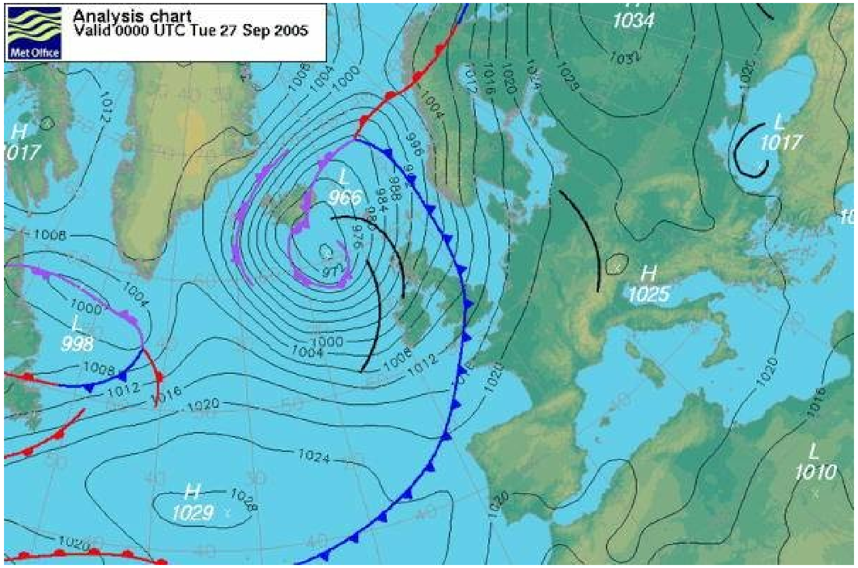
Using Model Reduction in Data Assimilation



Met Office Website

Nancy Nichols, Amos Lawless

The University of Reading

Caroline Boess, Angelika Bunse-Gerstner

Cerfacs & The University of Bremen

Outline

- Introduction to data assimilation
- 4D Variational assimilation (4DVar)
- Model reduction using balanced truncation
- Balanced truncation in incremental 4DVar
- Numerical experiments
- Conclusions

Initial data required (in grid boxes):

- Temperature
- Pressure
- Velocities (3)
- Density
- Humidity
- Chemical constituents
- Physical parameters

Picture Courtesy of Met Office © Crown Copyright

Initial data required (in grid boxes):

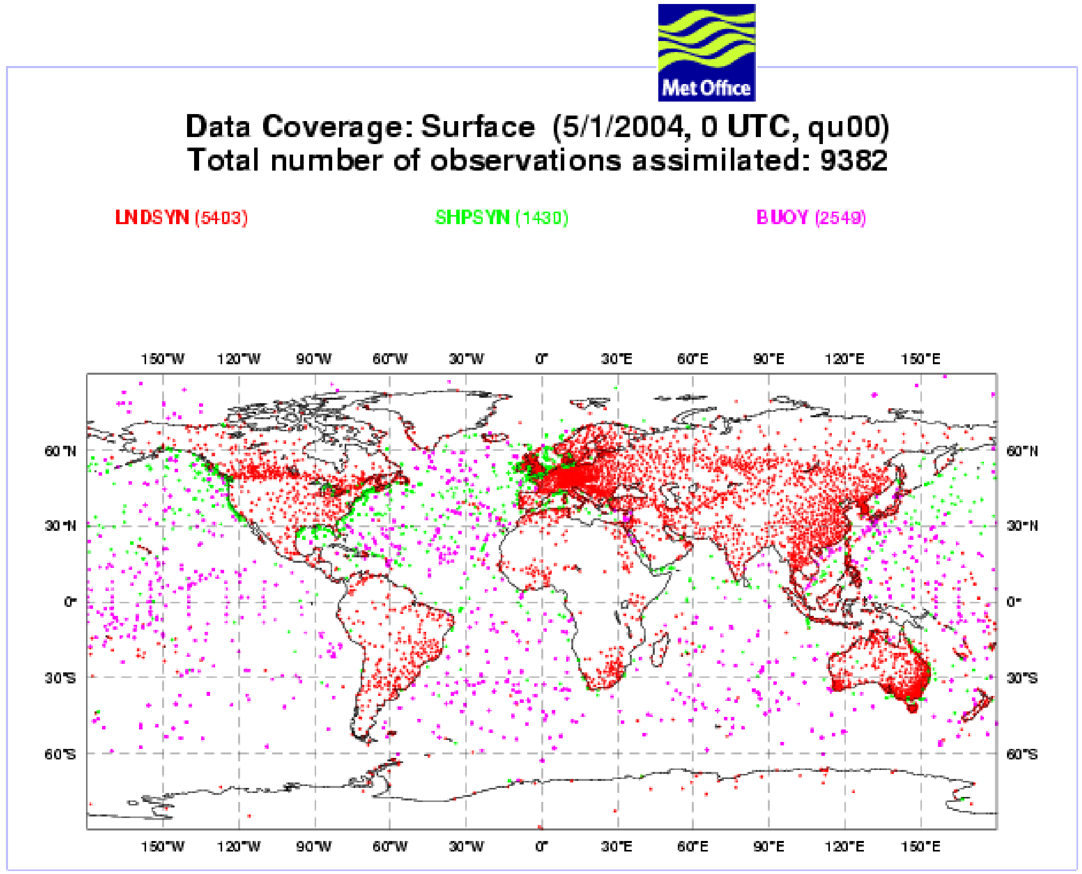
- Temperature
- Pressure
- Velocities (3)
- Density
- Humidity
- Chemical constituents
- Physical parameters

