Using Overlapping and Filtering Techniques for Parallel Preconditoners

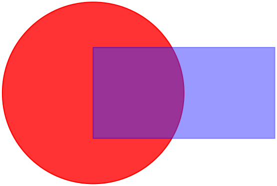
Long Qu

PhD supervisor L.Grigori

Collaborator F. Nataf and R.Fezzani

June 25, 2012

Introduction



- Schwarz type preconditoner $\approx$ Block Jacobi + overlap
- $\Longrightarrow$ overlap helps convergence

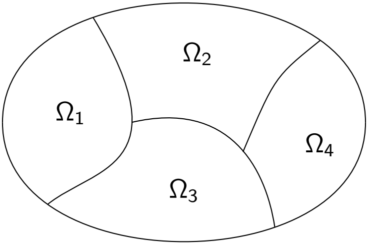
Objectif

use the overlapping technique from DDM in incomplete LU type preconditioners

Plan

Plan

K-way partition

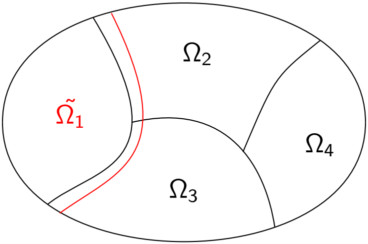


iLU type preconditioners are based on nested dissection

- Block incomplete LDU preconditioner
- Nested preconditoners

Overlapping in Domain Decomposition Method

K-way partition



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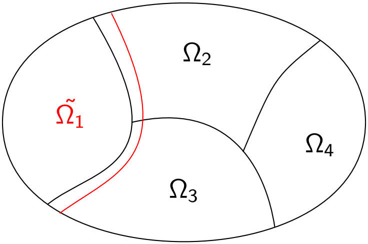

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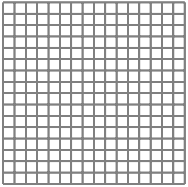
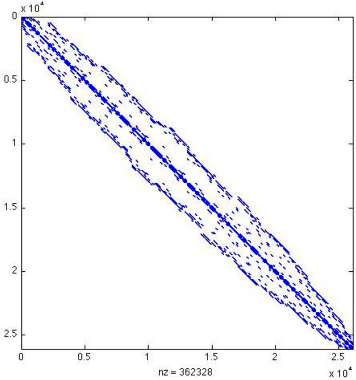
Basic Overlapping Strategy





- Given a domain
- Choose a separator
- Split into 3 parts
- Find connections
- Extend each part
- Rearrange the matrix

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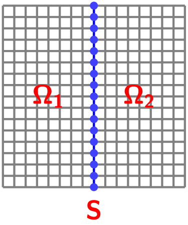

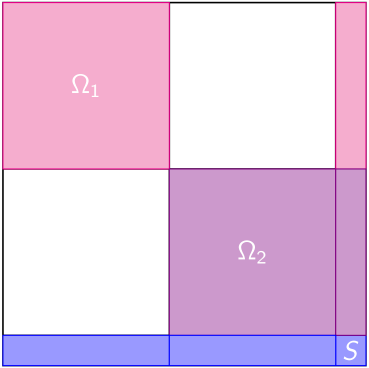

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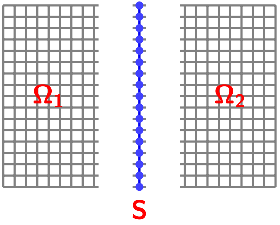

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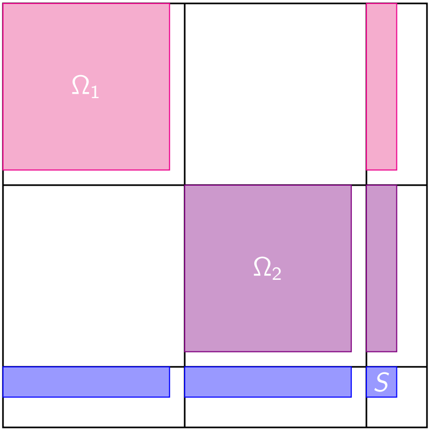

