Finding a well-centred point for a set of polyhedral constraints

Nick Gould (RAL)

joint with

Coralia Cartis (OUCL)

$$Ax = b, \quad x \geq 0$$

Sparse Days 2006
CERFACS, Toulouse, France
15th June 2006

Motivation — feasible points

- many optimization problems require $x \in \mathbb{R}^n$:

$$Ax = b \text{ and } x \geq 0$$

for given A, b

Motivation — feasible points

- many optimization problems require $x \in \mathbb{R}^n$:

$$Ax = b \quad \text{and} \quad x \geq 0$$

  for given A, b

  - usually under determined $\Longrightarrow$ select other important characteristics:

Motivation — feasible points

- many optimization problems require $x \in \mathbb{R}^n$:

$$Ax = b \quad \text{and} \quad x \geq 0$$

for given A, b

  - usually under determined $\Longrightarrow$ select other important characteristics:

    minimize $c^T x$: linear programming

    minimize $\frac{1}{2} x^T H x + c^T x$: quadratic programming

    minimize $f(x)$ & maybe $c(x) \geq 0$: nonlinear programming

Motivation — feasible points

- many optimization problems require $x \in \mathbb{R}^n$:

$$Ax = b \quad \text{and} \quad x \geq 0$$

for given A, b

  - usually under determined $\Longrightarrow$ select other important characteristics:

    minimize $c^T x$: linear programming

    minimize $\frac{1}{2} x^T H x + c^T x$: quadratic programming

    minimize $f(x)$ & maybe $c(x) \geq 0$: nonlinear programming

- everything we say applies more generally to

$$Ax = b \quad \text{and} \quad l \leq x \leq u$$

for given A, b, l, u . . .

Motivation — interior points

- often desirable to find **interior-point** (IP) $\boldsymbol{x}$:

$$\boldsymbol{Ax = b} \;\; \text{and} \;\; \boldsymbol{x > 0}$$

Motivation — interior points

- often desirable to find **interior-point** (IP) x:

$$Ax = b \quad \text{and} \quad x > 0$$

Why?

Motivation — interior points

- often desirable to find **interior-point** (IP) x:

$$Ax = b \text{ and } x > 0$$

Why?

  - good starting point for **feasible-point** IP methods for convex problems
    - low computational complexity of feasible-point IP methods

Motivation — interior points

- often desirable to find **interior-point** (IP) x:

$$Ax = b \quad \text{and} \quad x > 0$$

Why?

- good starting point for **feasible-point** IP methods for convex problems
  - low computational complexity of feasible-point IP methods
- reduces dimension of feasible region for more general non-convex problems
  - subsequent iterates $x + \Delta x$ satisfy

$$A\Delta x = 0 \quad \text{and} \quad \Delta x \geq -x$$

$\Longrightarrow$ all iterates lie in null-space of A

  - often lessens influence of non-convexity

Motivation — best interior point

What is the "best" interior point?

- in the absence of any objective (bounded case):

Motivation — best interior point

What is the “best” interior point?

- in the absence of any objective (bounded case):
  - **centroid** ? — expensive

Motivation — best interior point

What is the “best” interior point?

- in the absence of any objective (bounded case):
  - **centroid** ? — expensive
  - **analytic center** ?

$$\arg\max \prod_i x_i \equiv \arg\min - \sum_i \log x_i$$

such that $Ax = b$ and $x > 0$

Best interior point II

If there is an objective, e.g. $c^T x$, **balance** analytic center with objective $\Longrightarrow$

$$\min_{x>0} \quad c^T x - \mu \sum_i \log x_i \quad \text{such that} \quad Ax = b$$

for some **fixed** target $\mu > 0$

Best interior point II

If there is an objective, e.g. $c^T x$, **balance** analytic center with objective $\Longrightarrow$

$$\min_{x>0} \quad c^T x - \mu \sum_i \log x_i \ \text{ such that } \ Ax = b$$

for some **fixed** target $\mu > 0 \Longrightarrow$ **optimality conditions**

$$c - \mu \sum_i x_i^{-1} - A^T y = 0 \ \text{ and } \ Ax = b \ \text{ where } \ x > 0$$