Picture Courtesy of Met Office © Crown Copyright

The global model at the UK Met Office has over 5 million boxes!

Available observations:

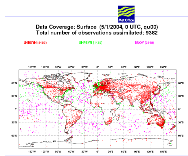
at the surface:



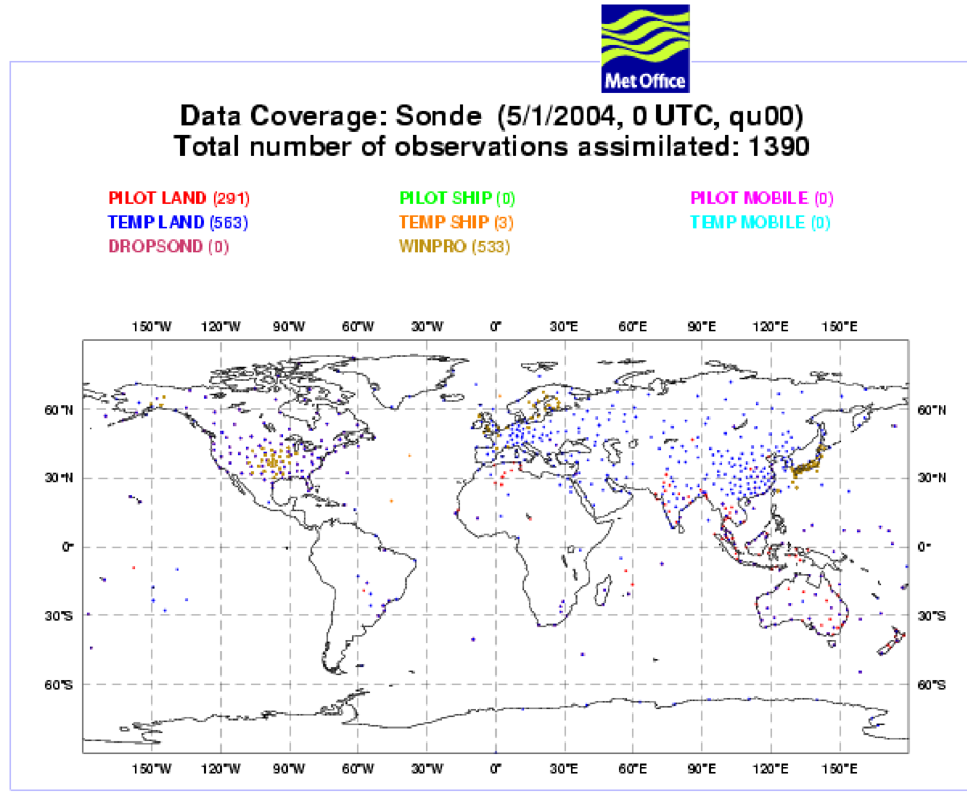
Picture Courtesy of Met Office © Crown Copyright

Available observations:

at the surface:



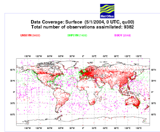
from radio-sondes:



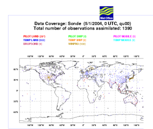
Picture Courtesy of Met Office © Crown Copyright

Available observations:

at the surface:

