The Bitter Truth About Interior-Point Methods

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Outline

- Interior-point methods for QP
- Three little linear systems (and their properties)
- Regularization
- Numerical illustration
- Conclusions and future work

Interior-Point Methods for Quadratic Programming

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \;\; c^T x + \tfrac{1}{2} x^T H x \quad \text{subject to} \;\; Ax = b, \;\; x \geq 0 \qquad \text{(QP)}$$

At each iteration, must solve

$$\begin{bmatrix} H & A^T & -I \\ A & & \\ -Z & & -X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \qquad \text{(U3x3)}$$

$$X = \operatorname{diag}(x) \succ 0 \qquad Z = \operatorname{diag}(z) \succ 0$$

- Large (but sparse)
- Not symmetric
- Structure?

Optimizers Have Bad Habits

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$$\begin{bmatrix} H & A^T & -I \\ A & & \\ -I & & -Z^{-1}X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} p \\ q \\ Z^{-1}r \end{bmatrix}$$

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$$\begin{bmatrix} H + X^{-1}Z & A^T \\ A & \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} p + X^{-1}r \\ q \end{bmatrix} \tag{S2x2}$$

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$$A(H + X^{-1}Z)^{-1}A^T\Delta y = (H + X^{-1}Z)^{-1}(p + X^{-1}r) - q \quad \text{(S1x1)}$$

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IF

- A has full rank
- No free variables

Interior-Point Software

- Most libraires use the normal equations formulation (S1x1) (for NLP, SDP, SOCP, ...): PCx, GLPK, BPMPD, HOPDM, LIPSOL, CPLEX, Gurobi, SDPT3, SeDuMi, ...
- A few use (S2x2): OOQP, QPB, KNITRO, IPOPT, BPMPD, HOPDM, ...
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Which one is "best?"

Related Work

- Nonsingularity, good conditioning of 3×3 matrix approaches has been observed by Forsgren, Saunders, ...
- Wright (1998): ill-conditioning of 2×2 matrix is “benign”
- A partially reduced system approach—multiply the third block equation by Z^{-1}, partition $Z^{-1}X$ and eliminate according to the size of elements. See Gill, Murray, Ponceleon, and Saunders (1992); Benzi, Haber and Taralli (2009).
- Forsgren, Gill (1998), Forsgren, Gill, Griffin (2005) : doubly augmented formulation, SPD on the central path
- Spectral observations by by Armand & Benoist, Wright, Forsgren, Korzak, ...

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Theorem

$$\lambda(K_1) \subset \left[\frac{\sigma_{\min}(A)^2}{\lambda_{\max}(H + X^{-1}Z)}, \frac{\sigma_{\max}(A)^2}{\lambda_{\min}(H + X^{-1}Z)}\right]$$

and these bounds are tight.

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Corollary

$$\mathrm{cond}_2(K_1) \leq \mathrm{cond}_2(A)^2 \, \mathrm{cond}_2(H + X^{-1}Z)$$

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- Common case: $\operatorname{cond}_2(K_1) = \operatorname{cond}_2(A)^2\ O(1/\mu^2)$

Augmented System (S2x2): $\mathbf{K_2}$

- Advocated by Fourer, Mehrotra and (subsequently) others
- Sparse, symmetric, indefinite: LBL^T, MINRES, SYMMLQ, ...

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Theorem (Gould, 1985?)

If $H \succeq 0$, A has full rank and $(x, z) > 0$, $\text{In}(K_2) = (n, m, 0)$.

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Theorem (Rusten & Winther, 1992)

If $H \succeq 0$, A has full rank and $(x, z) > 0$,

$$\lambda_{\min} \leq \lambda^{+} \leq \tfrac{1}{2}\left(\lambda_{\max} + \sqrt{\lambda_{\max}^2 + 4\sigma_{\max}^2}\right)$$

$$\tfrac{1}{2}\left(\lambda_{\min} - \sqrt{\lambda_{\min}^2 + 4\sigma_{\max}^2}\right) \leq \lambda^{-} \leq \tfrac{1}{2}\left(\lambda_{\max} - \sqrt{\lambda_{\max}^2 + 4\sigma_{\min}^2}\right)$$

where $\lambda_{\min\,/\,\max} := \lambda_{\min\,/\,\max}(H + X^{-1}Z)$.

