A Subspace Decomposition Framework for Nonlinear Optimization: Global Convergence and Global Rate

Zaikun Zhang
University of Coimbra
(Joint work with S. Gratton and L. N. Vicente)

July 25, Toulouse

http//www.mat.uc.pt/~zhang

Opening

Axiom

师徒如父子；一日为师，终身为父。

(Teacher and student are like father and son; if he/she is your teacher for one day, then he/she is your father for one life.)

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生日快乐，Toint 师伯！

Happy birthday, Academic (Older) Uncle Toint!

Joyeux anniversaire, Oncle Toint !

Outline

In this talk, to make things simple:

- we consider unconstrained optimization problem

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- we suppose that
  - f is smooth, but the derivatives are unavailable.

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- Important & difficult

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Introduction to Derivative-Free Optimization

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Introduction to Derivative-Free Optimization

> *Why work on derivative-free optimization? Because the problems are important and cool.*

— J. Dennis
July 24th, 2013, Toulouse

Existing methods

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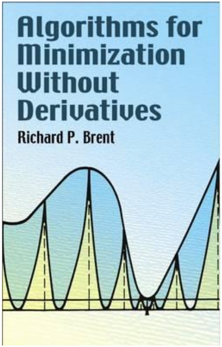
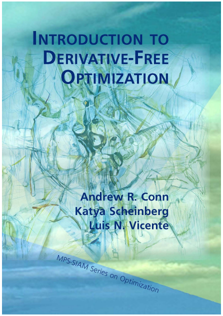
- Two main classes of rigorous methods in DFO
  - Directional methods, like direct search
  - Model-based methods, like trust-region methods

Books



R. P. Brent, Algorithms for Minimization Without Derivatives, Prentice-Hall, Englewood Cliffs, NJ, 1973



A. R. Conn, K. Scheinberg, and L. N. Vicente, Introduction to Derivative-Free Optimization, MOS-SIAM Series on Optimization, SIAM, Philadelphia, 2009

Difficulty of large-scale problems

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  - difficult to exploit problem structure

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  - divide a difficult problem into a sequence of easy problems, and solve each of them;

  more specifically,
  - divide a large problem into a sequence of small problems, and solve each of them.

An old idea, very old

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分而治之 Divide and conquer

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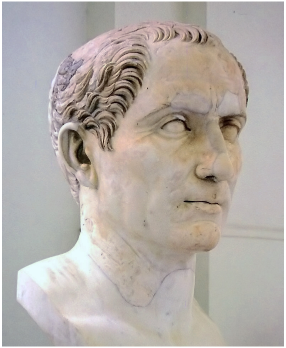

Divide et impera.

— Julius Caesar
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Subspace and decomposition techniques in optimization

- Subspace techniques
  - Gould, Nick, A. Sartenaer, and Ph L. Toint. On iterated-subspace minimization methods for nonlinear optimization. Rutherford Appleton Laboratory, 1994.
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- Coordinate-search . . .

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- Trust-region:

$$\begin{aligned} \min \ & f(x_k + d) \\ \text{s.t. } & d \in \mathcal{S}_k^{(i)} \\ & \|d\| \le \Delta_k \end{aligned}$$

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$$\rho_k = \frac{f(x_k) - f(x_k + d_k)}{\sum_{i=1}^{m_k} \left[f(x_k) - f(x_k + d_k^{(i)}) \right]}$$

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2. *The sequence $\{m_k\}$ is bounded.*
3. *The smallest eigenvalues of $\sum_{i=1}^{m_k} P_k^{(i)}$ are bounded away from zero, where $P_k^{(i)}$ is the orthogonal projection matrix from $\mathbb{R}^n$ onto $\mathcal{S}_k^{(i)}$.*

Global convergence

Theorem

Suppose that the assumptions stated before hold, then the iterates $\{x_k\}$ *generated by either of the frameworks satisfy*

$$\lim_{k\to\infty} \|\nabla f(x_k)\| = 0.$$

Theorem

Suppose that the assumptions stated before hold, and additionally

$$\Delta_{k+1} \geq \alpha \Delta_k$$

for some constant $\alpha \in (0, 1]$, *then the iterates* $\{x_k\}$ *generated by the trust-region framework satisfy*

$$\min_{0 \leq \ell \leq k} \|\nabla f(x_\ell)\| \leq C_1 \sqrt{\frac{m}{k}},$$

where m *is an upper bound of* $\{m_k\}$.