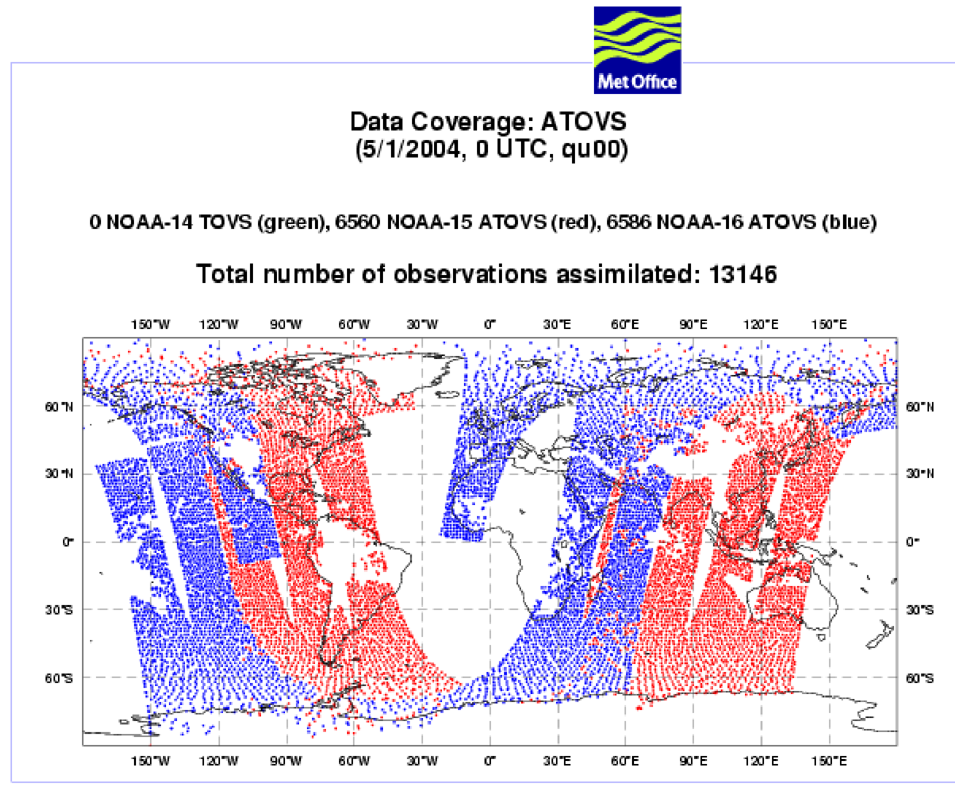


from radio-sondes:



from satellites:

Picture Courtesy of Met Office © Crc



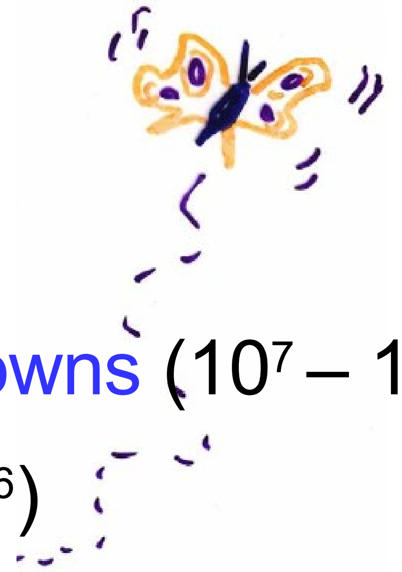
Data Assimilation

Aim:

Find the best estimate (analysis) of the true state of a system, consistent with both observations and the system dynamics given:

- Numerical prediction model
- Observations of the system (over time)
- Background state (first guess)
- Estimates of the errors