Asymptotic Behavior of K_2

Typically, at least one $x_i \to 0$, $\lambda_{\min} \to 0$ and $\lambda_{\max} \to +\infty$ so that we have the estimates

$$\begin{aligned} \lambda_{\min} &\le \lambda^+ \lesssim \lambda_{\max} \\ -\sigma_{\max} &\lesssim \lambda^- \lesssim -\sigma_{\min}^2/\lambda_{\max}. \end{aligned}$$

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The situation is only slightly better than with K_1.

An Alternative Symmetric 3×3 System: $\mathbf{K_3}$

$$K_3 := \begin{bmatrix} H & A^T & -Z^{\frac{1}{2}} \\ A & & \\ -Z^{\frac{1}{2}} & & -X \end{bmatrix} = \begin{bmatrix} I & & \\ & I & \\ & & Z^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} H & A^T & -I \\ A & & \\ -Z & & -X \end{bmatrix} \begin{bmatrix} I & & \\ & I & \\ & & Z^{\frac{1}{2}} \end{bmatrix}$$

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Fact: K_3 has the same eigenvalues as (U3x3).

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Theorem

Assume H is positive definite on $\text{Null}(A)$ and A has full rank. Then $\text{In}(K_3) = (n, n+m, 0)$.

Spectral Analysis

Theorem

The positive eigenvalues of K_3 are bounded in

$$\left[\min_{j\in\mathcal{I}} \frac{1}{2}\left(\lambda_n - x_j + \sqrt{(\lambda_n + x_j)^2 + 4z_j}\right), \frac{1}{2}\left(\lambda_1 + \sqrt{\lambda_1^2 + 4(\sigma_1^2 + z_{\max})}\right)\right].$$

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$$\left. \frac{1}{2}\left(\lambda_1 + \sqrt{\lambda_1^2 + 4(\sigma_1^2 + z_{\max})}\right)\right].$$

Lower bound is useful if $\lambda_n = 0$, but if $H \succ 0$, an alternative uniform lower bound can be used:

Corollary

λ_n is a uniform lower bound for the positive eigenvalues of K_3.

Bounds: Negative Eigenvalues

Theorem

Suppose $-x_i < 0$ is not an eigenvalue of K_3 for any $i = 1, \ldots, n$. The negative eigenvalues of K_3 are bounded in $I_- = [\zeta, 0)$, where

$$\zeta := \min \left\{ \frac{1}{2} \left(\lambda_n - \sqrt{\lambda_n^2 + 4\sigma_1^2} \right), \min_{\{j | \theta + x_j < 0\}} \theta_j^* \right\}$$

and θ_j^* is the smallest negative root of the cubic equation

$$\theta^3 + (x_j - \lambda_n)\theta^2 - (\sigma_1^2 + z_j + x_j \lambda_n)\theta - \sigma_1^2 x_j = 0.$$

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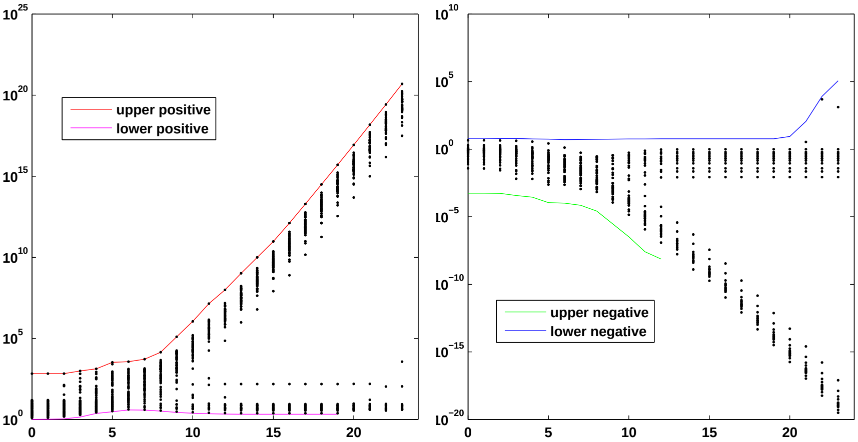
In practice, the condition number of K_3 can be substantially better than that of K_2, though we have not been able to establish a nonzero upper bound on λ^-.