Best interior point II

If there is an objective, e.g. $c^T x$, **balance** analytic center with objective $\Longrightarrow$

$$\min_{x>0} \quad c^T x - \mu \sum_i \log x_i \quad \text{such that} \quad Ax = b$$

for some **fixed** target $\mu > 0 \Longrightarrow$ **optimality conditions**

$$c - \mu \sum_i x_i^{-1} - A^T y = 0 \quad \text{and} \quad Ax = b \quad \text{where} \quad x > 0$$

$\Longrightarrow$ **central path**

$$\begin{aligned} Ax - b &= 0 \\ A^T y + s - c &= 0 \\ \text{and} \quad XSe &= \mu e \end{aligned}$$

- $(x, s) > 0$, parameter $\mu > 0$, $X = \text{diag } x$, $S = \text{diag } s$, e = vector of 1s

Summary of the talk

Aim: given $\mu > 0$, find $x > 0$ (along with $s > 0$ and y):

$$\begin{aligned} Ax - b &= 0 \\ A^T y + s - c &= 0 \\ \text{and} \quad XSe &= \mu e \end{aligned}$$

Summary of the talk

Aim: given $\mu > 0$, find $x > 0$ (along with $s > 0$ and y):

$$\begin{aligned} Ax - b &= 0 \\ A^T y + s - c &= 0 \\ \text{and} \quad XSe &= \mu e \end{aligned}$$

- assumptions
- "obvious" Newton method with safeguards
- convergence behaviour
- numerical experience
- extensions and future work

Assumptions

$$Ax - b = 0, \quad A^T y + s - c = 0 \text{ and } XSe = \mu e$$

Assume (to start with)

- A is full rank
- $\exists (x, s) > 0 : Ax = b$ and $A^T y + s = c$

$\Longrightarrow$ central path well defined

Newton's method

$$Ax - b = 0, \quad A^T y + s - c = 0 \text{ and } XSe = \mu e$$

Given $(x_k, s_k) > 0$ & y_k, the primal-dual **Newton** correction satisfies

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S_k & 0 & X_k \end{pmatrix} \begin{pmatrix} \dot{x}_k \\ \dot{y}_k \\ \dot{s}_k \end{pmatrix} = - \begin{pmatrix} Ax_k - b \\ A^T y_k + s_k - c \\ X_k S_k e - \mu e \end{pmatrix}$$

Newton's method

$$Ax - b = 0, \quad A^T y + s - c = 0 \text{ and } XSe = \mu e$$

Given $(x_k, s_k) > 0$ & y_k, the primal-dual **Newton** correction satisfies

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S_k & 0 & X_k \end{pmatrix} \begin{pmatrix} \dot{x}_k \\ \dot{y}_k \\ \dot{s}_k \end{pmatrix} = - \begin{pmatrix} Ax_k - b \\ A^T y_k + s_k - c \\ X_k S_k e - \mu e \end{pmatrix}$$

Linesearch $(x_{k+1}, y_{k+1}, s_{k+1}) = v_k(\alpha_k)$:

$$v_k(\alpha) \equiv \begin{pmatrix} x_k(\alpha) \\ y_k(\alpha) \\ s_k(\alpha) \end{pmatrix} = \begin{pmatrix} x_k \\ y_k \\ s_k \end{pmatrix} + \alpha \begin{pmatrix} \dot{x}_k \\ \dot{y}_k \\ \dot{s}_k \end{pmatrix}$$

for some suitable $\alpha_k \in (0, 1]$

Safeguards

$$(x_k(\alpha), y_k(\alpha), s_k(\alpha)) = (x_k + \alpha\dot{x}_k, y_k + \alpha\dot{y}_k, s_k + \alpha\dot{s}_k)$$

Pick $\alpha_k \in (0, 1]$ to

- ensure $(x_k(\alpha_k), s_k(\alpha_k)) > 0$

Safeguards

$$(x_k(\alpha), y_k(\alpha), s_k(\alpha)) = (x_k + \alpha\dot{x}_k, y_k + \alpha\dot{y}_k, s_k + \alpha\dot{s}_k)$$

Pick $\alpha_k \in (0, 1]$ to

- ensure $(x_k(\alpha_k), s_k(\alpha_k)) > 0$
  - given $\omega \in (0, 1)$, insist
    $$X_k(\alpha)s_k(\alpha) \geq \omega\mu e$$
    for all $0 \leq \alpha \leq \alpha_k$

