Sparse inverse incidence matrices in the factorization of indefinite matrices

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Where innovation starts

Outline

- Motivation and goals
- Incidence matrix
- Sparse inverse of permuted incidence matrix
- Numerical examples
- Concluding remarks

Motivation and goals

Consider indefinite linear systems of the form

$$\underbrace{\begin{array}{c@{\;}c} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \end{array}}_{\mathcal{A}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{u} = \underbrace{\begin{bmatrix} f \\ g \end{bmatrix}}_{b} \tag{1}$$

where A is symmetric and positive (semi) definite, and B^T is of maximal column rank of m with $m \leq n$.

- For very large $\mathcal{A}$, iterative methods take too much time.
- Preconditioning techniques to substantially reduce the number of iterations have become very important.
- Schilders' factorization can be used as a basis for such preconditioners.

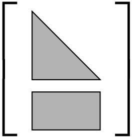
Motivation and goals (cont.)

In Schilders' factorization, a possibly permuted $\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{A} & \tilde{B}^T \\ \tilde{B} & 0 \end{bmatrix}$ is split into a block 3×3 structure and factorized as

$$\begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_1^T \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{B}_2^T \\ \tilde{B}_1 & \tilde{B}_2 & 0 \end{bmatrix} = \begin{bmatrix} \tilde{B}_1^T & 0 & L_1 \\ \tilde{B}_2^T & L_2 & M \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} D_1 & 0 & I \\ 0 & D_2 & 0 \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 & 0 \\ 0 & L_2^T & 0 \\ L_1^T & M^T & I \end{bmatrix} \tag{2}$$

The permutation should be such that

$$\tilde{B}^T = \begin{bmatrix} \tilde{B}_1^T \\ \tilde{B}_2^T \end{bmatrix} =$$



, a lower trapezoidal form. (3)

Motivation and goals (cont.)

$\mathcal{A}$ is sparse and so is $\tilde{\mathcal{A}}$.

But $$\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{B}_1^T & 0 & L_1 \\ \tilde{B}_2^T & L_2 & M \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} D_1 & 0 & I \\ 0 & D_2 & 0 \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 & 0 \\ 0 & L_2^T & 0 \\ L_1^T & M^T & I \end{bmatrix} \qquad (4)$$

can have blocks which are more dense.

Sparsity of these blocks depends on the sparsity of $\tilde{B}_1^{-T}$.

Motivation and goals (cont.)

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But $$\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{B}_1^T & 0 & L_1 \\ \tilde{B}_2^T & L_2 & M \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} D_1 & 0 & I \\ 0 & D_2 & 0 \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 & 0 \\ 0 & L_2^T & 0 \\ L_1^T & M^T & I \end{bmatrix} \tag{4}$$

can have blocks which are more dense.

Sparsity of these blocks depends on the sparsity of $\tilde{B}_1^{-T}$.

For example:

$$D_1 = \text{diag}\left(\tilde{B}_1^{-T} \tilde{A}_{11} \tilde{B}_1^{-1}\right) \tag{5}$$

$$L_1 = \tilde{B}_1^T \text{ lower}\left(\tilde{B}_1^{-T} \tilde{A}_{11} \tilde{B}_1^{-1}\right) \tag{6}$$

$$M = \left(\tilde{A}_{21} - \tilde{B}_2^T L_1^T\right) \tilde{B}_1^{-1} - \tilde{B}_2^T D_1 \tag{7}$$

Motivation and goals (cont.)

$\mathcal{A}$ is sparse and so is $\tilde{\mathcal{A}}$.

$$\text{But} \quad \tilde{\mathcal{A}} = \begin{bmatrix} \tilde{B}_1^T & 0 & L_1 \\ \tilde{B}_2^T & L_2 & M \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} D_1 & 0 & I \\ 0 & D_2 & 0 \\ I & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 & 0 \\ 0 & L_2^T & 0 \\ L_1^T & M^T & I \end{bmatrix} \tag{4}$$

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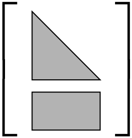
$$M = \left(\tilde{A}_{21} - \tilde{B}_2^T L_1^T\right) \tilde{B}_1^{-1} - \tilde{B}_2^T D_1 \tag{7}$$

Motivation and goals (cont.)

Different applications lead to different types of B^T.

Requirement is the lower trapezoidal form

$$\tilde{B}^T = \begin{bmatrix} \tilde{B}_1^T \\ \tilde{B}_2^T \end{bmatrix} =$$



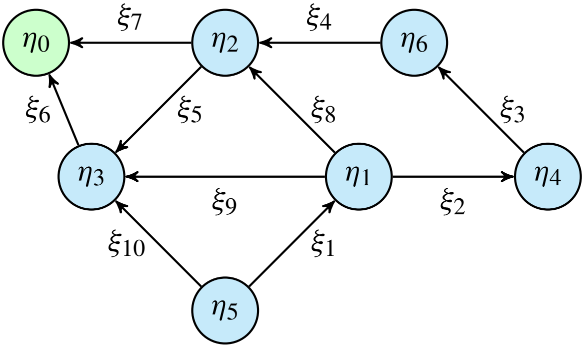
which can be achieved by:

1. permuting rows and columns of B^T if it is an <u>incidence matrix</u>.
2. performing LQ decomposition on B^T if it is of more general form.

Incidence matrix: definition by example

Consider a loopless digraph $G(V, E)$ with the

- **node set** $V = \{\eta_0, \eta_1, \ldots, \eta_6\}$
- **arc set** $E = \{\xi_1, \xi_2, \ldots, \xi_{10}\}$



- η_0 is called as **ground** or **reference** node.