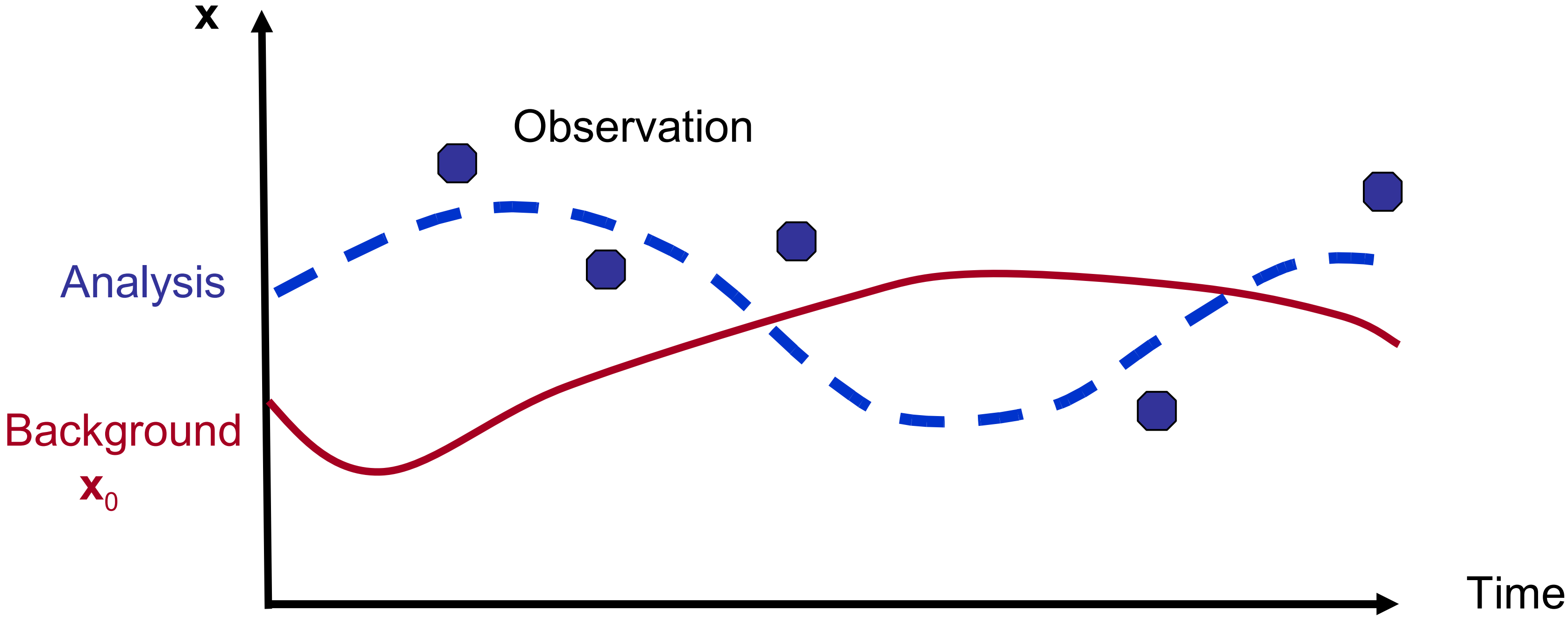
Significant Properties:



- Very large number of unknowns (10^7 – 10^8)
- Few observations (10^5 – 10^6)
- System nonlinear unstable/chaotic
- Multi-scale dynamics

Data Assimilation - 4DVar

Aim: Find the initial state x_0 such that the distance between the state trajectory and the observations is minimized, subject to x_0 remaining close to the prior estimate x_b .