Safeguards

$$(x_k(\alpha), y_k(\alpha), s_k(\alpha)) = (x_k + \alpha\dot{x}_k, y_k + \alpha\dot{y}_k, s_k + \alpha\dot{s}_k)$$

Pick $\alpha_k \in (0, 1]$ to

- ensure $(x_k(\alpha_k), s_k(\alpha_k)) > 0$
  - given $\omega \in (0, 1)$, insist
    $$X_k(\alpha)s_k(\alpha) \geq \omega\mu e$$
    for all $0 \leq \alpha \leq \alpha_k$
- (sufficiently) decrease some measure of optimality

Safeguards

$$(x_k(\alpha), y_k(\alpha), s_k(\alpha)) = (x_k + \alpha\dot{x}_k, y_k + \alpha\dot{y}_k, s_k + \alpha\dot{s}_k)$$

Pick $\alpha_k \in (0, 1]$ to

- ensure $(x_k(\alpha_k), s_k(\alpha_k)) > 0$
  - given $\omega \in (0, 1)$, insist
    $$X_k(\alpha)s_k(\alpha) \geq \omega\mu e$$
    for all $0 \leq \alpha \leq \alpha_k$
- (sufficiently) decrease some measure of optimality
  - obvious merit function

$$\begin{aligned} \Phi(x, y, s) = \; & \|Xs - \mu e\| + r(x, y, s) : \\ & r(x, y, s) = \|Ax - b\| + \|A^T y + s - c\| \end{aligned}$$

Nonzero stepsizes

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A^T & I \\ S_k & 0 & X_k \end{pmatrix} \begin{pmatrix} \dot{x}_k \\ \dot{y}_k \\ \dot{s}_k \end{pmatrix} = - \begin{pmatrix} Ax_k - b \\ A^T y_k + s_k - c \\ X_k s_k - \mu e \end{pmatrix}$$

Require

$$X_k(\alpha) s_k(\alpha) \geq \omega \mu e$$

for all $0 \leq \alpha \leq \alpha_k$ for given $\omega \in (0, 1)$

$$\begin{aligned} & X_k(\alpha)\, s_k(\alpha) - \omega \mu e \\ & \quad = \quad X_k s_k + \alpha(S_k \dot{x}_k + X_k \dot{s}_k) + \alpha^2 \dot{X}_k \dot{s}_k - \omega \mu e \\ & \quad = \quad (1-\alpha)(X_k s_k - \omega \mu e) + \alpha(1-\omega)\mu e + \alpha^2 \dot{X}_k \dot{s}_k \end{aligned}$$

At least one of the first two terms $> 0 \Longrightarrow$ nonzero stepsize

Decrease in merit functions

$$\begin{aligned} \Phi(v) \equiv \quad & \Phi(x, y, s) = \|Xs - \mu e\| + r(x, y, s) : \\ & r(v) \equiv r(x, y, s) = \|Ax - b\| + \|A^T y + s - c\| \end{aligned}$$

- primal-dual residual r decreases linearly with α:

$$r(v_k(\alpha)) = (1 - \alpha) r(v_k)$$

Decrease in merit functions

$$\Phi(v) \equiv \quad \Phi(x,y,s) = \|Xs - \mu e\| + r(x,y,s):$$
$$r(v) \equiv r(x,y,s) = \|Ax - b\| + \|A^T y + s - c\|$$

- primal-dual residual r decreases linearly with α:

$$r(v_k(\alpha)) = (1-\alpha) r(v_k)$$

- shifted complementarity behaves predictably:

$$X_k(\alpha) s_k(\alpha) - \mu e = (1-\alpha)(X_k s_k - \mu e) + \alpha^2 \dot{X}_k \dot{s}_k$$

$$\Longrightarrow$$

$$\Phi(v_k(\alpha)) \le (1-\alpha)\Phi(v_k) + \alpha^2 \|\dot{X}_k \dot{s}_k\| \quad \forall \alpha \in [0,1]$$

Algorithm

Given target μ, initial point $v_0 : (x_0, s_0) > 0$, constant $\omega \leq \min x_{i0} s_{i0}/\mu$, stopping tolerance $\epsilon > 0$ and $k = 0$