Incidence matrix: definition by example

The node-arc incidence matrix of G is defined as

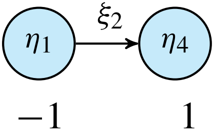
$$
\begin{array}{c}
\\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10}
\end{array}
\begin{array}{c}
\begin{array}{rrrrrrr} \eta_0 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0
\end{bmatrix}
\end{array}
$$

Incidence matrix: definition by example

The **node-arc incidence matrix** of G is defined as

$$
\begin{array}{c|ccccccc|}
 & \eta_0 & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \\
\hline
\xi_1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
\xi_2 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
\xi_3 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
\xi_4 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
\xi_5 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
\xi_6 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
\xi_7 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
\xi_8 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
\xi_9 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
\xi_{10} & 0 & 0 & 0 & 1 & 0 & -1 & 0
\end{array}
$$

for instance



Incidence matrix: definition by example

The **node-arc incidence matrix** of G is defined as

$$
\begin{array}{c}
\\
\xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \boxed{\xi_6} \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10}
\end{array}
\begin{array}{c}
\begin{array}{ccccccc}
\boxed{\eta_0} & \eta_1 & \eta_2 & \boxed{\eta_3} & \eta_4 & \eta_5 & \eta_6
\end{array} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
\boxed{1} & 0 & 0 & \boxed{-1} & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0
\end{bmatrix}
\end{array}
$$

for instance

$$
\begin{array}{ccc}
\eta_0 & \xleftarrow{\ \xi_6\ } & \eta_3 \\
1 & & -1
\end{array}
$$

Incidence matrix: definition by example

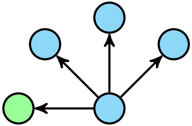
Deleting the column η_0 gives the incidence matrix

$$B^T = \begin{array}{c} \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

which can be permuted to a lower trapezoidal form.

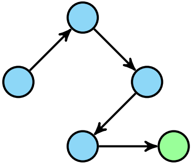
Inverse of permuted B_1^T

Different graphs lead to different $\tilde{B}_1^T$ and $\tilde{B}_1^{-T}$.



star digraph

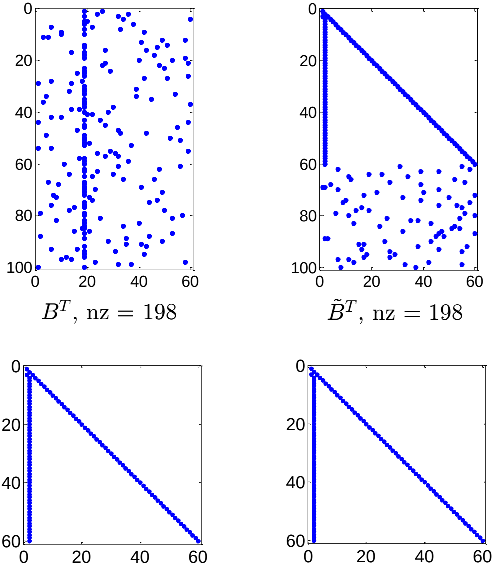
$$\tilde{B}_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \quad \tilde{B}_1^{-T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$



Hamiltonian path

$$\tilde{B}_1^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \quad \tilde{B}_1^{-T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Case I: B^T of size 100×60



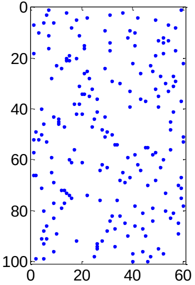
B^T, $\mathrm{nz} = 198$

$\tilde{B}^T$, $\mathrm{nz} = 198$

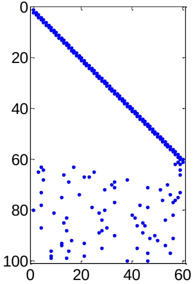
$\tilde{B1}^T$, $\mathrm{nz} = 118$

$\tilde{B1}^{-T}$, $\mathrm{nz} = 118$

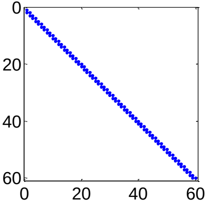
Case II (a): B^T of size 100×60



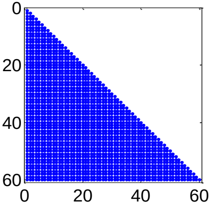
B^T, nz = 199



$\tilde{B}^T$, nz = 199

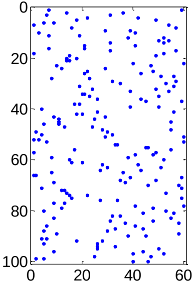


$\tilde{B}_1^T$, nz = 119

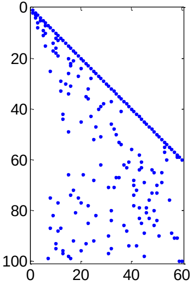


$\tilde{B}_1^{-T}$, nz = 1830

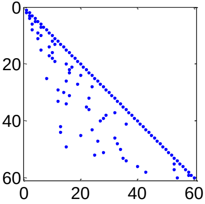
Case II (b): B^T of size 100×60



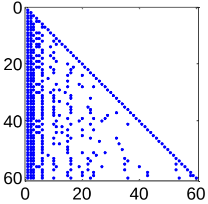
B^T, nz $= 199$



$\tilde{B}^T$, nz $= 199$



$\tilde{B}_1^T$, nz $= 119$



$\tilde{B}_1^{-T}$, nz $= 406$

The nilpotent part N of $\tilde{B}_1^T$

$N = \text{lower}\left(\tilde{B}_1^T\right)$ is a nilpotent part of $\tilde{B}_1^T$.

