Global Rates for Zero-Order Methods

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My talk will focus on: Derivative-Free Optimization (DFO), zero-order methods.

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- Model-based methods, like trust-region methods.

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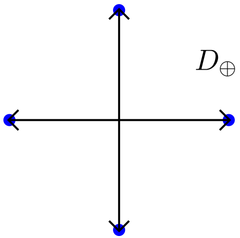

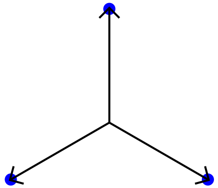

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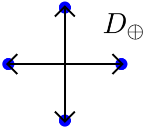

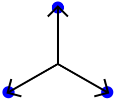

Our problem setting

Unconstrained optimization

$$\min_{x \in \mathbb{R}^n} f(x)$$

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In most of the talk, we take $p = 2$: $\rho(\alpha) = \alpha^2$.

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- Update the step size α_{k+1}.
  Possible increase if iteration is successful. Decrease otherwise.

Behavior of the step size parameter

Assumption

The level set $L(x_0) = \{x \in \mathbb{R}^n : \; f(x) \leq f(x_0)\}$ *is bounded.*

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There exists a point x_* *and a subsequence* K *of unsuccessful iterations:*

$$\lim_{k \in K} x_k = x_* \quad \textit{and} \quad \lim_{k \in K} \alpha_k = 0.$$

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Assumption

The directions in D_k *are bounded above and away from zero.*

The cosine measure of D_k *is bounded away from zero.*

Behavior of unsuccessful iterations

Theorem (Lewis, Tolda, and Torczon 2003)

Let D_k be a PSS.

Assume $f \in \mathcal{C}^1_\nu$.

If the iterate k is unsuccessful, i.e.,

$$f(x_k + \alpha_k d) \geq f(x_k) - \rho(\alpha_k), \qquad \text{for all } d \in D_k,$$

then

$$\|\nabla f(x_k)\| \leq \frac{C(\nu) \times \alpha_k}{\mathrm{cm}(D_k)} \qquad \text{... since } \rho(\alpha) = \alpha^2.$$

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Note that global convergence is deduced from here: $\|\nabla f(x_k)\| \underset{K}{\longrightarrow} 0$.

The question that interests us (smooth case)

Question

Given $\epsilon \in (0,1)$, *how many iterations* $\bar{k}$ *are needed to reach*

$$\|\nabla f(x_{\bar{k}})\| \leq \epsilon \quad ?$$

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The number of successful iterations between k_0 and $\bar{k}$ is

$$\mathcal{S}(k_0, \bar{k}) \;\leq\; \left\lceil C(\nu^2)(f(x_{k_0}) - f_*)\,\frac{1}{\epsilon^2} \right\rceil .$$

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The number of successful iterations between k_0 and $\bar{k}$ is

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$$\mathcal{U}(k_0, \bar{k}) \;=\; \mathcal{O}(\mathcal{S}(k_0, \bar{k})).$$

The number of suc. iterations until k_0 is at most

$$\left\lceil \frac{f(x_0) - f_*}{\alpha_0^2} \right\rceil.$$

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Any direct-search method (based on sufficient decrease) takes at most

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iterations to reduce the gradient below $\epsilon \in (0,1)$.

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- L. N. Vicente, Worst case complexity of direct search, to appear in EURO J. on Computational Optimization, Vol. 1, Num. 1, 2013.

Assumption in the smooth, convex case

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There exists a positive constant R *such that*

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One needs this assumption because

$$\|x_k - x_*\| \;\leq\; \|x_0 - x_*\|, \qquad \forall k \geq 0,$$

does NOT hold as in the gradient method (because $d_k \neq -\nabla f(x_k)$).

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Lemma (Decrease rate for f)

Any direct-search method (based on sufficient decrease) generates a sequence $\{x_k\}_{k \geq k_0}$ *such that*

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- M. Dodangeh and L. N. Vicente, Worst case complexity of direct search under convexity, 2013.

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Proposition

Let f be continuous and strongly convex with constant μ. Then

$$\sup_{y \in L(x_0)} \operatorname{dist}(y, X_*^f) \leq \sqrt{\frac{2}{\mu}(f(x_0) - f_*)}.$$

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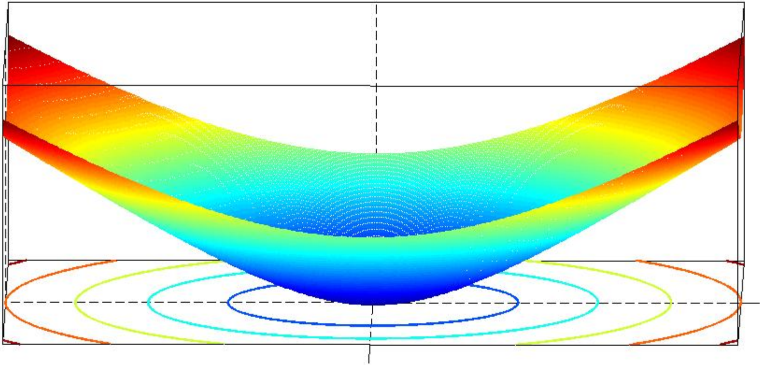

An example where R is UNBOUNDED

Objective function

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$$f(x, y) \ = \ y^2 + \frac{1}{2}(\epsilon x)^2 + \epsilon x.$$