- if $\Phi(v_k) \leq \epsilon$, **stop**
- compute the Newton correction $\dot{v}_k$
- compute $\alpha_k^{\mathrm{L}} \in (0, 1] : X_k(\alpha) s_k(\alpha) \geq \omega \mu e \quad \forall \alpha \in [0, \alpha_k^{\mathrm{L}}]$
- Find α_k: global minimizer $\Phi(v_k(\alpha))$ in $[0, \alpha_k^{\mathrm{L}}]$
- $v_{k+1} = v_k + \alpha_k \dot{v}_k$
- $k \leftarrow k + 1$

Algorithm

Given target μ, initial point $v_0 : (x_0, s_0) > 0$, constant $\omega \leq \min x_{i0} s_{i0} / \mu$, stopping tolerance $\epsilon > 0$ and $k = 0$

- if $\Phi(v_k) \leq \epsilon$, **stop**
- compute the Newton correction $\dot{v}_k$
- compute $\alpha_k^{\mathrm{L}} \in (0, 1] : X_k(\alpha) s_k(\alpha) \geq \omega \mu e \quad \forall \alpha \in [0, \alpha_k^{\mathrm{L}}]$
- Find α_k: global minimizer $\Phi(v_k(\alpha))$ in $[0, \alpha_k^{\mathrm{L}}]$
- $v_{k+1} = v_k + \alpha_k \dot{v}_k$
- $k \leftarrow k + 1$

Dominant cost / iteration: Newton correction

Convergence analysis

Require additionally that $(x_0, s_0) \geq (u_0, w_0)$ for some (u_0, t_0, w_0):

$$Au_0 = b \text{ and } A^T t_0 + w_0 = c$$ (Zhang)

Convergence analysis

Require additionally that $(x_0, s_0) \geq (u_0, w_0)$ for some (u_0, t_0, w_0):

$Au_0 = b$ and $A^T t_0 + w_0 = c$ (Zhang)

Results:

(x_k, s_k) bounded above and away from zero

Convergence analysis

Require additionally that $(x_0, s_0) \geq (u_0, w_0)$ for some (u_0, t_0, w_0):

$$Au_0 = b \text{ and } A^T t_0 + w_0 = c$$ (Zhang)

Results:

(x_k, s_k) bounded above and away from zero

$\Longrightarrow$ (Newton system)

$\|\dot{x}_k\|, \|\dot{s}_k\| \leq \kappa_1 \Phi(v_k)$ for some constant $\kappa_1 > 0$

Convergence analysis

Require additionally that $(x_0, s_0) \geq (u_0, w_0)$ for some (u_0, t_0, w_0):

$$Au_0 = b \text{ and } A^T t_0 + w_0 = c \qquad \text{(Zhang)}$$

Results:

(x_k, s_k) bounded above and away from zero

$\Longrightarrow$ (Newton system)

$\|\dot{x}_k\|, \|\dot{s}_k\| \leq \kappa_1 \Phi(v_k)$ for some constant $\kappa_1 > 0$

$\Longrightarrow$ $(X_k(\alpha)s_k(\alpha) - \omega\mu e \geq \alpha(1-\omega)\mu e - \alpha^2\|\dot{X}_k\dot{s}_k\|)$

$\alpha_k^{\mathrm{L}} \geq \min\left\{1, \dfrac{\kappa_2}{[\Phi(v_k)]^2}\right\}$ for some constant $\kappa_2 > 0$ ...

Convergence analysis II

$$\Phi(v_k(\alpha)) \leq (1-\alpha)\Phi(v_k) + \alpha^2\|\dot{X}_k\dot{s}_k\| \quad \forall\alpha \in [0,1]$$

$$\Longrightarrow \qquad \Phi(v_{k+1}) \leq \min_{\alpha\in[0,\alpha_k^{\mathrm{L}}]} (1-\alpha)\Phi(v_k) + \alpha^2\|\dot{X}_k\dot{s}_k\|$$

