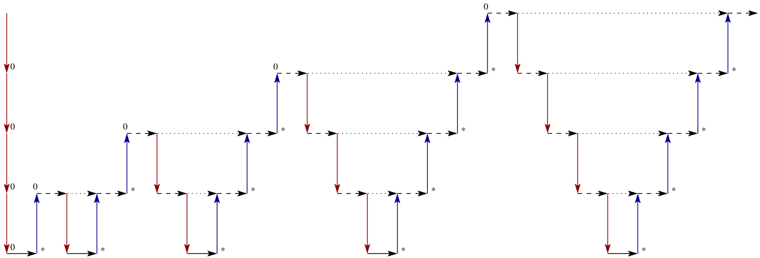
1. Smoothing	Reduces oscillatory error
2. Restriction of the problem	Smooth error appears more oscillatory
3. Smoothing on the coarse grid	Reduces coarse oscillatory error $\Rightarrow$ Reduces smooth fine error
4. Prolongate the solution	Oscillatory error reappears
5. Smoothing again	Reduces this oscillatory error

Recursive use ⇒ V-cycle



Full multilevel scheme

- FMG : Combination of mesh refinement and V-cycles



Outline

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Why trust-region methods

- Newton method : local quadratic convergence
- Trust-region methods : Convergence for all starting point (Global convergence)
- Reduces to the Newton method when close enough to the solution ⇒ Quadratic convergence
- Overview of convergence results and algorithms [A. Conn, N. Gould and Ph. Toint, 2000]

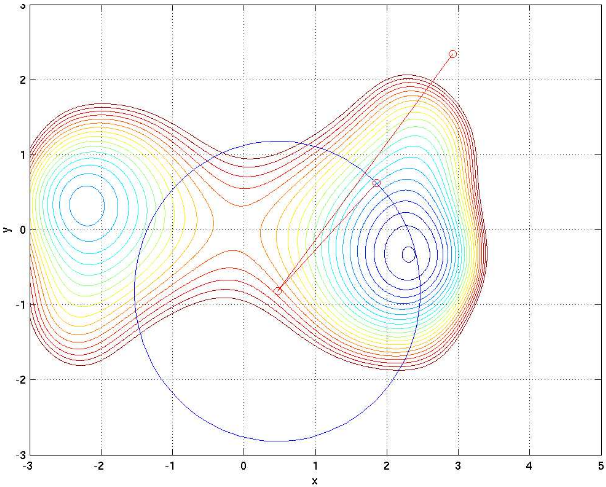
Trust-region mechanism (Bound-constrained optimization)

- Define a model m_k of the objective function f
- Define a trust region where the model is supposed to represent well the objective function
- Compute a step (TR subproblem)
  - inside the TR
  - that sufficiently reduces m_k
  - such that $x_k + s_k \in \{x : l \leq x \leq u\}$
- Step acceptance and TR radius Δ update related to the ratio

$$\frac{f(x_{k+1}) - f(x_k)}{m_k(x_{k+1}) - m_k(x_k)}$$

  - Refuse the step and shrink the TR when the ratio is smaller than a constant
  - Accept the step and possibly enlarge the TR when the ratio is large enough

Trust-region mechanism



Criticality measure and sufficient decrease condition

- Criticality measure : $\chi_k = \left| \min_{\substack{x_k + d \in \mathcal{C} \\ ||d|| \leq 1}} g(x_k)^T d \right|$
  - Unconstrained $\Rightarrow$ Reduces to gradient norm
  - $\chi_k = 0$ at the exact solution
- Stopping criterion : $\chi_k < tol$
- Sufficient decrease condition on the model :

$$m(x_{k+1}) - m(x_k) \geq \kappa \chi_k \min \left[\frac{\chi_k}{1 + ||\nabla^2 f(x_k)||}, \Delta_k, 1 \right]$$

Globally convergent algorithm

Outline

Multigrid ideas in RMTR

- Suppose you have a set of discretizations of the objective function f :

$$\{f_i\}_{i=0}^{r}$$

  with $f_r = f$.

- Transfert operators :

$$\begin{array}{llcll} R_i : & \mathbb{R}^{n_i} & \rightarrow & \mathbb{R}^{n_{i-1}} & \text{Restriction} \\ P_i : & \mathbb{R}^{n_{i-1}} & \rightarrow & \mathbb{R}^{n_i} & \text{Prolongation} \end{array}$$

- Coarse model : Need to be first order coherent

$$\Leftrightarrow$$

"tau correction" in multigrid

Smoothing

In multigrid : Smoothing = Solving the equations of the linear system one by one

In optimization :

- Smoothing = Solving the minimization problem along the coordinate axes j.
- Bound-constrained unidirectional problem (Trust-region and possibly original bounds constraints)
- The final step is defined by $s = \sum_j s_j$