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Algorithmic choices

Using $\gamma = 1$ (suc. iterates), $x_0 = (-\epsilon^{-1}, \frac{\sqrt{6}}{2})$, $\alpha_0 = 1$, $\rho(\alpha) = \epsilon\alpha^2$, and

$$D \ = \begin{bmatrix} 1 & 0 & -1 \\ -\frac{\sqrt{6}}{2}\epsilon & \frac{\sqrt{6}}{2}\epsilon & 0 \end{bmatrix}.$$

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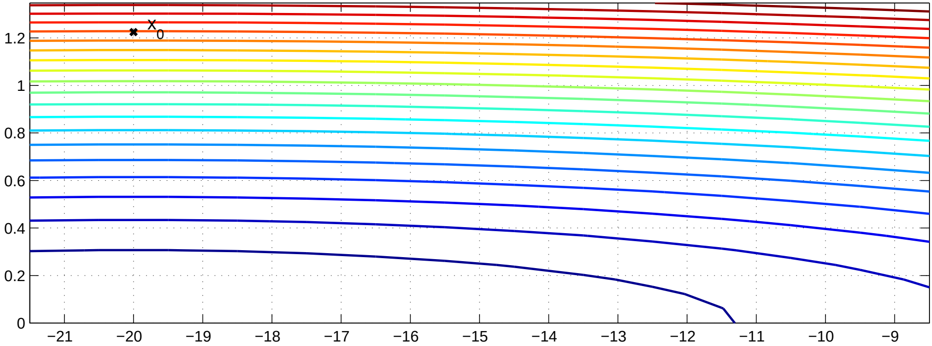

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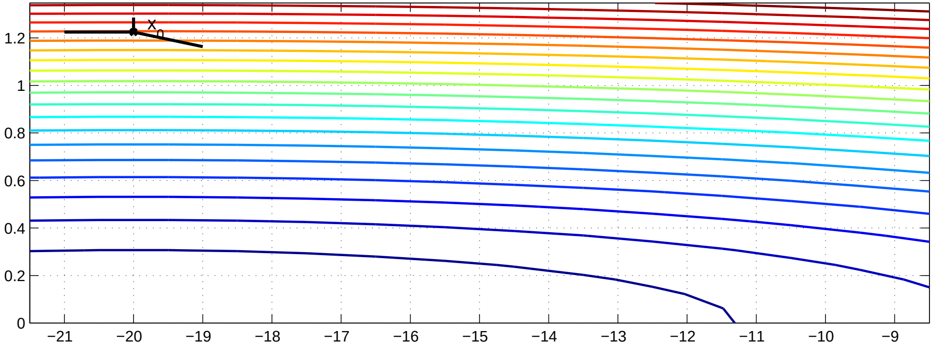

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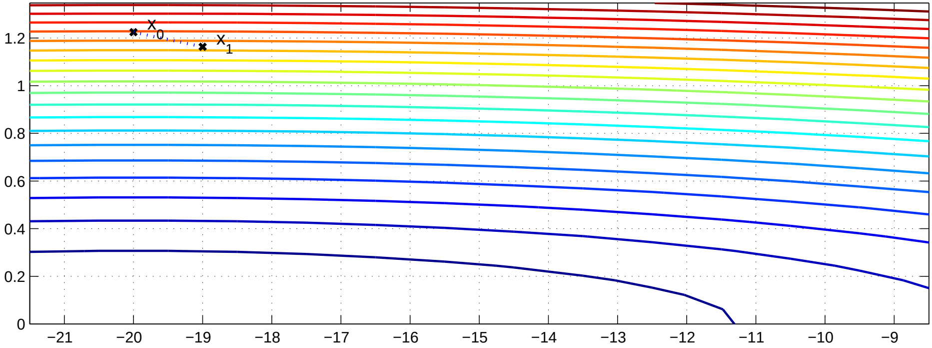

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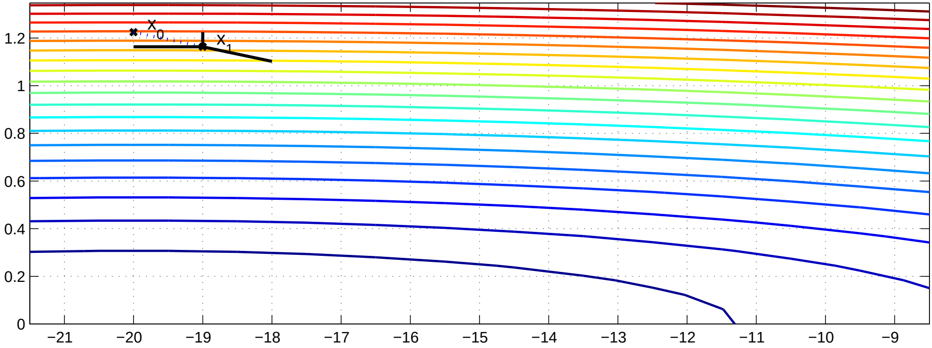

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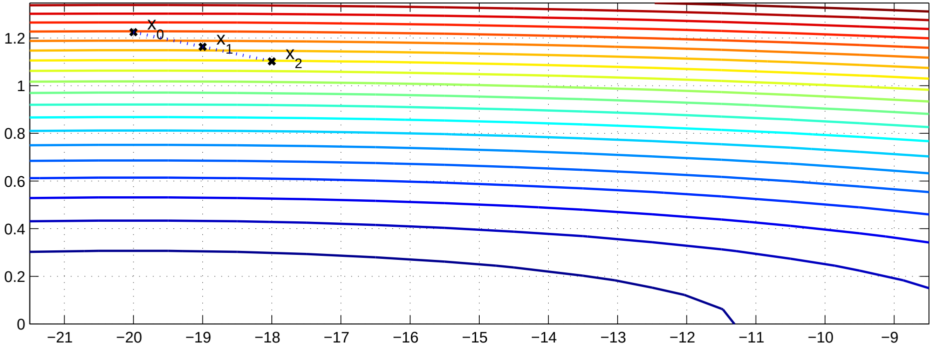