4D-Var Nonlinear Problem

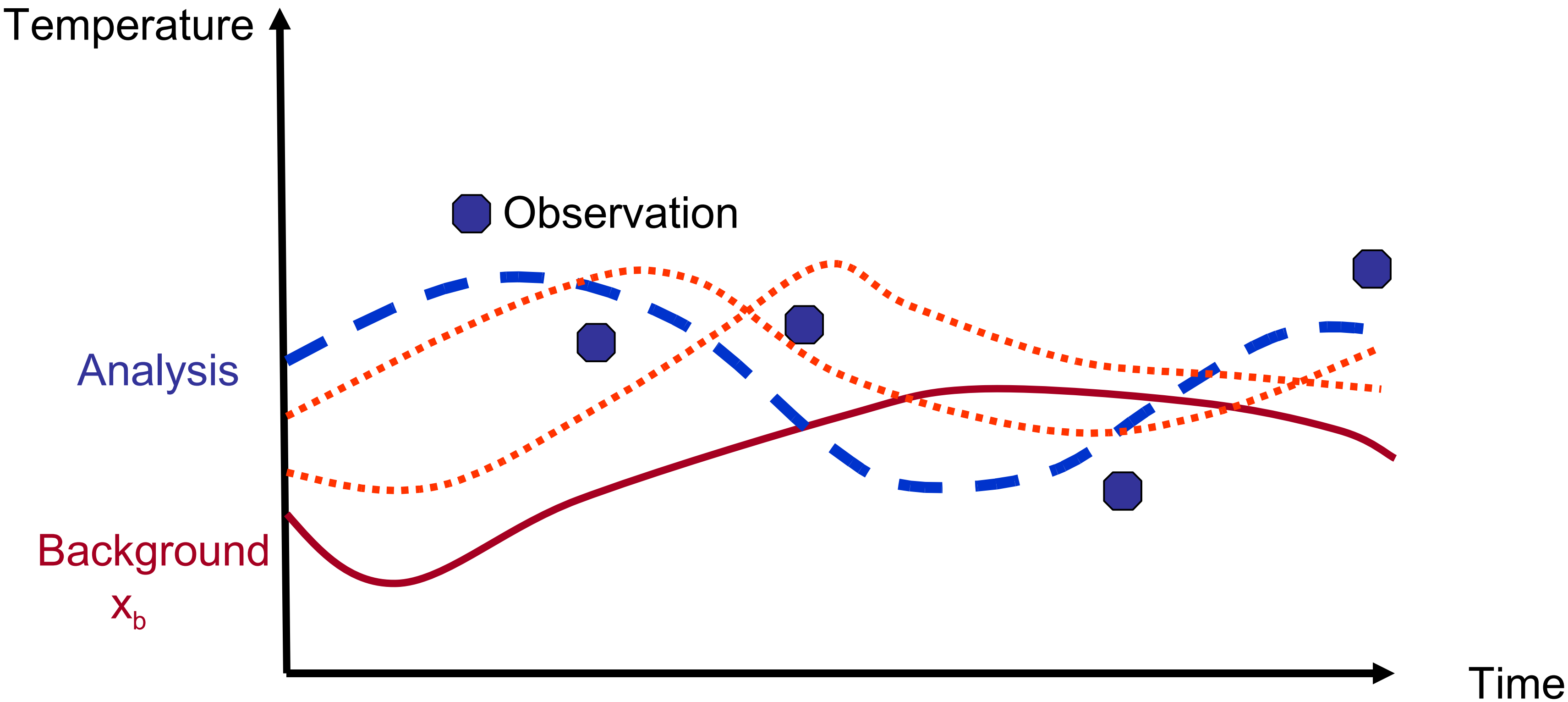
$$\min \; J(\mathbf{x}_0) = \frac{1}{2}(\mathbf{x}_0 - \mathbf{x}_b)^T \mathbf{B}_0^{-1}(\mathbf{x}_0 - \mathbf{x}_b)$$

$$+ \sum_{i=0}^{n} (H_i[\mathbf{x}_i] - \mathbf{y}_i)^T \mathbf{R}_i^{-1}(H_i[\mathbf{x}_i] - \mathbf{y}_i)$$

subject to $\mathbf{x}_i = S(t_i, t_0, \mathbf{x}_0)$

- $\mathbf{x}_b$ - Background state
- $\mathbf{y}_i$ - Observations
- H_i - Observation operator
- $\mathbf{B}$ - Background error covariance matrix
- $\mathbf{R}_i$ - Observation error covariance matrix

Incremental 4D-Var



Solve by iteration a sequence of linear least squares problems that approximate the nonlinear problem.

Incremental 4D-Var

Set $\mathbf{x}_0^{(0)}$ (usually equal to background)

For k = 0, …, K find: $\mathbf{x}_i^{(k)} = S(t_i, t_0, \mathbf{x}_0^{(k)})$

Solve inner loop minimization problem:

$$\begin{aligned}\tilde{\mathcal{J}}^{(k)}[\delta\mathbf{x}_0^{(k)}] &= \frac{1}{2}(\delta\mathbf{x}_0^{(k)} - [\mathbf{x}^b - \mathbf{x}_0{}^{(k)}])^{\mathrm{T}}\mathbf{B}_0^{-1}(\delta\mathbf{x}_0^{(k)} - [\mathbf{x}^b - \mathbf{x}_0{}^{(k)}]) \\ &+ \frac{1}{2}\sum_{i=0}^{N}(\mathbf{H}_i\delta\mathbf{x}_i^{(k)} - \mathbf{d}_i^{(k)})^{\mathrm{T}}\mathbf{R}_i^{-1}(\mathbf{H}_i\delta\mathbf{x}_i^{(k)} - \mathbf{d}_i^{(k)})\end{aligned}$$

with $\delta\mathbf{x}_i^{(k)} = \mathbf{L}(t_i, t_0, \mathbf{x}_0^{(k)})\delta\mathbf{x}_0^{(k)}$, $\mathbf{d}_i = \mathbf{y}_i - H_i[\mathbf{x}_i^{(k)}]$

Update: $\mathbf{x}_0^{(k+1)} = \mathbf{x}_0^{(k)} + \delta\mathbf{x}_0^{(k)}$

Previous Results

- Incremental 4D-Var without approximations is equivalent to a Gauss-Newton iteration for nonlinear least squares problems.
- In operational implementation we usually approximate the solution procedure:
  - Truncate inner loop iterations
  - Use approximate linear system model
- Theoretical convergence results obtained by reference to Gauss-Newton method.