If k is the degree of nilpotency, then

$$\tilde{B}_1^{-T} = \left(I - N\right)^{-1} = I + N + N^2 + N^3 + \cdots + N^{k-1} \tag{4}$$

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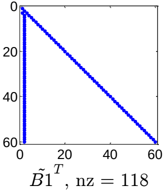
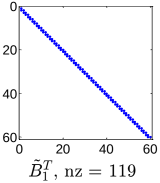


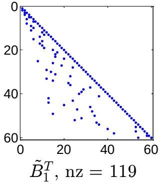
I

$N^2 = 0$



II (a)

$N^{60} = 0$



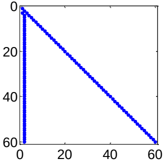
II (b)

$N^9 = 0$

$N = \text{lower}\left(\tilde{B}_1^T\right)$ is a nilpotent part of $\tilde{B}_1^T$.

If k is the degree of nilpotency, then

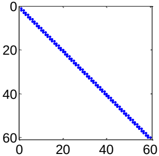
$$\tilde{B}_1^{-T} = (I - N)^{-1} = I + N + N^2 + N^3 + \cdots + N^{k-1} \qquad (4)$$



$\tilde{B}1^T$, nz = 118

I

$N^2 = 0$



$\tilde{B}_1^T$, nz = 119

II (a)

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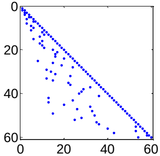


$\tilde{B}_1^T$, nz = 119

II (b)

$N^9 = 0$

We aim for a permutation which leads to a small k.

Consider

$$B^T = \begin{array}{c} \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Permutation algorithm

Search for a node connected to η_0

$$B^T = \begin{array}{c c} & \begin{array}{cccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \end{array} \\ \begin{array}{c} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Permutation algorithm

Search for a node connected to η_0

$$B^T = \begin{array}{c} \\ \boxed{\xi_1} \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & \boxed{-1} & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Search for a node connected to η_0

$$B^T = \begin{array}{c} \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ \boxed{-1} & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Permutation algorithm

Search for a node connected to η_0

$$
B^T = \begin{array}{c} \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \boxed{\xi_6} \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{array}
\begin{array}{c}
\begin{array}{cccccc} \eta_1 & \eta_2 & \boxed{\eta_3} & \eta_4 & \eta_5 & \eta_6 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & \boxed{-1} & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0
\end{bmatrix}
\end{array}
$$

Permutation algorithm

Search for a node connected to η_0

$$
B^T = \begin{array}{c} \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \boxed{\xi_6} \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{array}
\begin{array}{c}
\begin{array}{cccccc} \eta_1 & \eta_2 & \boxed{\eta_3} & \eta_4 & \eta_5 & \eta_6 \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & \boxed{-1} & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0
\end{bmatrix}
\end{array}
$$

row

$r_1 \leftrightarrow r_6$

column

$c_1 \leftrightarrow c_3$

Permutation algorithm

$$B^T \sim \begin{array}{c} \\ \boldsymbol{\xi_6} \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \boldsymbol{\eta_3} & \eta_2 & \eta_1 & \eta_4 & \eta_5 & \eta_6 \end{array} \\ \begin{bmatrix} \mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Permutation algorithm

Search for all the nodes connected to η_3 or η_0.

$$B^T \sim \begin{array}{c} \\ \color{blue}{\boldsymbol{\xi_6}} \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \color{blue}{\boldsymbol{\eta_3}} & \eta_2 & \eta_1 & \eta_4 & \eta_5 & \eta_6 \end{array} \\ \begin{bmatrix} \color{blue}{\mathbf{-1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Permutation algorithm

Search for all the nodes connected to η_3 or η_0.

$$B^T \sim \begin{array}{c} \\ \boldsymbol{\xi_6} \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \boxed{\xi_5} \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \boldsymbol{\eta_3} & \boxed{\eta_2} & \eta_1 & \eta_4 & \eta_5 & \eta_6 \end{array} \\ \begin{bmatrix} \mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ \boxed{1} & \boxed{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

Permutation algorithm

Search for all the nodes connected to η_3 or η_0.

$$
B^T \sim
\begin{array}{c}
\\ \xi_6 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \boxed{\xi_5} \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \xi_{10}
\end{array}
\begin{array}{c}
\begin{array}{cccccc} \boldsymbol{\eta_3} & \boxed{\eta_2} & \eta_1 & \eta_4 & \eta_5 & \eta_6 \end{array} \\
\begin{bmatrix}
\mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & -1 \\
\boxed{1} & \boxed{-1} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0
\end{bmatrix}
\end{array}
\quad
\begin{array}{l}
r_2 \leftrightarrow r_5 \\
c_2 \leftrightarrow c_2
\end{array}
$$

Permutation algorithm

Search for all the nodes connected to η_3 or η_0.

$$B^T \sim \begin{array}{c} \\ \boldsymbol{\xi}_6 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_1 \\ \boxed{\xi_7} \\ \xi_8 \\ \xi_9 \\ \xi_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \boldsymbol{\eta}_3 & \eta_2 & \eta_1 & \eta_4 & \eta_5 & \eta_6 \end{array} \\ \begin{bmatrix} \mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & \boxed{-1} & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