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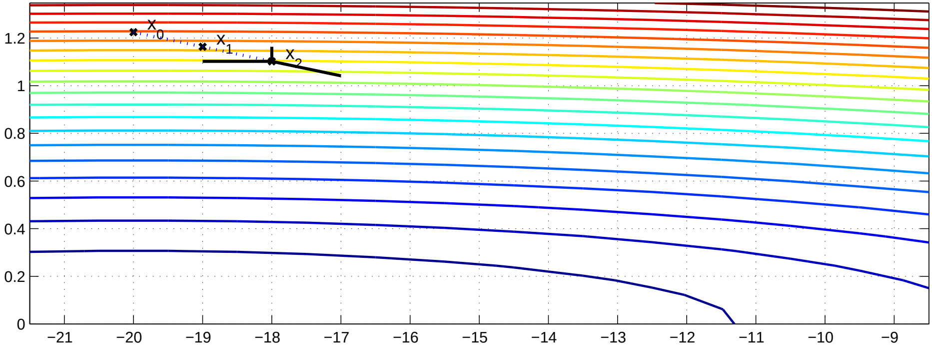

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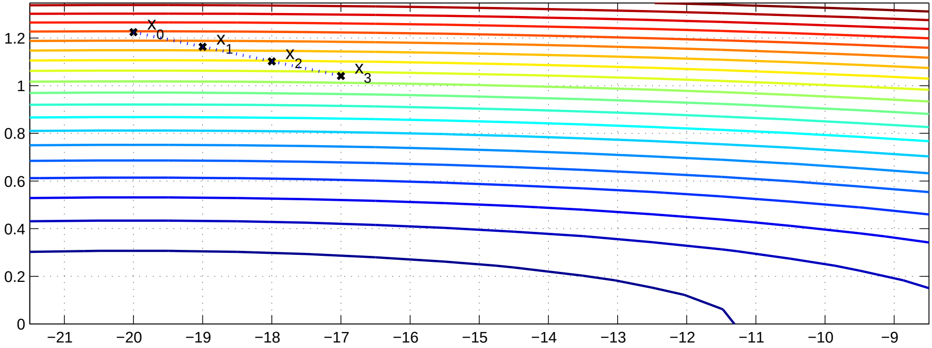

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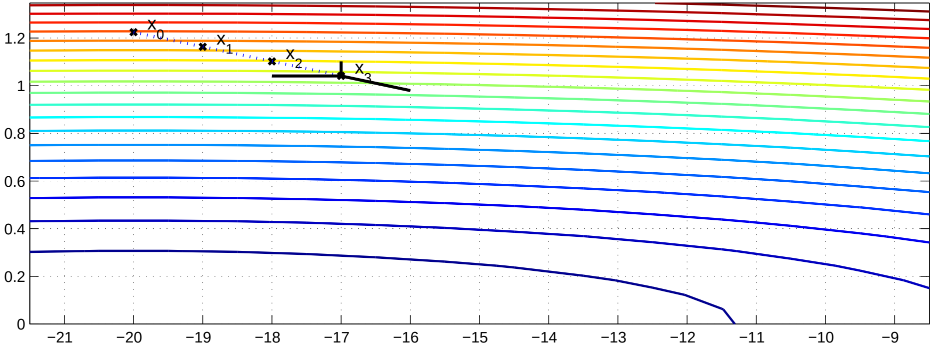

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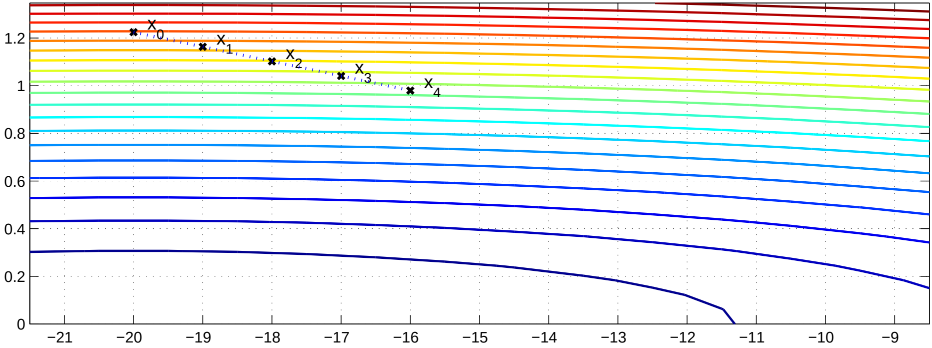

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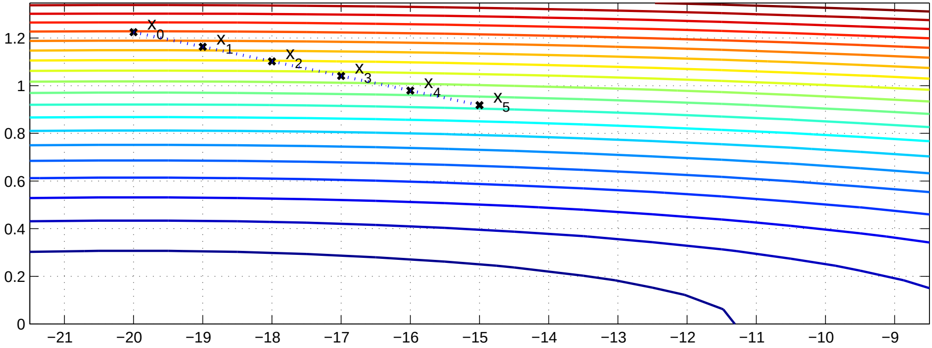