Convergence analysis II

$$\Phi(v_k(\alpha)) \le (1-\alpha)\Phi(v_k) + \alpha^2\|\dot{X}_k\dot{s}_k\| \quad \forall\alpha \in [0,1]$$

$$\Longrightarrow \qquad \Phi(v_{k+1}) \le \min_{\alpha\in[0,\alpha_k^{\mathrm{L}}]} (1-\alpha)\Phi(v_k) + \alpha^2\|\dot{X}_k\dot{s}_k\|$$

$$\Longrightarrow$$

$$\Phi(v_{k+1}) \le \begin{cases} (1-\frac{1}{2}\alpha_k^{\mathrm{L}})\Phi(v_k) & \text{if } \alpha_k^{\mathrm{L}} \le \overline{\alpha}_k \equiv \frac{1}{2}\Phi(v_k)/\|\dot{X}_k\dot{s}_k\| \\ (1-\frac{1}{2}\overline{\alpha}_k)\Phi(v_k) & \text{otherwise} \end{cases}$$

Convergence analysis II

$$\Phi(v_k(\alpha)) \leq (1-\alpha)\Phi(v_k) + \alpha^2 \|\dot{X}_k \dot{s}_k\| \quad \forall \alpha \in [0,1]$$

$$\Longrightarrow \qquad \Phi(v_{k+1}) \leq \min_{\alpha \in [0, \alpha_k^{\mathrm{L}}]} (1-\alpha)\Phi(v_k) + \alpha^2 \|\dot{X}_k \dot{s}_k\|$$

$$\Longrightarrow$$

$$\Phi(v_{k+1}) \leq \begin{cases} (1 - \frac{1}{2}\alpha_k^{\mathrm{L}})\Phi(v_k) & \text{if } \alpha_k^{\mathrm{L}} \leq \overline{\alpha}_k \equiv \frac{1}{2}\Phi(v_k)/\|\dot{X}_k \dot{s}_k\| \\ (1 - \frac{1}{2}\overline{\alpha}_k)\Phi(v_k) & \text{otherwise} \end{cases}$$

$$\Longrightarrow \qquad \Phi(v_{k+1}) \leq \kappa_3 \Phi(v_k) \ \text{ for some } \ \kappa_3 \in [0,1)$$

Convergence analysis II

$$\Phi(v_k(\alpha)) \leq (1-\alpha)\Phi(v_k) + \alpha^2 \|\dot{X}_k \dot{s}_k\| \quad \forall \alpha \in [0,1]$$

$$\Longrightarrow \quad \Phi(v_{k+1}) \leq \min_{\alpha \in [0, \alpha_k^{\mathrm{L}}]} (1-\alpha)\Phi(v_k) + \alpha^2 \|\dot{X}_k \dot{s}_k\|$$

$$\Longrightarrow$$

$$\Phi(v_{k+1}) \leq \begin{cases} (1 - \frac{1}{2}\alpha_k^{\mathrm{L}})\Phi(v_k) & \text{if } \alpha_k^{\mathrm{L}} \leq \overline{\alpha}_k \equiv \frac{1}{2}\Phi(v_k)/\|\dot{X}_k \dot{s}_k\| \\ (1 - \frac{1}{2}\overline{\alpha}_k)\Phi(v_k) & \text{otherwise} \end{cases}$$

$$\Longrightarrow \quad \Phi(v_{k+1}) \leq \kappa_3 \Phi(v_k) \ \text{ for some } \ \kappa_3 \in [0,1)$$

$\Longrightarrow$ **global linear** convergence

Convergence analysis II

$$\Phi(v_k(\alpha)) \leq (1-\alpha)\Phi(v_k) + \alpha^2 \|\dot{X}_k \dot{s}_k\| \quad \forall \alpha \in [0,1]$$

$$\Longrightarrow \qquad \Phi(v_{k+1}) \leq \min_{\alpha \in [0,\alpha_k^{\mathrm{L}}]} (1-\alpha)\Phi(v_k) + \alpha^2 \|\dot{X}_k \dot{s}_k\|$$

$\Longrightarrow$

$$\Phi(v_{k+1}) \leq \begin{cases} (1-\frac{1}{2}\alpha_k^{\mathrm{L}})\Phi(v_k) & \text{if } \alpha_k^{\mathrm{L}} \leq \overline{\alpha}_k \equiv \frac{1}{2}\Phi(v_k)/\|\dot{X}_k \dot{s}_k\| \\ (1-\frac{1}{2}\overline{\alpha}_k)\Phi(v_k) & \text{otherwise} \end{cases}$$

$$\Longrightarrow \qquad \Phi(v_{k+1}) \leq \kappa_3 \Phi(v_k) \quad \text{for some} \quad \kappa_3 \in [0,1)$$