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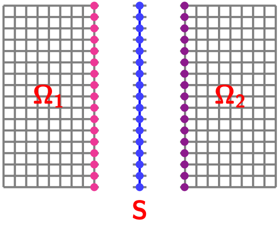

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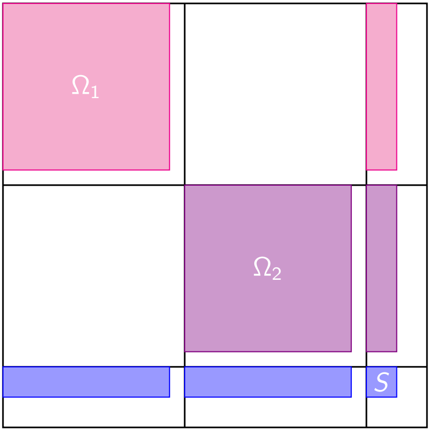

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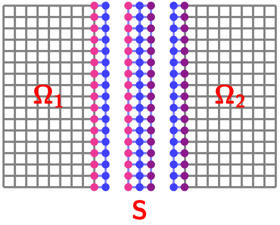

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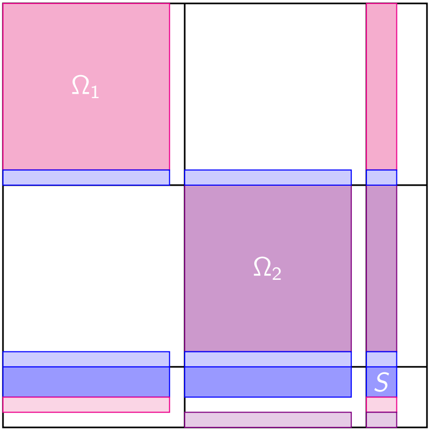

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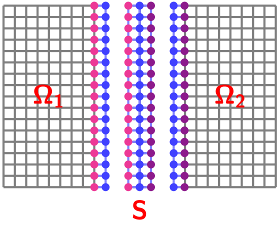

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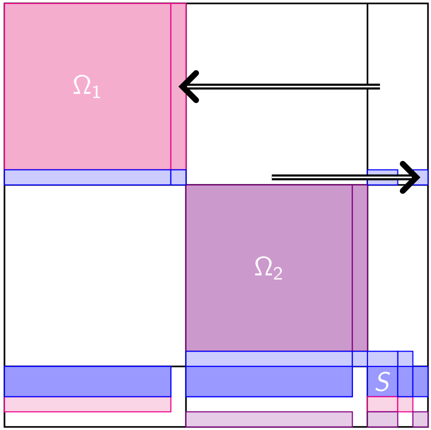
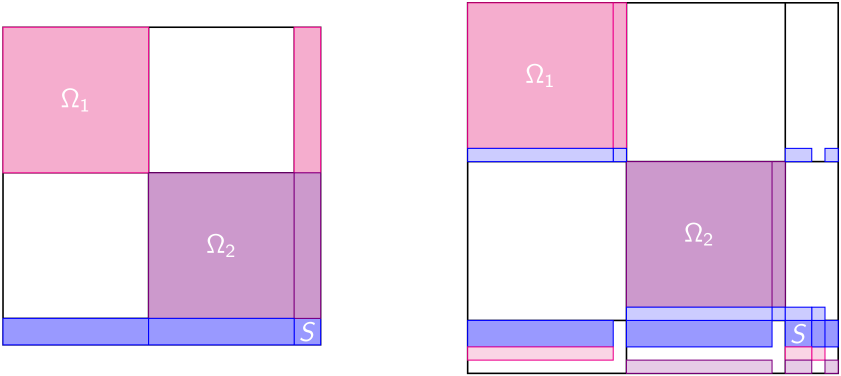
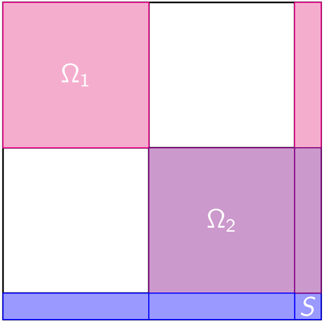


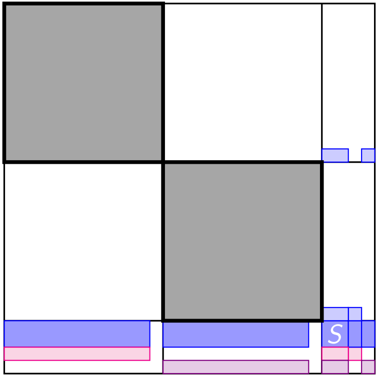
Multi-level Overlapping Strategy



- A Parallel Multi-Level Overlapping Strategy ! !
- How much we have to pay ? How many vertices we have to add ?