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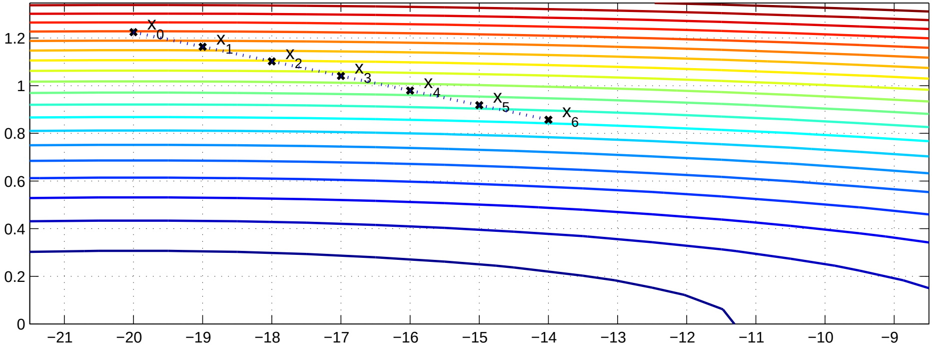

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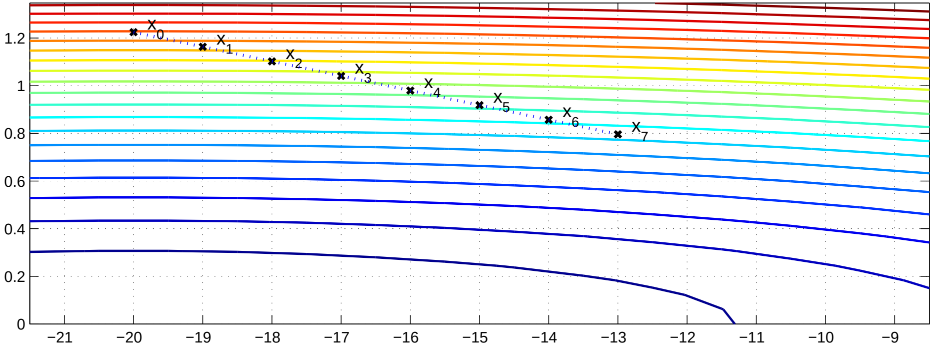

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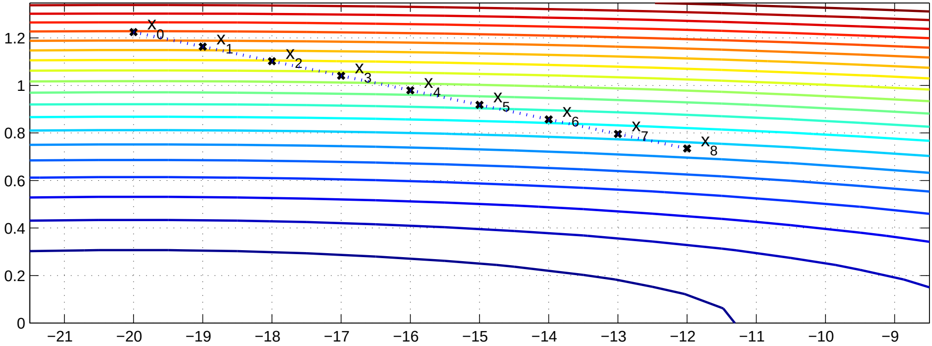

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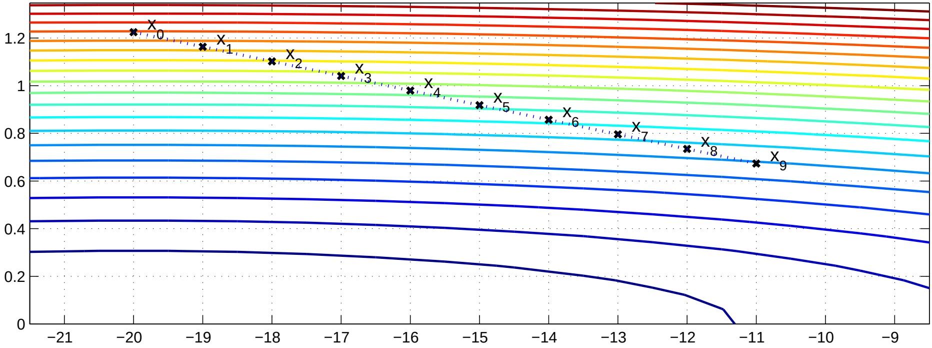

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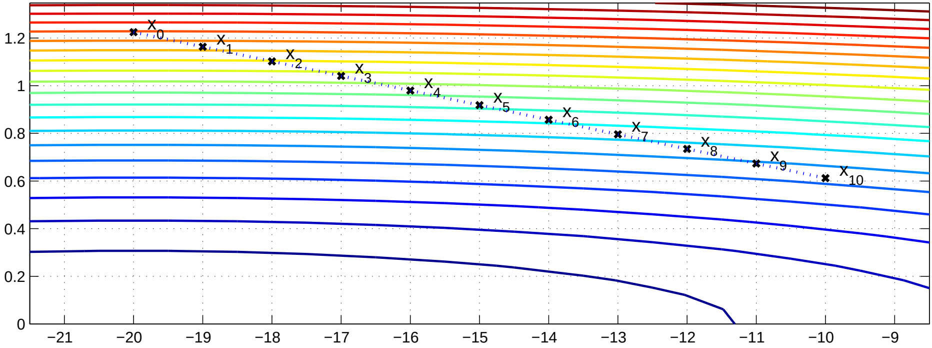