Numerical Illustration: Eigenvalues

Problem AFIRO from CUTEr collection : K_2



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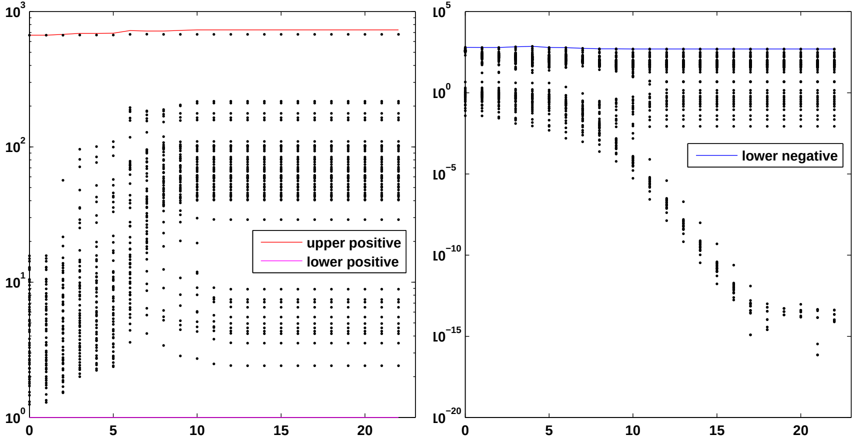
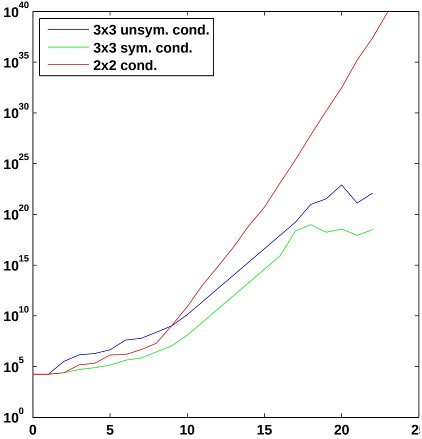


Numerical Illustration: Condition Numbers

Problem AFIRO from CUTEr collection



Regularization

Friedlander and Orban (2012) suggest to solve

$$\begin{aligned}&\underset{x\in\mathbb{R}^n, r\in\mathbb{R}^m}{\text{minimize}}\ c^T x + \tfrac{1}{2}x^T Hx + \tfrac{1}{2}\rho\|x - x_k\|^2 + \tfrac{1}{2}\delta\|w + y_k\|^2 \\ &\text{subject to } Ax + \delta w = b,\ x \geq 0\end{aligned} \tag{RQP}$$

to find a solution of (QP).

At each iteration, must now solve

$$\begin{bmatrix} H + \rho I & A^T & -I \\ A & \delta I & \\ -Z & & -X \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{RU3x3}$$

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No more assumptions that A has full rank or that there are no free variables.

Three Little Linear Systems, Revisited

1. Regularized symmetric 3×3:

$$K_{3R} := \begin{bmatrix} H + \rho I & A^T & -Z^{\frac{1}{2}} \\ A & -\delta I & \\ -Z^{\frac{1}{2}} & & -X \end{bmatrix}$$

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2. Regularized 2×2:

$$K_{2R} := \begin{bmatrix} H + X^{-1}Z + \rho I & A^T \\ A & -\delta I \end{bmatrix}$$

3. Regularized normal equations:

$$K_{1R} := A(H + X^{-1}Z + \rho I)^{-1}A^T + \delta I$$

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$$\lambda(K_{1R}) \subset \left[\frac{\sigma_{\min}(A)^2}{\lambda_{\max}(H + X^{-1}Z) + \rho} + \delta,\ \frac{\sigma_{\max}(A)^2}{\lambda_{\min}(H + X^{-1}Z) + \rho} + \delta \right]$$

and these bounds are tight.