$r_2 \leftrightarrow r_5$

$c_2 \leftrightarrow c_2$

Permutation algorithm

Search for all the nodes connected to η_3 or η_0.

$$B^T \sim \begin{array}{c} \\ \boldsymbol{\xi_6} \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \boxed{\xi_9} \\ \xi_{10} \end{array} \begin{array}{c} \begin{array}{cccccc} \boldsymbol{\eta_3} & \eta_2 & \boxed{\eta_1} & \eta_4 & \eta_5 & \eta_6 \end{array} \\ \begin{bmatrix} \boldsymbol{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ \boxed{1} & 0 & \boxed{-1} & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

$r_2 \leftrightarrow r_5$

$c_2 \leftrightarrow c_2$

Permutation algorithm

Search for all the nodes connected to η_3 or η_0.

$$
B^T \sim \begin{array}{c} \\ \xi_6 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \boxed{\xi_9} \\ \xi_{10} \end{array}
\begin{array}{c} \begin{array}{cccccc} \boldsymbol{\eta_3} & \eta_2 & \boxed{\eta_1} & \eta_4 & \eta_5 & \eta_6 \end{array} \\
\begin{bmatrix}
\mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
\boxed{1} & 0 & \boxed{-1} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & -1 & 0
\end{bmatrix} \end{array}
\begin{array}{l} r_2 \leftrightarrow r_5 \\ c_2 \leftrightarrow c_2 \\ \\ r_3 \leftrightarrow r_9 \\ c_3 \leftrightarrow c_3 \end{array}
$$

Permutation algorithm

Search for all the nodes connected to η_3 or η_0.

$$B^T \sim \begin{array}{c} \\ \boldsymbol{\xi_6} \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \boxed{\xi_{10}} \end{array} \begin{array}{c} \begin{array}{cccccc} \boldsymbol{\eta_3} & \eta_2 & \eta_1 & \eta_4 & \boxed{\eta_5} & \eta_6 \end{array} \\ \begin{bmatrix} \mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ \boxed{1} & 0 & 0 & 0 & \boxed{-1} & 0 \end{bmatrix} \end{array}$$

$r_2 \leftrightarrow r_5$
$c_2 \leftrightarrow c_2$

$r_3 \leftrightarrow r_9$
$c_3 \leftrightarrow c_3$

Permutation algorithm

Search for all the nodes connected to η_3 or η_0.

$$
B^T \sim \begin{array}{c} \\ \xi_6 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_9 \\ \boxed{\xi_{10}} \end{array}
\begin{array}{c} \begin{array}{cccccc} \boldsymbol{\eta_3} & \eta_2 & \eta_1 & \eta_4 & \boxed{\eta_5} & \eta_6 \end{array} \\
\begin{bmatrix}
\mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
\boxed{1} & 0 & 0 & 0 & \boxed{-1} & 0
\end{bmatrix} \end{array}
\begin{array}{c}
r_2 \leftrightarrow r_5 \\
c_2 \leftrightarrow c_2 \\
\\
r_3 \leftrightarrow r_9 \\
c_3 \leftrightarrow c_3 \\
\\
r_4 \leftrightarrow r_{10} \\
c_4 \leftrightarrow c_5
\end{array}
$$

Permutation algorithm

$$B^T \sim \begin{array}{c} \\ \xi_6 \\ \xi_5 \\ \xi_9 \\ \xi_{10} \\ \xi_2 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_3 \\ \xi_4 \end{array} \begin{array}{c} \begin{array}{cccccc} \eta_3 & \eta_2 & \eta_1 & \eta_5 & \eta_4 & \eta_6 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

Permutation algorithm

Search from $\{\eta_4, \eta_6\}$ the one connected to η_2 else η_1 else η_5.

$$
B^T \sim \begin{array}{c} \\ \boldsymbol{\xi}_6 \\ \boldsymbol{\xi}_5 \\ \boldsymbol{\xi}_9 \\ \boldsymbol{\xi}_{10} \\ \xi_2 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_3 \\ \xi_4 \end{array}
\begin{array}{c}
\begin{array}{cccccc} \boldsymbol{\eta}_3 & \boldsymbol{\eta}_2 & \boldsymbol{\eta}_1 & \boldsymbol{\eta}_5 & \eta_4 & \eta_6 \end{array} \\
\begin{bmatrix}
\mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\
\mathbf{1} & \mathbf{-1} & 0 & 0 & 0 & 0 \\
\mathbf{1} & 0 & \mathbf{-1} & 0 & 0 & 0 \\
\mathbf{1} & 0 & 0 & \mathbf{-1} & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
0 & 1 & 0 & 0 & 0 & -1
\end{bmatrix}
\end{array}
$$

Permutation algorithm

Search from $\{\eta_4, \eta_6\}$ the one connected to η_2 else η_1 else η_5.

$$
B^T \sim \begin{array}{c} \\ \xi_6 \\ \xi_5 \\ \xi_9 \\ \xi_{10} \\ \xi_2 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_3 \\ \boxed{\xi_4} \end{array}
\begin{array}{c}
\begin{array}{cccccc} \eta_3 & \eta_2 & \eta_1 & \eta_5 & \eta_4 & \boxed{\eta_6} \end{array} \\
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
0 & \boxed{1} & 0 & 0 & 0 & \boxed{-1}
\end{bmatrix}
\end{array}
$$

Permutation algorithm

Search from $\{\eta_4, \eta_6\}$ the one connected to η_2 else η_1 else η_5.