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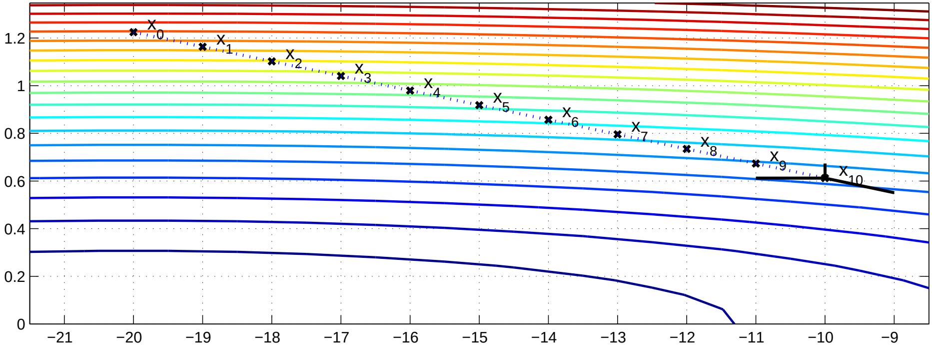

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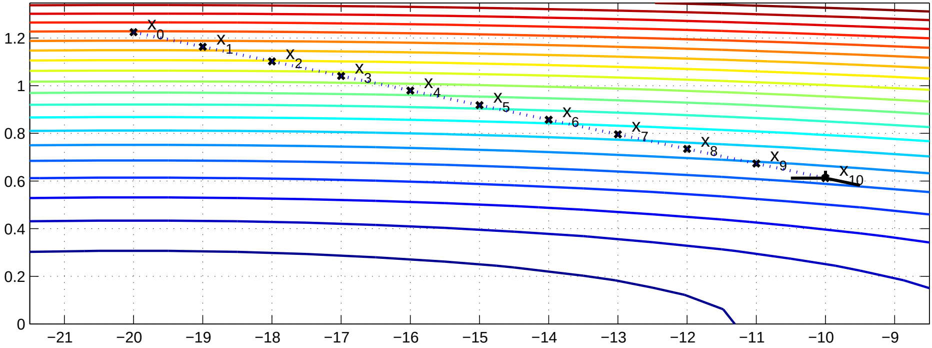

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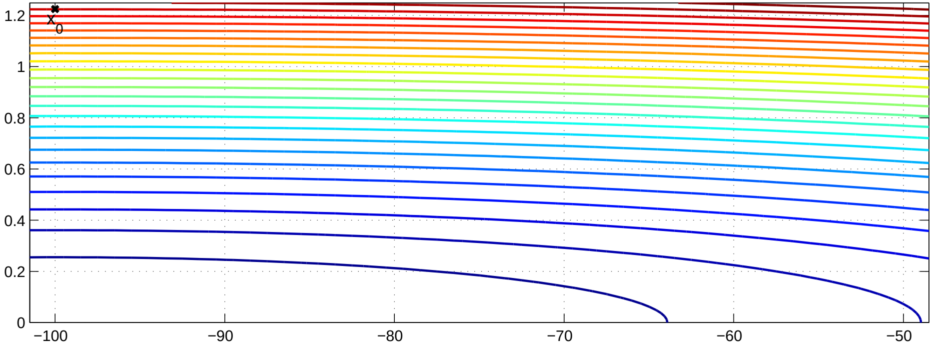

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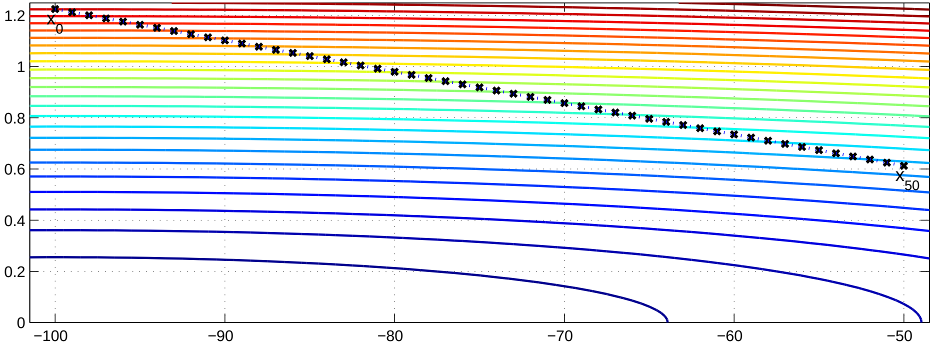

$$\begin{cases} \epsilon = 0.01, & \|x_0 - x_*\| = \frac{\sqrt{6}}{2}, \\ k_0 = 50 \geq \frac{\epsilon^{-1}-1}{2}, & \|x_k - x_*\| \geq k. \end{cases}$$