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Corollary

$$\text{cond}_2(K_{1R}) \leq \sigma_{\max}(A)^2/(\rho\delta)$$

and this bound is (quite) tight.

Symmetric 2×2: $\mathbf{K_{2R}}$

Asymptotically, we have the following corollary from a theorem of Friedlander and Orban (2012):

Corollary

$$\begin{array}{rcccl} \rho & \leq & \lambda^+ & \lesssim & \lambda_{\mathsf{max}}(H + X^{-1}Z + \rho I) \\ -\sigma_{\mathsf{max}} & \lesssim & \lambda^- & \lesssim & -\delta \end{array}$$

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Corollary

$$\mathrm{cond}_2(K_{2R}) \lesssim \lambda_{\max}(H + X^{-1}Z + \rho I) / \min(\rho, \delta)$$

Symmetric 3×3: $\mathsf{K_{3R}}$

Asymptotically and if the problem is appropriately scaled,

Theorem

$$\begin{array}{ccccc} \rho & \leq & \lambda^+ & \lesssim & \eta \\ -\sigma_{\mathsf{max}} & \lesssim & \lambda^- & \lesssim & -\delta \end{array}$$

where η is the largest root of

$$\theta^3 + (\delta - (\lambda_1 + \rho))\theta^2 - (\delta(\lambda_1 + \rho) + \sigma^2_{\mathsf{max}} + z_{\mathsf{max}})\theta - z_{\mathsf{max}}\delta.$$

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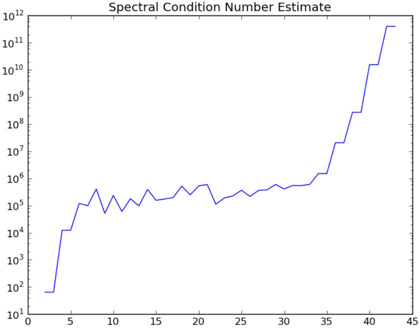
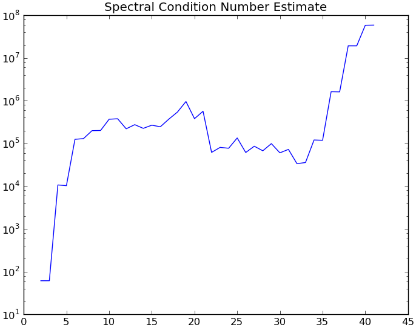
Corollary

$$\mathrm{cond}_2(K_{3R}) \lesssim \eta / \min(\rho, \delta) < \infty$$

Numerical Illustration

Problem CVXQP1M from the CUTEr collection

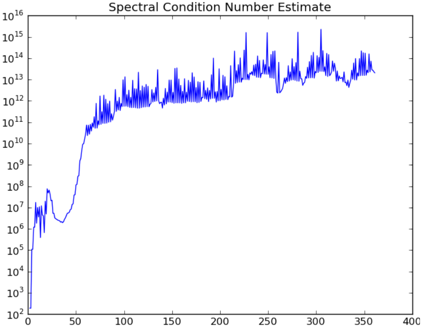


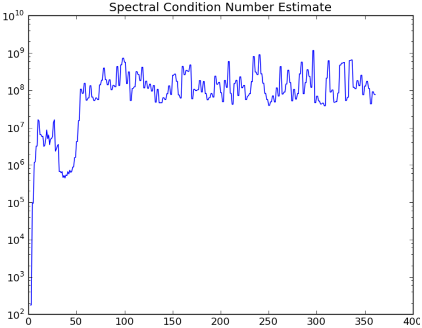


2×2 3×3

Numerical Illustration

Problem CVXQP1L from the CUTEr collection





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Bottom Line

DON'T REDUCE YOUR SYSTEM!

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REGULARIZE!

Conclusions and Future Work

- Impact on performance in factorization-based IPM seems minor
- Accuracy?
- Expect much more impact on matrix-free IPM!
- Study preconditioners
- Adequate iterative methods

You Can Now Resume Normal Activity

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