$$B^T \sim \begin{array}{c} \\ \boldsymbol{\xi}_6 \\ \boldsymbol{\xi}_5 \\ \boldsymbol{\xi}_9 \\ \boldsymbol{\xi}_{10} \\ \xi_2 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_3 \\ \boxed{\xi_4} \end{array} \begin{array}{c} \begin{array}{cccccc} \boldsymbol{\eta}_3 & \boldsymbol{\eta}_2 & \boldsymbol{\eta}_1 & \boldsymbol{\eta}_5 & \eta_4 & \boxed{\eta_6} \end{array} \\ \begin{bmatrix} \mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{-1} & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{-1} & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & \mathbf{-1} & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & \boxed{1} & 0 & 0 & 0 & \boxed{-1} \end{bmatrix} \end{array} \begin{array}{c} r_5 \leftrightarrow r_{10} \\ c_5 \leftrightarrow c_6 \end{array}$$

Permutation algorithm

$$B^T \sim \begin{array}{c} \\ \xi_6 \\ \xi_5 \\ \xi_9 \\ \xi_{10} \\ \xi_4 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_3 \\ \xi_2 \end{array} \begin{array}{c} \begin{array}{cccccc} \eta_3 & \eta_2 & \eta_1 & \eta_5 & \eta_6 & \eta_4 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Permutation algorithm

Search for a connection from η_4 to η_1 else η_5 else η_6.

$$
B^T \sim \begin{array}{c} \\ \boldsymbol{\xi}_6 \\ \boldsymbol{\xi}_5 \\ \boldsymbol{\xi}_9 \\ \boldsymbol{\xi}_{10} \\ \boldsymbol{\xi}_4 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_3 \\ \xi_2 \end{array}
\begin{array}{c}
\begin{array}{cccccc} \boldsymbol{\eta}_3 & \boldsymbol{\eta}_2 & \boldsymbol{\eta}_1 & \boldsymbol{\eta}_5 & \boldsymbol{\eta}_6 & \eta_4 \end{array} \\
\begin{bmatrix}
\mathbf{-1} & 0 & 0 & 0 & 0 & 0 \\
\mathbf{1} & \mathbf{-1} & 0 & 0 & 0 & 0 \\
\mathbf{1} & 0 & \mathbf{-1} & 0 & 0 & 0 \\
\mathbf{1} & 0 & 0 & \mathbf{-1} & 0 & 0 \\
0 & \mathbf{1} & 0 & 0 & \mathbf{-1} & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & -1 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

Permutation algorithm

Search for a connection from η_4 to η_1 else η_5 else η_6.

$$B^T \sim \begin{array}{c} \\ \xi_6 \\ \xi_5 \\ \xi_9 \\ \xi_{10} \\ \xi_4 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_3 \\ \boxed{\xi_2} \end{array} \begin{array}{c} \begin{array}{cccccc} \eta_3 & \eta_2 & \eta_1 & \eta_5 & \eta_6 & \boxed{\eta_4} \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & \boxed{-1} & 0 & 0 & \boxed{1} \end{bmatrix} \end{array}$$

Permutation algorithm

Search for a connection from η_4 to η_1 else η_5 else η_6.

$$
B^T \sim \begin{array}{c} \\ \xi_6 \\ \xi_5 \\ \xi_9 \\ \xi_{10} \\ \xi_4 \\ \xi_1 \\ \xi_7 \\ \xi_8 \\ \xi_3 \\ \boxed{\xi_2} \end{array}
\begin{array}{c} \begin{array}{cccccc} \eta_3 & \eta_2 & \eta_1 & \eta_5 & \eta_6 & \boxed{\eta_4} \end{array} \\
\begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & \boxed{-1} & 0 & 0 & \boxed{1}
\end{bmatrix} \end{array}
\quad \begin{array}{l} r_6 \leftrightarrow r_{10} \\ c_6 \leftrightarrow c_6 \end{array}
$$

Permutation algorithm

$$\tilde{B}^T = \begin{array}{c} \\ \xi_6 \\ \xi_5 \\ \xi_9 \\ \xi_{10} \\ \xi_4 \\ \xi_2 \\ \xi_7 \\ \xi_8 \\ \xi_3 \\ \xi_1 \end{array} \begin{array}{c} \begin{array}{cccccc} \eta_3 & \eta_2 & \eta_1 & \eta_5 & \eta_6 & \eta_4 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

Permutation algorithm

There is a row permutation matrix

$$P_r = [e_6,\ e_5,\ e_9,\ e_{10},\ e_4,\ e_2,\ e_7,\ e_8,\ e_3,\ e_1]^T$$

and a column permutation matrix

$$P_c = [e_3,\ e_2,\ e_1,\ e_5,\ e_6,\ e_4]$$

such that

$$P_r B^T P_c = \tilde{B}^T$$

Permutation algorithm

There is a row permutation matrix

$$P_r = [e_6,\ e_5,\ e_9,\ e_{10},\ e_4,\ e_2,\ e_7,\ e_8,\ e_3,\ e_1]^T$$

and a column permutation matrix

$$P_c = [e_3,\ e_2,\ e_1,\ e_5,\ e_6,\ e_4]$$

such that

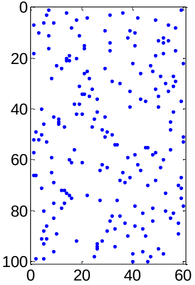
$$P_r B^T P_c = \tilde{B}^T$$

Note

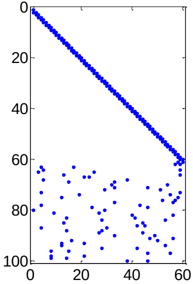
Permutation is not unique.