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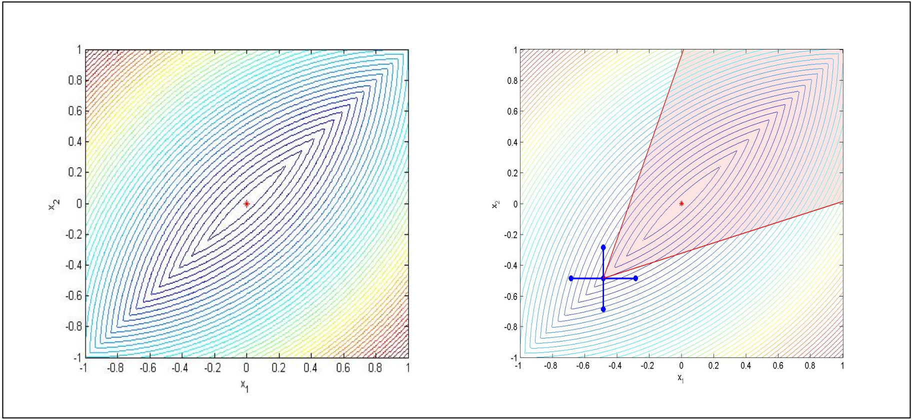
and one expects the global rate $\mathcal{O}(\epsilon^{-1})$ to hold for gradient methods.

Difficulties in the nonsmooth case



The cone of descent directions at the poll center is shaded.

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- L. N. Vicente and A. L. Custódio Analysis of direct searches for discontinuous functions, Math. Programming, 133 (2012) 299-325.

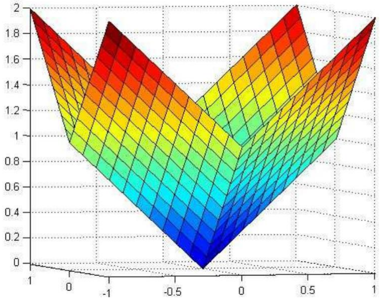
  Dense generation is waived of rules when imposing sufficient decrease.

Another possible fix: Smoothing functions

Definition

We call $\tilde{f} : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ *a* *smoothing function* *of* f *if,* $\forall \mu \in (0, +\infty)$, $\tilde{f}(\cdot, \mu)$ *is* $\mathcal{C}^1$ *and,* $\forall x \in \mathbb{R}^n$,

$$\lim_{z \to x, \mu \downarrow 0} \tilde{f}(z, \mu) \;=\; f(x).$$



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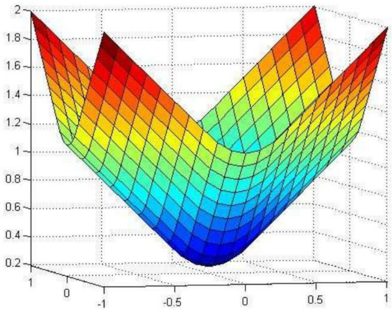

A class of smoothing DS methods

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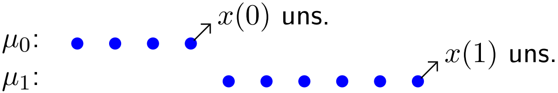

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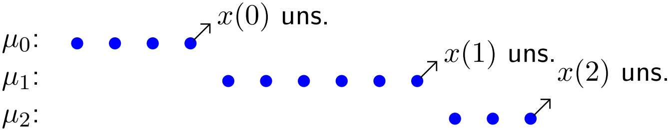

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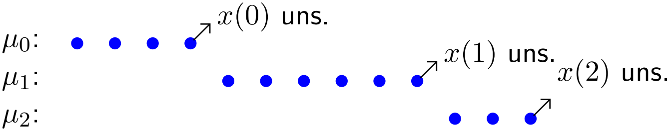

Initialization: Choose a function $r(\cdot)$ such that $\lim_{\mu \downarrow 0} r(\mu) = 0$.

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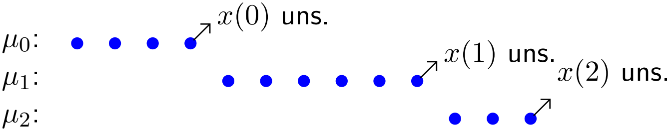

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For $k = 0, 1, 2 \ldots$ (Until μ_k is suff. small)

- Apply DS to $\tilde{f}(\cdot, \mu_k)$ until step size $< r(\mu_k)$.
- Decrease the smoothing parameter: $\mu_{k+1} = \sigma \mu_k$.

Global convergence of smoothing DS (behavior of μ)

Assumption

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Theorem

1. $\lim_{k\to+\infty} \alpha(k) = 0.$
2. $\exists x_*$ *and a subsequence* $K \subseteq \{(0),(1),\ldots\}$ *of unsucc. DS iterates such that* $x(k) \underset{K}{\longrightarrow} x_*$.

Now, $\|\nabla \tilde{f}(x(k), \mu_k)\| \;\leq\; C(\tilde{\nu}(\mu_k))\,\alpha(k)$

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We say that x_* is a stationary point associated with the smoothing function $\tilde{f}$ if $0 \in G_{\tilde{f}}(x_*)$, where

$$G_{\tilde{f}}(x_*) \;=\; \{\textit{all limits of}\, \nabla \tilde{f}(x, \mu) \textit{ when } x \to x_* \textit{ and } \mu \to 0\}.$$

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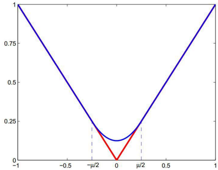

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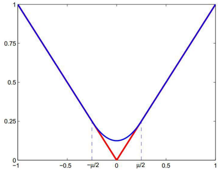

Then we obtain $\tilde{F}(x,\mu) = \sum_{i=1}^{m} \tilde{s}(F_i(x),\mu)$ for $\|F\|_1 = \sum_{i=1}^{m} |F_i|$.

WCC of smoothing DS (to reduce μ)

Theorem

Let $\rho(\alpha) = \alpha^p$ *and* $r(\alpha) = \alpha^q$, *with* $p, q > 1$.