Multi-level Overlapping Strategy





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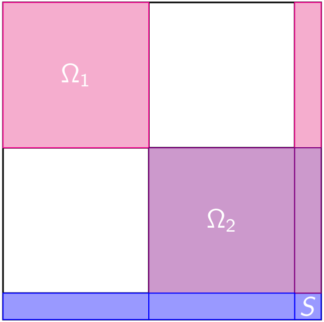

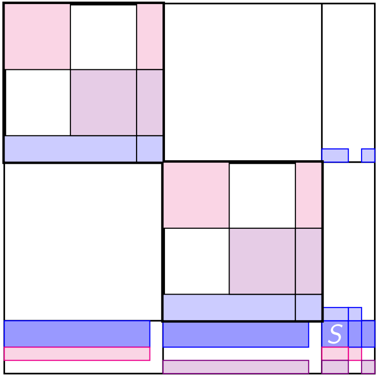

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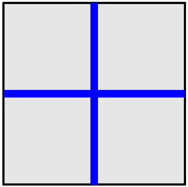
Analysis of Overlapping on a Regular Grid



- **On a $n \times n$ 2D grid, for p parts,** $V_{add} \approx 8(\sqrt{p} - 1)n$
  - example : $p = 4 \times 4 = 16$, $V_{add} \approx 24n$
- **for each part,** $v_{add} \approx \frac{4(\sqrt{p}-1)p}{n^2}$**, it's affordable !**

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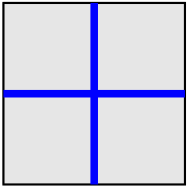
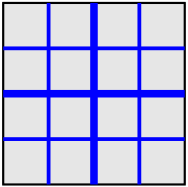


4 parts



4×4 parts

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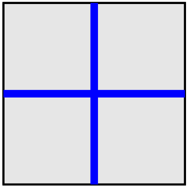
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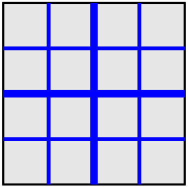
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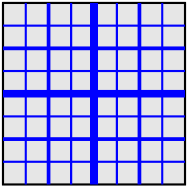
$4 \times 4 \times 4$ parts

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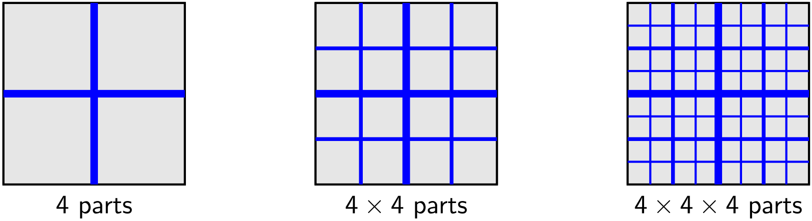

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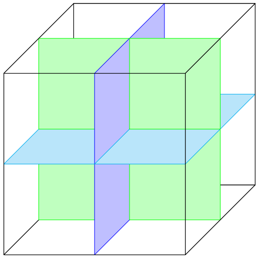
Analysis of Overlapping on a Regular Grid

- **On a $n \times n \times n$ 3D grid, for p parts, $V_{add} \approx 12(\sqrt[3]{p} - 1)n^2$**
  - **example : $p = 8$, $V_{add} \approx 12n^2$**
- **for each part, $v_{add} \approx \frac{6(\sqrt[3]{p}-1)p}{n^2}$, it's affordable !**



Plan

Block iLDU Preconditioner based on Nested Dissection

$$A = \begin{pmatrix} A_{11} & & A_{13} & & & & A_{17} \\ & A_{22} & A_{23} & & & & A_{27} \\ A_{31} & A_{32} & A_{33} & & & & A_{37} \\ & & & A_{44} & & A_{46} & A_{47} \\ & & & & A_{55} & A_{56} & A_{57} \\ & & & A_{64} & A_{65} & A_{66} & A_{67} \\ A_{71} & A_{72} & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} \end{pmatrix}$$

$$= (L + D)D^{-1}(D + U)$$

- Approximation on off-diagonal blocks.