Theorem

Suppose that the assumptions stated before hold, and additionally

$$\sigma_{k+1} \leq \beta \sigma_k$$

for some constant $\beta \geq 1$, then the iterates $\{x_k\}$ generated by the Levenberg-Marquardt framework satisfy

$$\min_{0 \leq \ell \leq k} \|\nabla f(x_\ell)\| \leq C_2 \sqrt{\frac{m}{k}},$$

where m is an upper bound of $\{m_k\}$.

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Using this and the WCC $O(n^2\varepsilon^{-2})$ for subproblems,

- a reasonable choice for m is $O(\sqrt{n})$
- a reasonable subproblem solution accuracy is $O(n^{-\frac{1}{4}})$

Applications to derivative-free optimization

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$$\Downarrow$$

Our goal

Parallel and multilevel algorithms without using derivatives and capable of solving relatively large problems.

Very preliminary numerical results

- Use the Levenberg-Marquardt framework
- Subproblem solver: `NEWUOA`
- Number of subspaces: $\sqrt{n/2}$
- Benchmark: `NEWUOA`
- Very preliminary: not parallel, not multilevel, not large-scale . . .
- Dimension of test problems: 25, 30, 35, 40
- Denote our code as `SSD`

VARDIM

Table : Numerical results of VARDIM

n	25	30	35	40	
$\#f$	8343	8926	12689	17741	NEWUOA
	3592	6222	7507	16653	SSD
f_{final}	1.61E-11	4.08E-11	4.93E-11	1.76E-10	NEWUOA
	9.74E-11	6.85E-10	5.74E-11	7.89E-13	SSD

$$f(x) = \sum_{i=1}^{n} (x_i - 1)^2 + \left[\sum_{i=1}^{n} i(x_i - 1) \right]^2 + \left[\sum_{i=1}^{n} i(x_i - 1) \right]^4$$

Table : Numerical results of PENALTY1

n	25	30	35	40	
$\#f$	9532	10947	14427	13577	NEWUOA
	2089	2784	2348	2812	SSD
f_{final}	2.03E-04	2.48E-04	2.93E-04	3.39E-04	NEWUOA
	2.04E-04	2.50E-04	2.95E-04	3.41E-04	SSD

$$f(x) = 10^{-15}\sum_{i=1}^{n}(x_i - 1)^2 + \left(\frac{1}{4} - \sum_{i=1}^{n} x_i^2\right)^2$$

SBRYBND

Table : Numerical results of SBRYBND

n	25	30	35	40	
$\#f$	968	576	2052	2363	`NEWUOA`
	27889	53103	90304	206608	`SSD`
f_{final}	235	326	342	395	`NEWUOA`
	3.08	3.08	3.08	3.08	`SSD`
	134	284	233	229	

$$f(x) = \sum_{i=1}^{n} \left[(2 + 5p_i^2 x_i^2) p_i x_i + 1 - \sum_{j \in J_i} p_j x_j (1 + p_j x_j) \right]^2,$$

where $J_i = \{ j \mid j \neq i, \max\{1, i-5\} \leq j \leq \min\{n, j+1\}\}$, and $p_i = \exp\left(6 \frac{i-1}{n-1}\right)$.

CHROSEN

Table : Numerical results of CHROSEN

n	25	30	35	40	
$\#f$	1123	1445	1717	1859	NEWUOA
	96040	103296	127726	142272	SSD
f_{final}	8.94E-12	1.07E-11	1.13E-11	3.14E-11	NEWUOA
	2.95E-10	5.49E-10	7.26E-10	8.09E-10	SSD

$$f(x) = \sum_{i=1}^{n-1} \left[4(x_i - x_{i+1}^2)^2 + (1 - x_{i+1})^2 \right]$$

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Merci! 谢谢！

zhang@mat.uc.pt