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Let $\rho(\alpha) = \alpha^p$ and $r(\alpha) = \alpha^q$, with $p, q > 1$.

Any smoothing DS (based on sufficient decrease) takes at most

$$\mathcal{O}\left((-\log(\xi))\xi^{-pq}\right)$$

DS inner iterations to reduce μ below $\xi \in (0,1)$.

WCC of smoothing DS (to reduce smoothing gradient)

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Therefore, the number of iterations needed to reach $\|\nabla \tilde{f}\| \leq \epsilon$ and $\mu \leq \xi = \mathcal{O}(n^{-\frac{1}{2}}\epsilon)$ is

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This compares to $\mathcal{O}\left(n^3\epsilon^{-3}\right)$ using Gaussian densities (Nesterov, 2011).

WCC of smoothing DS (function evaluations)

Therefore, the number of iterations needed to reach $\|\nabla \tilde{f}\| \leq \epsilon$ and $\mu \leq \xi = \mathcal{O}(n^{-\frac{1}{2}}\epsilon)$ is

$$\mathcal{O}\left((-\log(\xi))\xi^{-pq}\right) \overset{p=\frac{3}{2},q=2}{=} \mathcal{O}\left(n^{\frac{3}{2}}[-\log(\epsilon)+\log(n)]\epsilon^{-3}\right).$$

In terms of function evaluations:

$$\mathcal{O}\left(n^{\frac{5}{2}}[-\log(\epsilon)+\log(n)]\epsilon^{-3}\right).$$

This compares to $\mathcal{O}\left(n^3\epsilon^{-3}\right)$ using Gaussian densities (Nesterov, 2011).

Reference:

- R. Garmanjani and L. N. Vicente, Smoothing and worst-case complexity for direct-search methods in nonsmooth optimization, to appear in IMA Journal of Numerical Analysis .

Summary of DS global rates

Imposing sufficient decrease to accept new iterates, as in derivative-based optimization:

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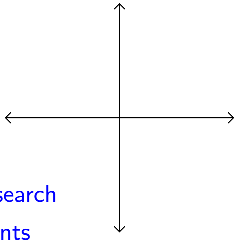
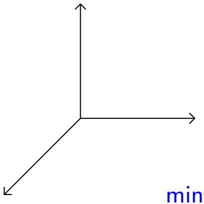
- $\mathcal{O}(\epsilon^{-1})$ smooth, convex (under sort of strong convexity...).
- $\mathcal{O}(\epsilon^{-2})$ smooth, non-convex.
- $\mathcal{O}(\epsilon^{-3})$ non-smooth, non-convex (using smoothing techniques...).

Positive spanning sets



coordinate search

$2n$ elements



minimal case

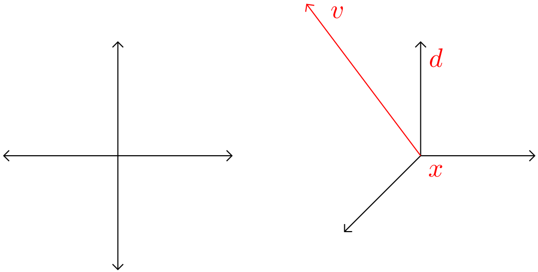
$n+1$ elements

$\mathrm{cm}(D_\oplus) = \frac{1}{\sqrt{n}}$

$\mathcal{O}\left(\frac{1}{\mathrm{cm}(D_\oplus)^2}\epsilon^{-2}\right) = \mathcal{O}\left(n\epsilon^{-2}\right)$ iterations

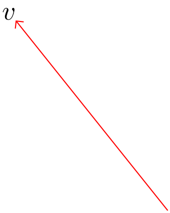
$$\mathrm{cm}(D) \; = \; \min_{0 \neq v \in \mathbb{R}^n} \max_{d \in D} \frac{v^\top d}{\|v\|\|d\|}$$

Positive spanning sets



If $v = -\nabla f(x)$ then d is a descent direction.

Randomly generating 'positive spanning sets' ...



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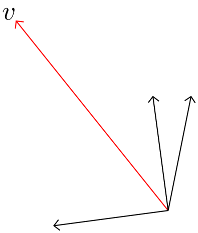

$n+1$ random polling directions

in this case not a PSS

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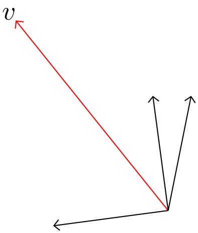

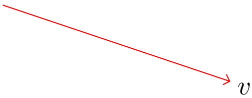

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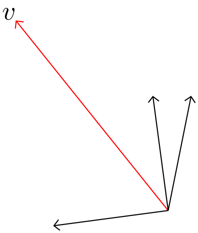

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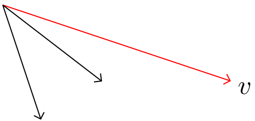

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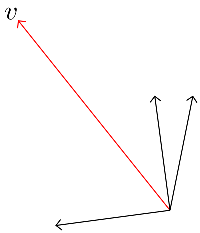

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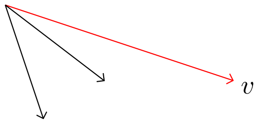