Nested Exact Factorization [Grigori et al., 2010]

$$
\begin{aligned}
A &= \begin{pmatrix} T_1^1 & & U_1^1 \\ & T_1^2 & U_1^2 \\ L_1^1 & L_1^2 & S_1^1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & & \\ & 0 & \\ L_1^1 & L_1^2 & 0 \end{pmatrix}}_{L_1} + \underbrace{\begin{pmatrix} T_1^1 & & \\ & T_1^2 & \\ & & S_1^1 \end{pmatrix}}_{D} + \underbrace{\begin{pmatrix} 0 & & U_1^1 \\ & 0 & U_1^2 \\ & & 0 \end{pmatrix}}_{U_1} \\
&= (L_1 + D) D^{-1} (D + U_1) - L_1 D^{-1} U_1 \\
&= (L_1 + F_1) F_1^{-1} (F_1 + U_1)
\end{aligned}
$$

where

$$
\begin{aligned}
F_1 &= D - L_1 F_1^{-1} U_1 \\
&= \begin{pmatrix} T_1^1 & & \\ & T_1^2 & \\ & & S_1^1 - L_1^1 (T_1^1)^{-1} U_1^1 - L_1^2 (T_1^2)^{-1} U_1^2 \end{pmatrix}
\end{aligned}
$$

Parallism in each recursion level !

Nested Preconditioners [Grigori et al., 2010]

- Nested exact factorization

$$\begin{aligned} A &= F_0 \\ F_k &= (L_{k+1} + F_{k+1})F_{k+1}^{-1}(F_{k+1} + U_{k+1}),\ \textit{for}\ k = 0 \ldots K-1 \\ F_K &= D - \sum_{k=1}^{K} L_k F_k^{-1} U_k \end{aligned}$$

- NSSOR Preconditioner

$$F_K = D$$

- NMILUR Preconditioner

$$F_K = D - \sum_{k=1}^{K} rowsum(L_k F_k^{-1} U_k) = D - \sum_{k=1}^{K} Diag(L_k F_k^{-1} U_k \mathbf{1})$$

$A\mathbf{1} = P_{NMILUR}\mathbf{1}$ (filtering on vector $\mathbf{1}$.)

Nested Preconditioners [Grigori et al., 2010]

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\begin{aligned}
A &= F_0 \\
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Plan

Overlapping techniques

Parallel preconditioners

Numerical results

Conclusion

Test Matrices

Boundary value problem

(provided by Achdou, Nataf)

- Advection-diffusion
- Non-homogeneous
- Convective Skyscraper
- Anisotropic

$$\begin{aligned} div(a(x)u) - div(\kappa(x)\nabla u) = f & \quad \text{in } \Omega \\ u = 0 & \quad \text{on } \partial\Omega_D \\ \frac{\partial u}{\partial n} = 0 & \quad \text{on } \partial\Omega_N \end{aligned}$$

discretized on a cartesian grid

Simulation of Black Oil Model

(provided by R. Masson IFP)

- BO60x60x32
  - compositional triphase Darcy flow simulator (oil, water and gas).