Has the permutation improved the sparsity of $\tilde{B}_1^{-T}$?

Case II (a): B^T of size 100×60

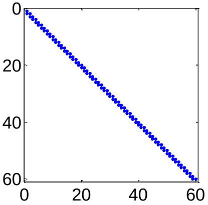


B^T, nz $= 199$

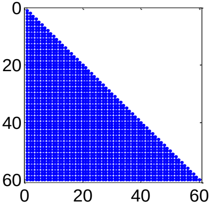


$\tilde{B}^T$, nz $= 199$

$k = 60$

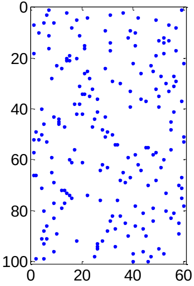


$\tilde{B}_1^T$, nz $= 119$

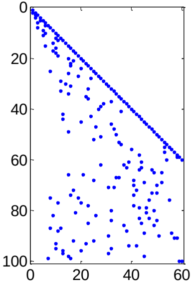


$\tilde{B}_1^{-T}$, nz $= 1830$

Case II (b): B^T of size 100×60

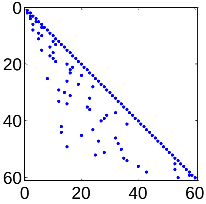


B^T, nz $= 199$

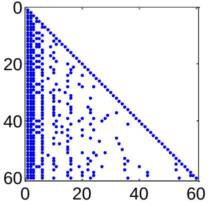


$\tilde{B}^T$, nz $= 199$

$k = 9$

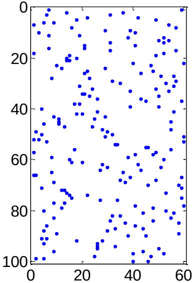


$\tilde{B}_1^T$, nz $= 119$

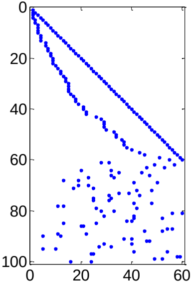


$\tilde{B}_1^{-T}$, nz $= 406$

Case II (c): B^T of size 100×60 (new permutation)