Smoothing ⇒ Sufficient decrease

For unconstrained optimization

If the minimization begins in the direction $\text{argmax}_j |g_j|$

Convergence theory available

[S. Gratton, A. Sartenaer and Ph. Toint, 2005]

For bound-constrained optimization

If the minimization begins in the direction $\text{argmax}_j g_j^T d_j$

where d is defined by $\left|\text{argmin}_{\substack{x_k + d \in \mathcal{C} \\ \|d\| \leq 1}} g(x_k)^T d\right|$

Coarse step

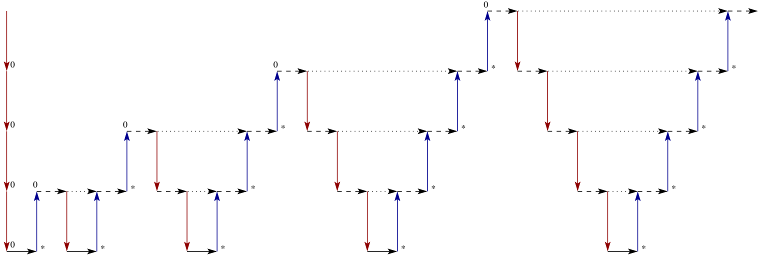
Step computation on a coarse level :

- Define $x_{i-1,0} = R_i x_{i,k}$
- Find a coarse step s_{i-1} (e.g. using smoothing)
  - inside a coarse version of the TR
  - inside a coarse version of the bounds, and
  - that reduces sufficiently a coarse model f_{i-1} of the objective function f_i
- Use P_i to obtain a fine step s_i by $s_i = P_i s_{i-1}$
- Coarse sufficient decrease $\Rightarrow$ fine sufficient decrease

Full multigrid (FMG)

As in multigrid methods :

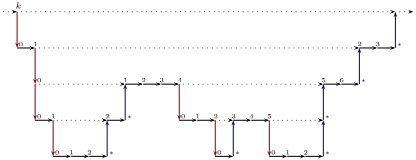
- Mesh refinement
- V-cycles



Outline

Free cycles

Solve at a coarse level i until $\chi_i < \varepsilon_i$



Example of recursion with 5 levels ($r = 4$)

Descent condition

Use coarse levels only if

$$\boxed{\chi_{i-1,0} \geq \kappa_{\chi} \chi_{i,k}}$$

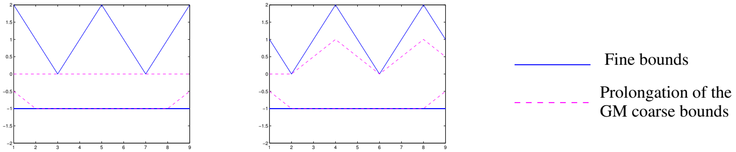
not worth working on coarser level
if the problem is already solved there

Coarse problem definition

- Coarse TR : Restriction of the fine TR using R_i
- Coarse bounds : Gelman & Mandel's definition

$$[l_{r-1}]_j = [R_r x_{r,k}]_j + \max_{\substack{t=1,\ldots,n_r \\ [P_r]_{tj}>0}} [l - x_{r,k}]_t$$

$$[u_{r-1}]_j = [R_r x_{r,k}]_j + \min_{\substack{t=1,\ldots,n_r \\ [P_r]_{tj}>0}} [u - x_{r,k}]_t$$



Nonlinear multigrid ?

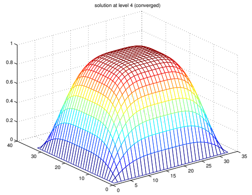
Multigrid : Solving a 1st order Taylor approximation
(e.g. a linear system)

$\Rightarrow$

RMTR is equivalent if a 2nd order Taylor model is used in the trust-region method (e.g. quadratic minimization problem)

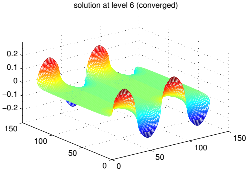
Outline

Some treated problems



- Quadratic minimization :

  $\min_u u^T A u - 2u^T b \Leftrightarrow Au = b$



- Minimal surface :

  $\min_u \int\int \sqrt{1 + u_x^2 + u_y^2} dxdy$

Quadratic minimization

	Mesh Refinement	$RMTR_\infty$
Nb iter	365	6
Nb eval f	2	11
Nb eval g	2	11
Nb eval H	1	1

Minimal surface with obstacle

	Mesh Refinement	RMTR_∞
Nb iter	1410	111
Nb eval f	95	182
Nb eval g	85	177
Nb eval H	58	29

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Conclusion

RMTR :

- Optimization algorithm inspired by multigrid ideas
- Adapts multigrid techniques
- Proved globally convergent
- Adapted to bound-constrained problems
- Very efficient on discretized problems

Thank you !