Gene meets Gronwall

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When Gene met Gronwall

- Oxford visit 1983
- Student symposium (Reading?)

Gronwall's Lemma

Simplest form of **Gronwall's lemma (1919)**:
Let $x \in C[0,T]$ and non-negative and let $a, b > 0$.
If

$$x(t) \leq a + b \int_0^t x(s)ds, \quad 0 \leq t \leq T,$$

then

$$x(t) \leq a exp(bt), \quad 0 \leq t \leq T.$$

In other words, x is bounded by y where y is solution of the integral equation

$$y(t) = a + b \int_0^t y(s)ds, \quad 0 \leq t \leq T.$$

- Used with Picard-Cauchy iterations to prove existence and uniqueness of solutions of initial value problems.
- Generalised and extended (sometimes called Gronwall-Bellman inequalities)
- Many applications (try googling Gronwall)

Discrete analogue

Lemma: If $x_i > 0$, $i = 0, 1, ..., N$ satisfies

$$x_0 \le \delta, \quad x_i \le \delta + Mh \sum_{j=0}^{i-1} x_j, \quad 1 \le i \le N,$$

where $\delta > 0$ and $M > 0$ is bounded independently of h ($Nh = T$), then

$$x_i \le \delta exp(Mih), \quad 1 \le i \le N.$$

Used to prove convergence of discretisation schemes
($N \to \infty$, $h \to 0$ with $Nh = T$ fixed).

Note:

$$x_i \le y(t_i)$$

where $t_i = ih$ and y satisfies

$$y(t) = \delta + M \int_0^t y(s) ds, \quad 0 \le t \le T,$$

and so discrete points are bounded by solution of the continuous problem.

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Simple example

$$x(t) = a + b\int_0^t x(s)ds, \quad 0 \le t \le T.$$

Simple discretisation: x_i approximates $x(t_i)$ $(t_i = ih)$

$$x_0 = a, \quad x_i = a + bh\sum_{j=0}^{i-1} x_j, 1 \le i \le N.$$

Clearly

$$\begin{aligned} x(t_i) &= a + b\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}\{x(t_j) + (x(s) - x(t_j))\}ds \\ &= a + bh\sum_{j=0}^{i-1} x(t_j) + T_i. \end{aligned}$$

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Example continued

Subtracting, follows that $e_i = |x(t_i) - x_i|$ satisfies

$$e_i \leq |T_i| + bh \sum_{j=0}^{i-1} e_j, \quad 1 \leq i \leq N,$$

Discrete Gronwall inequality yields

$$e_i \leq |T_i| exp(bih), \quad 1 \leq i \leq N,$$

and since $|T_i| \leq Ch$, **first order convergence** follows.

Volterra equations

Interested in equations of the form

$$y(t) = \phi(t) + Ky(t), \qquad (Ky)(t) = \int_0^t k(t,s)y(s)ds.$$

Lemma: For $0 \le t \le T$, let

$$x(t) \le \phi(t) + (Kx)(t)$$

If

- $k(t,s)$ is non-negative on $0 \le s \le t \le T$
- $k(t,s)$ is integrable as a function of s
- there exists ν such that $k^{(\nu)}(t,s)$ is continuous in t and s, where

$$k^{(\nu)}(t,s) = \int_s^t k(t,r)k^{(\nu-1)}(r,s)dr,$$

then

$$x(t) \le y(t), \quad 0 \le t \le T$$

Special case

$$k(t,s) = \frac{1}{(t-s)^{\alpha}}, \quad 0 \le \alpha < 1$$

Can show

$$y(t) = \phi(t) + Ky(t), \qquad (Ky)(t) = \int_0^t k(t,s)y(s)ds$$

has unique solution

$$y(t) = \phi(t)E_{1-\alpha}(\Gamma(1-\alpha)t^{1-\alpha})$$

where $E_{1-\alpha}$ is the Mittag-Leffler function (reduces to exponential if $\alpha = 0$).