$< n$ random polling directions

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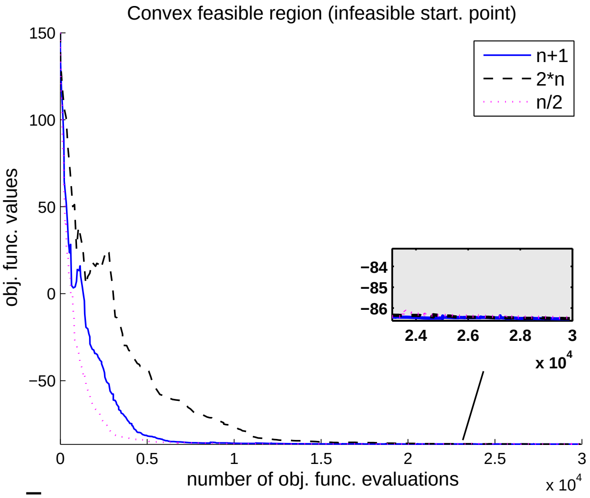
All we need is $\mathrm{cm}(D, v) \;=\; \max_{d \in D} \dfrac{v^\top d}{\|v\|\|d\|} \;\geq\; \kappa \in (0,1)$

Using $n/2$ random polling directions...



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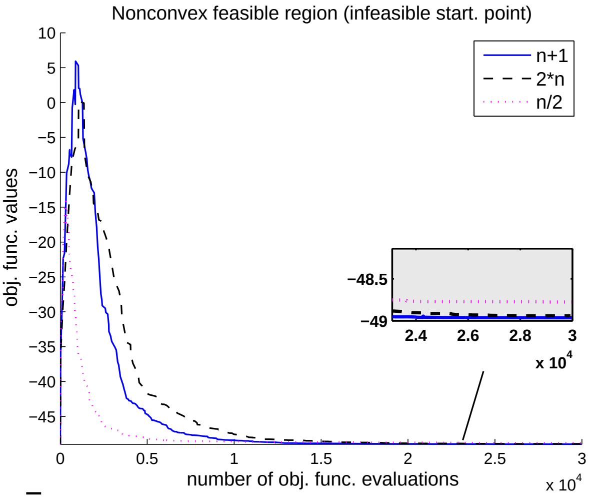

problem	$[Q_k \ -Q_k]$	$2n$	$n+1$	$n/2$
arglina	3958	2954	1681	943
arglinb	266	94	62	44
arwhead	3903	2874	1735	945
bdqrtic	1198	1088	682	369
broydn3d	4196	3491	2005	1202
dqrtic	2485	1533	873	493
engval1	1642	888	566	308
freuroth	4	4	5	6
integreq	3796	3100	1789	956
nondia	882	1162	884	764
nondquar	3105	2719	1694	1052
penalty1	1422	1439	832	462
penalty2	2425	1391	744	458
tquartic	- (100)	28059	20087	14848
vardim	6	17	19	16

fevals to reach an opt. accuracy of 10^{-3}.

Here $n = 20$ and averages where taken for 30 runs.

DS based on probabilistic descent

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We say that a sequence of polling directions $\{D_k\}$ is (p)-probabilistically κ-descent for corresponding sequences $\{X_k\}$, $\{\mathrm{Alpha}_k\}$ if the events

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Furthermore, if $p \geq \frac{1}{2}$, then we say that the polling directions are probabilistically κ-descent.

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Lemma

For every realization of the algorithm, $\lim_{k\to\infty} \alpha_k = 0$.

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The proof is based on the trust-region corresponding one:

- A. S. Bandeira, K. Scheinberg, and L. N. Vicente, Convergence of trust-region methods based on probabilistic models, submitted.

Worst case complexity of DS based on prob. descent

$p \nearrow$ when $|D| \nearrow$, but what is the effect of this on the performance/analysis of the algorithm?

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$\cdots$ is (roughly) the σ-algebra until the index corresponding to the smallest step size up to K.

- One can relax the lower bound on p to

$$p \;\geq\; \frac{\ln(C_{dec})}{\ln(C_{dec}/C_{inc})}.$$

DS based on probabilistic descent

- One can relax the lower bound on p to

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- When one imposes $p \geq \frac{1}{2}$, one must have $|D| \geq 2$.

problem	$[Q_k \ -Q_k]$	$2n$	$n+1$	$n/2$	$n/4$	2	1
arglina	8.60	6.42	3.65	2.05	1.23	1	- (100)
arglinb	11.08	3.92	2.58	1.83	1.46	1	4.17 (13)
arwhead	8.18	6.03	3.64	1.98	1.19	1	- (100)
bdqrtic	6.62	6.01	3.77	2.04	1.25	1	4.80 (80)
broydn3d	4.71	3.92	2.25	1.35	0.99	1	- (100)
dqrtic	8.28	5.11	2.91	1.64	1.06	1	4.67 (87)
engval1	11.09	6.00	3.82	2.08	1.36	1	4.60 (73)
freuroth	0.67	0.67	0.83	1	1	1	1
integreq	8.38	6.84	3.95	2.11	1.27	1	4.26 (93)
nondia	0.84	1.11	0.84	0.73	0.83	1	0.05 (13)
nondquar	4.27	3.73	2.33	1.45	1.02	1	- (100)
penalty1	5.51	5.58	3.22	1.79	1.17	1	3.82 (70)
penalty2	11.28	6.47	3.46	2.13	1.37	1	5.54 (90)
tquartic	- (100)	1.62	1.16	0.86	0.75	1	- (100)
vardim	0.46	1.31	1.46	1.23	1.08	1	5.54 (3.3)

Now, we display increase in # fevals relatively to using 2 directions.

References and support

References:

- S. Gratton, C. Royer, L. N. Vicente, and Z. Zhang, Direct search based on probabilistic descent, in preparation.
- S. Gratton and L. N. Vicente, A merit function approach for direct search, submitted.

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