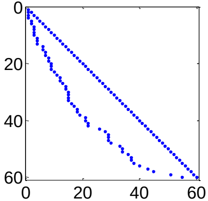


B^T, nz $= 199$

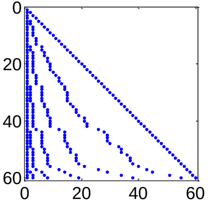


$\tilde{B}^T$, nz $= 199$

$k = 5$

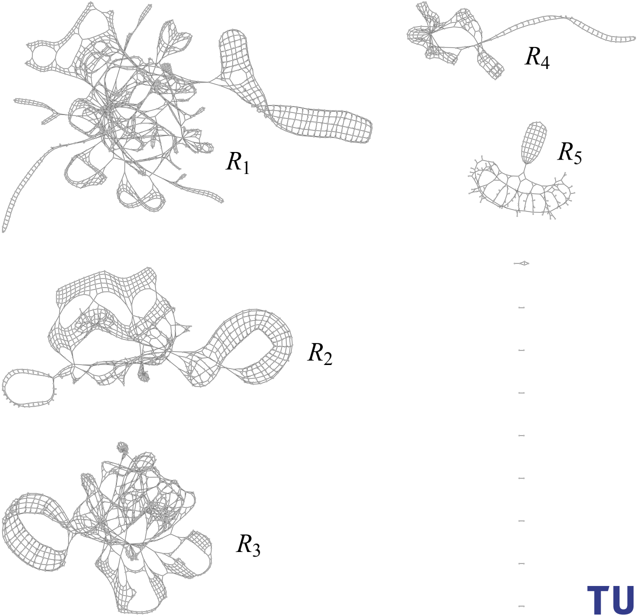


$\tilde{B}_1^T$, nz $= 119$



$\tilde{B}_1^{-T}$, nz $= 275$

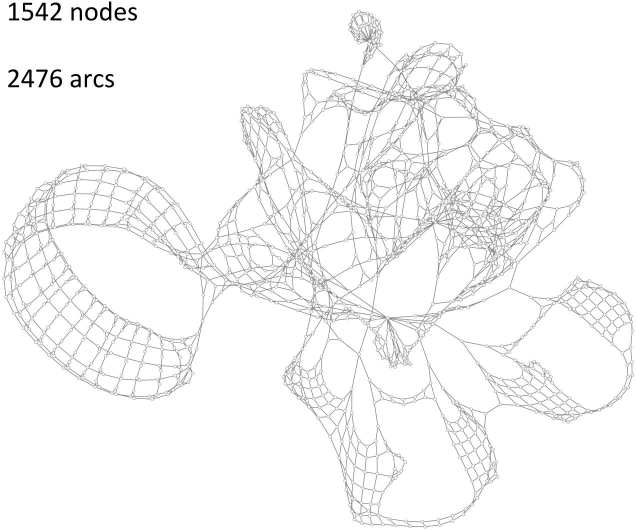
Numerical Example: A Resistor network



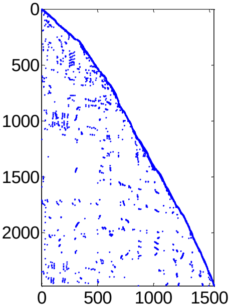
Numerical Example: Component R_3

1542 nodes

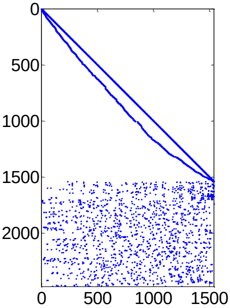
2476 arcs



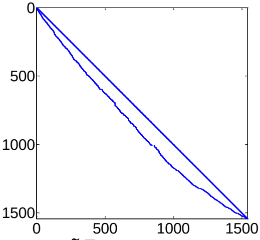
B^T of size 2476×1541 from R_3



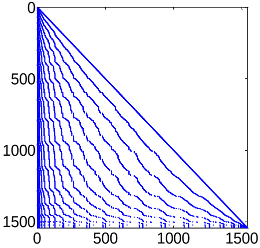
B^T, nz $= 4949$



$\tilde{B}^T$, nz $= 4949$



$\tilde{B}_1^T$, nz $= 3079$



$\tilde{B}_1^{-T}$, nz $= 19929$

Nilpotency degree $k = 23$

Blocks in Schilders' factorization

$$D_1 = \mathrm{diag}\left(\tilde{B}_1^{-T}\tilde{A}_{11}\tilde{B}_1^{-1}\right) \tag{5}$$

$$L_1 = \tilde{B}_1^T\,\mathrm{lower}\left(\tilde{B}_1^{-T}\tilde{A}_{11}\tilde{B}_1^{-1}\right) \tag{6}$$

$$M = \left(\tilde{A}_{21} - \tilde{B}_2^T L_1^T\right)\tilde{B}_1^{-1} - \tilde{B}_2^T D_1 \tag{7}$$

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Sparsity of blocks from component R_3

$$D_1 = \mathrm{diag}\left(\tilde{B}_1^{-T}\tilde{A}_{11}\tilde{B}_1^{-1}\right) \tag{5}$$

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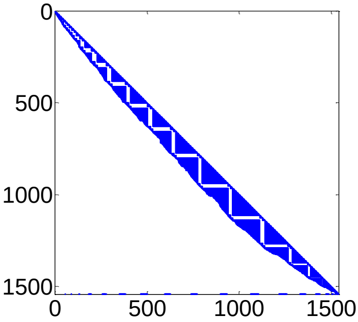
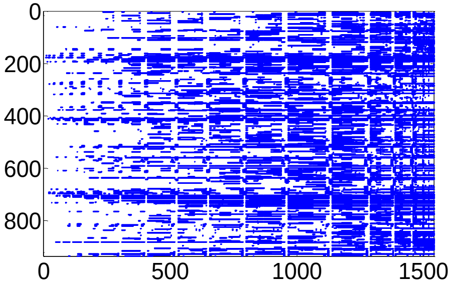
Note

For a resistor network, A is a diagonal matrix with entries of resistance values.

Sparsity of blocks from component R_3



L_1, nz = 112536



M, nz = 151673

Sparsity of blocks from component R_3

Block	Size	Nonzeros	Nonzeros (%)	Sparsity (%)
$\tilde{B}_1^T$	1541×1541	3079	0.26	99.74
$\tilde{B}_1^{-T}$	1541×1541	19929	1.68	98.32
L_1	1541×1541	112536	9.48	90.52
M	935×1541	151673	10.53	89.47

Sparsity of blocks from other components

	component	R_1	R_2	R_4	R_5
	nodes	2300	1210	341	243
	arcs	3682	1936	555	333
	nilpotency (k)	50	33	42	20
Block sparsity (%)	$\tilde{B}_1^T$	99.83	99.67	98.83	98.37
	$\tilde{B}_1^{-T}$	98.24	97.33	91.50	92.45
	L_1	92.11	96.42	90.69	88.86
	M	90.96	91.20	77.56	84.72

Concluding remarks

- For general incidence matrices, we can determine a permutation which leads to a sparse inverse.

- The determination of the most sparse inverse still remains to be a significant challenge.

Micro- and macro-block factoirzations for regularized saddle point systems

Joseph M.L. Maubach and Wil H.A. Schilders

Regularized saddle-point problems

Structure of indefinite X from regularized saddle point problems:

$$X = \begin{array}{c c} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \end{array}, \quad m \le n$$

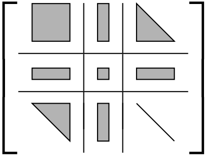
- A is $n \times n$ symmetric positive definite;
- B^T is $n \times m$ of full column rank;
- $C \neq 0$ (regularized saddle point problem) is $m \times m$ symmetric positive semi-definite;

Goal: Construct a preconditioner or implicit/explicit factorization.

Structure of X for factorization

We exploit the structure:

$$X = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} =$$



- A is $n \times n$ symmetric positive definite;
- B^T is $n \times m$ lower trapezoidal of full column rank;
- C is $m \times m$ positive diagonal matrix.

Results do extend to symmetric positive semi-definite C.