In the above lemma, choose $\nu = \rho + 1$ where

$$\frac{\rho - 1}{\rho} < \alpha \le \frac{\rho}{\rho + 1}.$$

Perturbation analysis

$$z'(t) = f(t, z(t)) + c\int_0^t \frac{z'(s)}{(t-s)^{1-\alpha}} ds + q(t)$$

with $z(0) = z_0$.

$\alpha = 1/2$ models motion of particle in turbulent fluid.

Suppose $z(0)$ perturbed to $z_0 + \delta z_0$, then (assuming some continuity conditions) $x(t) = |\delta z(t)|$ satisfies

$$x(t) \leq \phi(t) + (Kx)(t), \qquad (Kx)(t) = \int_0^t \frac{1}{(t-s)^{\alpha}} x(s) ds$$

Hence, from lemma,

$$x(t) \leq \phi(t) E_{1-\alpha}(\Gamma(1-\alpha)t^{1-\alpha}).$$

Discrete analogue

Let $y^h = (y_0, y_1, ..., y_n)^T$ and suppose

$$y^h = \phi^h + K^h y^h \qquad (K^h y^h)_i = h \sum_{j=0}^{i-1} k_{ij} y_j. \qquad (1)$$

Lemma: If

- k_{ij} is non-negative for $0 \leq j < i \leq N$
- for each i, $h \sum_{j=0}^{i-1} k_{ij}$ is bounded independently of h
- there exists ν such that $k_{ij}^{(\nu)}$ is bounded independently of h, where

$$k_{ij}^{(\nu)} = \sum_{j+1}^{i-1} k_{il} k_{lj}^{(\nu-1)},$$

Then (1) has a unique solution

$$y^h = \sum_{0}^{\infty} (K^h)^n \phi^h.$$

Discrete analogue

Moreover, if

$$x_i \leq \phi_i + h \sum_{j=0}^{i-1} k_{ij} x_j,$$

then

$$x_i \leq y_i$$

Corollary: If $\delta > 0$ and

$$x_0 \leq \delta, \qquad x_i \leq \delta + h^{1-\alpha} \sum_{j=0}^{i-1} \frac{x_j}{(i-j)^\alpha}, \quad 1 \leq i \leq N$$

then

$$x_i \leq \delta E_{1-\alpha}(\Gamma(1-\alpha)(ih)^{1-\alpha}), \quad 0 \leq i \leq N.$$

Example

$$y(t) = \phi(t) + \int_0^t \frac{y(s)}{(t-s)^\alpha} ds, \quad 0 \le t \le T.$$

Apply numerical scheme (such as trapeziodal-type scheme) so that

$$y_i = \phi_i + h \sum_{j=0}^{i-1} k_{ij} y_j, \quad 0 \le i \le N, (1)$$

and for some M, independent of h ($Nh = T$),

$$k_{ij} \le M(h(i-j))^{-\alpha}.$$

True solution satisfies

$$y(t_i) = \phi(t_i) + h \sum_{j=0}^{i-1} k_{ij} y(t_j) + T_i, \quad 0 \le i \le N, (2)$$

Hence, by subtracting (1) from (2) and using the discrete Gronwall inequality, convergence follows if $T_i \to 0$ as $h \to 0$.

What came out of this?

This idea of proving discrete results by mirroring what goes on in the continuous case led to

- convergence proofs for whole classes of numerical methods for Volterra equations (first and second kind and integro-differential equations)
- the development of new numerical methods
- generalised Gronwall inequalities for other related problems (eg repeated integral inequalities).

What came out of this?

I am not sure that Gene ever used Gronwall inequalities.

BUT try googling Golub and Gronwall and many connections found.

Moreover,

- Gene appreciated their importance and beauty
- he was very open to and excited by new ideas
- he respected the work of young people
- he provided encouragement and kindness.

Thank you, Gene!