$\Longrightarrow$ **global linear** convergence

Also $\alpha_k^{\mathrm{L}} \geq \min\left\{1, \dfrac{\kappa_2}{[\Phi(v_k)]^2}\right\} \to 1$ and $\overline{\alpha}_k \geq \dfrac{1}{2\kappa_1^2 \Phi(v_k)} \to \infty$

$\Longrightarrow$ **asymptotic superlinear** convergence

Convergence analysis III

If $(x_0, s_0) > 0$ such that $Ax_0 = b$ and $A^T y_0 + s_0 = c \Longrightarrow$ at most

$$\frac{1}{q}\log\left(\frac{\Phi(v_0)}{\epsilon}\right)$$

iterations to find $\Phi(v_k) \leq \epsilon$, where

$$q = \min\left(\frac{1}{2}, \frac{\omega\mu}{2\Phi(v_0)}, \frac{(1-\omega)\omega\mu^2}{\Phi(v_0)^2}\right)$$

depends only on v_0, μ and ω

$\Longrightarrow$ **polynomial** complexity

Good numerical experience . . .

```
Iter    p-feas   d-feas com-slk   merit     step       mu    time
   0  3.9E+01 1.9E+01 1.6E+01  1.9E+03     -     1.0E+00  0.00
   1r 3.5E+01 1.7E+01 3.1E+01  1.7E+03 1.0E-01  1.0E+00  0.00
   2r 2.9E+01 1.4E+01 2.6E+01  1.4E+03 1.6E-01  1.0E+00  0.00
   3r 2.1E+01 1.0E+01 1.8E+01  9.9E+02 3.0E-01  1.0E+00  0.00
   4r 1.1E+01 5.2E+00 9.3E+00  5.2E+02 4.8E-01  1.0E+00  0.00

====================== feasible point found =================

   5r 1.4E-15 1.7E-14 9.8E-02  1.4E+01 1.0E+00  1.0E+00  0.00
   6r 1.8E-15 6.8E-15 4.8E-04  3.5E+00 7.5E-01  1.0E+00  0.00
   7r 1.8E-15 1.5E-15 2.1E-04  1.7E-01 1.0E-00  1.0E+00  0.00
   8r 3.6E-15 0.0E+00 5.5E-07  7.6E-04 1.0E+00  1.0E+00  0.00
```

. . . and bad numerical experience

```
Iter    p-feas  d-feas com-slk   merit    step      mu    time
   0  2.1E+03 4.1E+01 1.2E+02  7.8E+03     -    1.0E+00  0.00
   1r 2.1E+03 4.1E+01 1.9E+01  7.7E+03 1.0E-02  1.0E+00  0.00
.....................
 377r 2.1E-06 4.1E-08 7.9E-05  1.1E-02 5.5E-03  1.0E+00  0.44
 378r 2.1E-06 4.1E-08 7.9E-05  1.1E-02 5.4E-03  1.0E+00  0.44
```

. . . and bad numerical experience

```
Iter    p-feas   d-feas com-slk    merit     step       mu    time
   0  2.1E+03 4.1E+01 1.2E+02  7.8E+03      -     1.0E+00  0.00
   1r 2.1E+03 4.1E+01 1.9E+01  7.7E+03 1.0E-02  1.0E+00  0.00
.....................
 377r 2.1E-06 4.1E-08 7.9E-05  1.1E-02 5.5E-03  1.0E+00  0.44
 378r 2.1E-06 4.1E-08 7.9E-05  1.1E-02 5.4E-03  1.0E+00  0.44
```

Why? No strict interior! $x_i \to 0$ and $s_i \approx \mu / x_i \to \infty \Longrightarrow \alpha_k \not\to 1$