Tim Davis Collection

- Dubcova2 (PDE solver)
- thermomech (FEM problem)

Impact of Overlap on matrix size and nnz

matrix	$n(A)$	$n(A_o)$	**increase**	$nnz(A)$	$nnz(A_o)$	**increase**
2dANI400 2dNH400 2dAD400	160000	169355	**5%**	798400	845131	**5%**
dubcova2	65025	74913	**15%**	1030225	1200940	**16%**
thermomech	102158	108984	**6%**	711558	759895	**6%**
3dCSKY40 3dANI40	64000	90643	**41%**	438400	622410	**41%**
BO60x60x32	115200	156109	**35%**	791482	1074100	**36%**

TABLE : analyse on a 4×4 partition

- On a 400×400 grid
  $V_{add} \approx 8n(\sqrt{p} - 1) = 8 \times 400(\sqrt{16} - 1) = 9600 \Longrightarrow 6\%$

Impact of Overlap on convergence rate for BILDU

GMRES (no restart, $maxiter = 200$, $tol \leq 10^{-8}$, $err = \frac{\|x - x_{exact}\|}{\|x\|}$)

	without overlap		with overlap		
matrix	iter	err	iter	err	**improvement**
2dANI400	200	e-4	200	e-4	
2dNHI400	114	e-7	82	e-7	**-28%**
2dAD400	114	e-7	82	e-7	**-28%**
dubcova2	39	e-7	35	e-8	**-10%**
thermo	5	e-10	4	e-11	**-20%**
3dCSKY40	145	e-6	93	e-7	**-36%**
3dANI40	200	e-8	46	e-8	**-75%**

TABLE : ilu(0)-like approximation on 4x4 partition

convergence rate for Nested Preconditioners

GMRES ($restart = 200, maxiter = 1000, tol \leq 10^{-8}, err = \frac{\|x - x_{exact}\|}{\|x\|}$)

	NSSOR			
	4×4		8×8	
matrix	iter	err	iter	err
2dANI400	fail		fail	
2dNH400	fail		fail	
2dAD400	fail		fail	
dubcova2	46	e-7	64	e-7
thermo	6	e-9	6	e-9
3dCSKY40	fail		fail	
3dANI40	fail		fail	
BO	fail		fail	

convergence rate for Nested Preconditioners

GMRES ($restart = 200$, $maxiter = 1000$, $tol \leq 10^{-8}$, $err = \frac{\|x - x_{exact}\|}{\|x\|}$)

	NSSOR				NSSOR + overlap			
	4×4		8×8		4×4		8×8	
matrix	iter	err	iter	err	iter	err	iter	err
2dANI400	fail		fail		301	e-6	378	e-6
2dNH400	fail		fail		84	e-7	100	e-7
2dAD400	fail		fail		81	e-7	104	e-7
dubcova2	46	e-7	64	e-7	33	e-8	39	e-8
thermo	6	e-9	6	e-9	3	e-9	4	e-11
3dCSKY40	fail		fail		93	e-7	94	e-7
3dANI40	fail		fail		40	e-8	45	e-8
BO	fail		fail		97	e-7	146	e-7

convergence rate for Nested Preconditioners

GMRES $(restart = 200, maxiter = 1000, tol \leq 10^{-8}, err = \frac{\|x - x_{exact}\|}{\|x\|})$

	NSSOR				NSSOR + overlap				NMILUR + overlap			
	4 × 4		8 × 8		4 × 4		8 × 8		4 × 4		8 × 8	
matrix	iter	err	iter	err	iter	err	iter	err	iter	err	iter	err
2dANI400	fail		fail		301	e-6	378	e-6	86	e-5	100	e-3
2dNH400	fail		fail		84	e-7	100	e-7	54	e-8	73	e-7
2dAD400	fail		fail		81	e-7	104	e-7	57	e-8	73	e-7
dubcova2	46	e-7	64	e-7	33	e-8	39	e-8	28	e-8	33	e-8
thermo	6	e-9	6	e-9	3	e-9	4	e-11	3	e-9	4	e-11
3dCSKY40	fail		fail		93	e-7	94	e-7	96	e-6	132	e-6
3dANI40	fail		fail		40	e-8	45	e-8	102	e-7	195	e-7
BO	fail		fail		97	e-7	146	e-7	59	e-7	80	e-7

Plan

Conclusion and Future Work

Conclusion

- Overlap
  - nested dissection, parallel
  - reasonable storage cost
- Convergence rate
  - BILDU
  - NSSOR and NMILUR

Future work

- parallel implementation

References (1)

Grigori, L., Kumar, P., Nataf, F., and Wang, K. (2010).
A class of multilevel parallel preconditioning strategies.
Research Report RR-7410, INRIA.