I Micro-block factorization

Define a permutation matrix Q and obtain

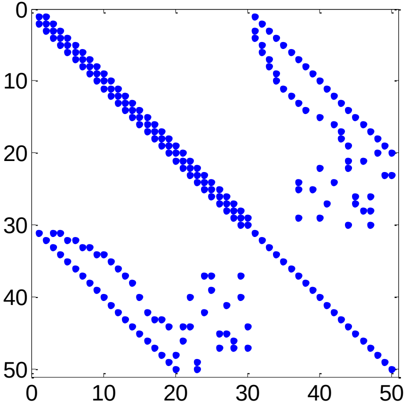
$$Y = Q^T X Q$$

Micro-block factorization:

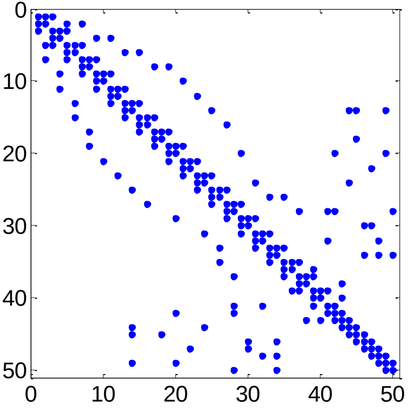
$$Y = L \,\mathrm{diag}\left(L^{-T}\right)\, L^T$$

- L is a micro-block lower triangular with 2×2 , 1×2 and 1×1 micro-blocks
- $\mathrm{diag}\big(L^{-T}\big)$ is a micro-block diagonal with 2×2 and 1×1 micro-blocks

I Micro-block factorization: Example

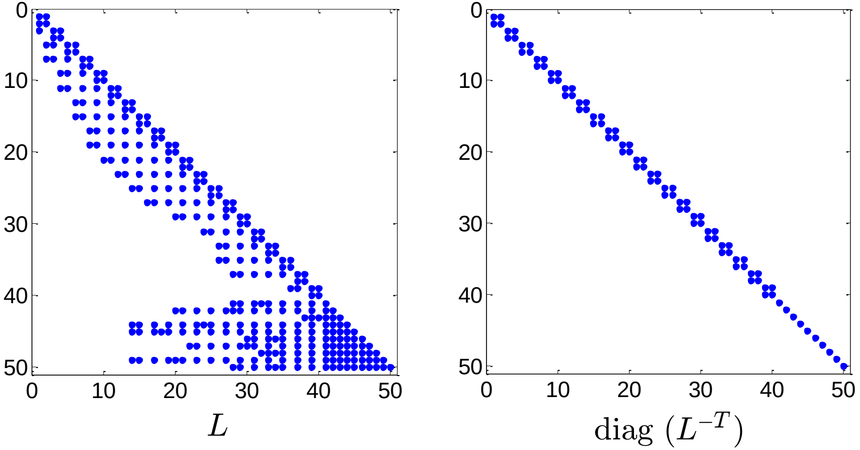


X



Y

I Micro-block factorization: Example



II Macro-block factorization

Back permutation of Y induces

$$
\begin{aligned}
X &= QYQ^T \\
&= Q\left(L\,\mathrm{diag}\left(L^{-T}\right)\,L^T\right)Q^T \\
&= \underbrace{\left(QLQ^T\right)}_{L_x}\underbrace{\left(Q\,\mathrm{diag}\left(L^{-T}\right)Q^T\right)}_{D_x}\underbrace{\left(QL^TQ^T\right)}_{L_x^T},
\end{aligned}
$$

$$
L_x = \begin{bmatrix} L_1 & 0 & B_1^T \\ M & L_2 & B_2^T \\ dB_1 & 0 & -C \end{bmatrix}, \quad D_x = \begin{bmatrix} FC & 0 & F\,dB_1 \\ 0 & D_2 & 0 \\ F\,dB_1 & 0 & -FD_1 \end{bmatrix},
$$

where

- L_1, L_2 are lower triangulars; D_1, D_2 are diagonals; and M a rectangular
- $dB_1 = \mathrm{diag}\left(B_1\right)$ and $F = \left(dB_1^2 + CD_1\right)^{-1}$

II Macro-block factorization(cont.)

Let $\mathcal{L}_x = L_x\, D_x$ and $\mathcal{D}_x = D_x$. Then there exists the macro-block factorization

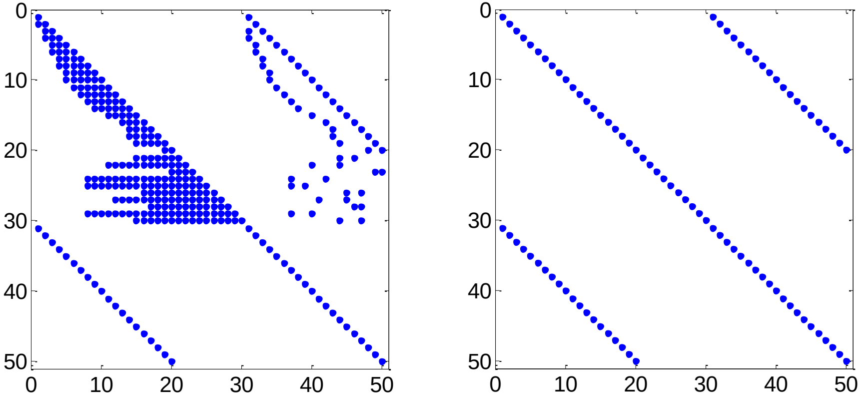
$$X = \mathcal{L}_x \mathcal{D}_x \mathcal{L}_x^T$$

where

$$\mathcal{L}_x = \begin{bmatrix} B_1^T F\, d B_1 + L_1 F C & 0 & -B_1^T F D_1 + L_1 F\, d B_1 \\ B_2^T F\, d B_1 + M F C & L_2 D_2 & -B_2^T F D_1 + M F\, d B_1 \\ 0 & 0 & I \end{bmatrix}.$$

This is an extension of Schilders' factorization to the case $C \neq 0$.

II Macro-block factorization: Example



L_x D_x

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Thank You