Controlled perturbations to cope with these difficulties

For $k = 1, 2, \ldots$,

- pick $(\theta_k^x, \theta_k^s) \geq 0$:

$$
\begin{aligned}
Ax - b &= 0 \\
A^T y + s - c &= 0 \\
\text{and} \quad (X + \Theta_k^x)(S + \Theta_k^s)e &= \mu e
\end{aligned}
$$

  has an interior solution (x_k, y_k, s_k)

Controlled perturbations to cope with these difficulties

For $k = 1, 2, \ldots$,

- pick $(\theta_k^x, \theta_k^s) \geq 0$:

$$\begin{aligned} Ax - b &= 0 \\ A^T y + s - c &= 0 \\ \text{and} \quad (X + \Theta_k^x)(S + \Theta_k^s)e &= \mu e \end{aligned}$$

  has an interior solution (x_k, y_k, s_k)
  - use previous algorithm to find (x_k, y_k, s_k)

Controlled perturbations to cope with these difficulties

For $k = 1, 2, \ldots,$

- pick $(\theta_k^x, \theta_k^s) \geq 0$:

$$\begin{aligned} Ax - b &= 0 \\ A^T y + s - c &= 0 \\ \text{and} \quad (X + \Theta_k^x)(S + \Theta_k^s)e &= \mu e \end{aligned}$$

  has an interior solution (x_k, y_k, s_k)
  - use previous algorithm to find (x_k, y_k, s_k)
- reduce $(\theta_k^x, \theta_k^s) \longrightarrow (\theta_{k+1}^x, \theta_{k+1}^s)$ via $(\zeta \in (0,1))$

$$(\theta_{k+1}^x)_i = \begin{cases} 0 & \text{if } (x_k)_i > 0 \\ (1-\zeta)(\theta_k^x)_i - \zeta (x_k)_i & \text{if } (x_k)_i \leq 0 \end{cases}$$

$$(\theta_{k+1}^s)_i = \begin{cases} 0 & \text{if } (s_k)_i > 0 \\ (1-\zeta)(\theta_k^s)_i - \zeta (s_k)_i & \text{if } (s_k)_i \leq 0 \end{cases}$$

Controlled perturbations to cope with these difficulties

For $k = 1, 2, \ldots,$

- pick $(\theta_k^x, \theta_k^s) \geq 0$:

$$\begin{aligned} Ax - b &= 0 \\ A^T y + s - c &= 0 \\ \text{and} \quad (X + \Theta_k^x)(S + \Theta_k^s)e &= \mu e \end{aligned}$$

  has an interior solution (x_k, y_k, s_k)
  - use previous algorithm to find (x_k, y_k, s_k)
- reduce $(\theta_k^x, \theta_k^s) \longrightarrow (\theta_{k+1}^x, \theta_{k+1}^s)$ via $(\zeta \in (0,1))$

$$(\theta_{k+1}^x)_i = \begin{cases} 0 & \text{if} \ \ (x_k)_i > 0 \\ (1-\zeta)(\theta_k^x)_i - \zeta(x_k)_i & \text{if} \ \ (x_k)_i \leq 0 \end{cases}$$

$$(\theta_{k+1}^s)_i = \begin{cases} 0 & \text{if} \ \ (s_k)_i > 0 \\ (1-\zeta)(\theta_k^s)_i - \zeta(s_k)_i & \text{if} \ \ (s_k)_i \leq 0 \end{cases}$$

  - $\Longrightarrow (x_k + \theta_{k+1}^x, s_k + \theta_{k+1}^s) > 0$

Details

- equivalent to enlarging and then systematically contracting the primal-dual feasible region

Details

- equivalent to enlarging and then systematically contracting the primal-dual feasible region
- initial (θ_1^x, θ_1^s)?
  - compute any (x_0, y_0, s_0):

$$Ax_0 = b \;\text{ and }\; A^T y_0 + s_0 = c$$

  - compute sufficiently large $(\theta_1^x, \theta_1^s) \geq 0$:

$$(x_0 + \theta_1^x, s_0 + \theta_1^s) > 0$$

Details

- equivalent to enlarging and then systematically contracting the primal-dual feasible region
- initial (θ_1^x, θ_1^s)?
  - compute any (x_0, y_0, s_0):

$$Ax_0 = b \ \text{ and } \ A^T y_0 + s_0 = c$$

  - compute sufficiently large $(\theta_1^x, \theta_1^s) \geq 0$:

$$(x_0 + \theta_1^x, s_0 + \theta_1^s) > 0$$

$\Longrightarrow$ all iterates satisfy

$$Ax_k = b \ \text{ and } \ A^T y_k + s_k = c$$

$\Longrightarrow$ polynomial complexity of each inner iteration

Theoretical results

- if there is a strictly feasible point, there will be a finite number $\sim 1/\mu$ of contractions before $(\theta_k^x, \theta_k^s) = 0$

Theoretical results

- if there is a strictly feasible point, there will be a finite number $\sim 1/\mu$ of contractions before $(\theta_k^x, \theta_k^s) = 0$
- if $\lim_{k\to\infty} \max_i (\theta_k^x)_i > 0$, there is no primal feasible point

Theoretical results

- if there is a strictly feasible point, there will be a finite number $\sim \mathbf{1}/\boldsymbol{\mu}$ of contractions before $(\boldsymbol{\theta_k^x}, \boldsymbol{\theta_k^s}) = \mathbf{0}$
- if $\lim_{k\to\infty} \max_i (\boldsymbol{\theta_k^x})_i > \mathbf{0}$, there is no primal feasible point
- if $\lim_{k\to\infty} \max_i (\boldsymbol{\theta_k^s})_i > \mathbf{0}$, there is no dual feasible point

Theoretical results

- if there is a strictly feasible point, there will be a finite number $\sim 1/\mu$ of contractions before $(\theta_k^x, \theta_k^s) = 0$
- if $\lim_{k\to\infty} \max_i (\theta_k^x)_i > 0$, there is no primal feasible point
- if $\lim_{k\to\infty} \max_i (\theta_k^s)_i > 0$, there is no dual feasible point
- if there is a primal feasible point, any index i for which
$$\lim_{k\to\infty} (x_k)_i \to 0 \ (\equiv \lim_{k\to\infty} (s_k)_i \to \infty)$$
$\Longleftrightarrow$ x_i is always zero — **implicit primal equality**
    - remove variable from problem

Theoretical results

- if there is a strictly feasible point, there will be a finite number $\sim 1/\mu$ of contractions before $(\theta_k^x, \theta_k^s) = 0$
- if $\lim_{k\to\infty} \max_i(\theta_k^x)_i > 0$, there is no primal feasible point
- if $\lim_{k\to\infty} \max_i(\theta_k^s)_i > 0$, there is no dual feasible point
- if there is a primal feasible point, any index i for which
  $$\lim_{k\to\infty}(x_k)_i \longrightarrow 0 \; (\equiv \lim_{k\to\infty}(s_k)_i \longrightarrow \infty)$$
  $\Longleftrightarrow$ x_i is always zero — **implicit primal equality**
  - remove variable from problem
- if there is a dual feasible point, any index i for which
  $$\lim_{k\to\infty}(s_k)_i \longrightarrow 0 \; (\equiv \lim_{k\to\infty}(x_k)_i \longrightarrow \infty)$$
  $\Longleftrightarrow$ s_i is always zero — **implicit dual equality**
  - remove bound on variable

Numerical experience — Netlib LP collection

problem	n	m	x_{imp}	s_{imp}	c_{imp}	y_{imp}
80BAU3B	9799	2262	10	6	38	4
ADLITTLE	97	56	1	0	0	0
AGG2	302	516	2	58	1	44
BCDOUT	5940	5414	64	0	543	0
FIT2P	13525	3000	0	0	0	0
GREENBEA	5405	2392	699	84	9	0
SCRS8	1169	490	32	14	6	17
STOCFOR3	15695	16675	212	4	0	0
WOODW	8405	1098	2511	0	0	0

n = # variables, m = # constraints
x_{imp} = # implied fixed variables, s_{imp} = # implied free variables
c_{imp} = # implied fixed constraints, y_{imp} = # implied free constraints

Conclusions

- simple method for finding a well-centered feasible point of strictly feasible region
- suitable for large-scale computation
- globally linearly and locally superlinearly convergent
- polynomial complexity
- controlled perturbations allow us to identify less favourable outcomes
- what is a good target value μ?
- easily extensible for more general problems (convex QP, horizontal complementarity)
- software package WCP coming as part of GALAHAD 2.0