New Research

Aims:

- Find approximate linear system models using optimal reduced order modeling techniques from control theory to improve the efficiency of the incremental 4DVar method.
- Test feasibility of approach in comparison with low resolution models using a simple shallow water flow model.

1D Shallow Water Model

Nonlinear continuous equations

$$\frac{\mathrm{D}u}{\mathrm{D}t} + \frac{\partial \varphi}{\partial x} = -g\frac{\partial \bar{h}}{\partial x}$$

$$\frac{\mathrm{D}(\ln \varphi)}{\mathrm{D}t} + \frac{\partial u}{\partial x} = 0$$

with

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

We discretize using a semi-implicit semi-Lagrangian scheme and linearize to get linear model (TLM).

Model Reduction via Oblique Projections

Given discrete-time linear system

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i, \\ y_i &= Cx_i \end{aligned}$$

where $A \in \mathbb{R}^{N \times N}$, $B \in \mathbb{R}^{N \times m}$, $C \in \mathbb{R}^{p \times N}$.
Task: Find projection matrices $U, V \in \mathbb{R}^{N \times l}$ with $U^T V = I_l$ and $l \ll N$ such that the reduced order system

$$\begin{aligned} \hat{x}_{i+1} &= U^T A V \hat{x}_i + U^T B u_i, \\ y_i &= C V \hat{x}_i \end{aligned}$$

and $x_i \approx V \hat{x}_i$ approximates the full order system.

Balanced Truncation

Balancing: Compute balancing state space transformation T with $x_i^{(B)} = T^{-1}x_i$ and get the transformed balanced system

$$\begin{aligned} x_{i+1}^{(B)} &= T^{-1}ATx_i^{(B)} + T^{-1}Bu_i, \\ y_i &= CTx_i^{(B)}. \end{aligned}$$

Truncating: Define truncation matrix $S = [I_l, 0]$. Then we get the reduced order system

$$\begin{aligned} \hat{x}_{i+1} &= ST^{-1}ATS^T\hat{x}_i + ST^{-1}Bu_i, \\ y_i &= CTS^T\hat{x}_i. \end{aligned}$$

By setting $U := T^{-T}S^T$, $V := TS^T$ you see that $U^TV = I_l$ and that balanced truncation is an oblique projection method.

Incremental 4DVar & Reduced Order Models

For initial tests, aim to minimize inner linear least square problem subject to the time-invariant linear system:

$$\begin{aligned} \delta x_{i+1} &= M\delta x_i, \\ d_{i+1} &= H\delta x_{i+1} \end{aligned}$$

with initial starting condition:

$$\delta x_0 = B_0^{\frac{1}{2}}\omega_0, \text{ with } \omega_0 \sim \mathcal{N}(0, I), \; \delta x_0 \sim \mathcal{N}(0, B_0)$$

Apply oblique projection to reduce order.

Projected Minimization Problem

The projected problem is to minimize:

$$\begin{aligned}\hat{\mathcal{J}}^{(k)}[\delta \hat{x}_0] &= \frac{1}{2}(\delta \hat{x}_0 - U^T[x^b - x_0])^T (U^T B_0 U)^{-1} (\delta \hat{x}_0 - U^T[x^b - x_0]) \\ &+ \frac{1}{2}\sum_{i=0}^{N}(HV\delta \hat{x}_i - d_i)^T R^{-1} (HV\delta \hat{x}_i - d_i)\end{aligned}$$

subject to:

$$\begin{aligned}\delta \hat{x}_{i+1} &= U^T M V \delta \hat{x}_i, \\ d_{i+1} &= HV\delta \hat{x}_{i+1}\end{aligned}$$

with $U, V \in \mathbb{R}^{N\times l}$, $U^T V = I_l$ and $\delta x_i \approx V\delta \hat{x}_i$.

Numerical Experiments -
Error Norms and Condition Numbers

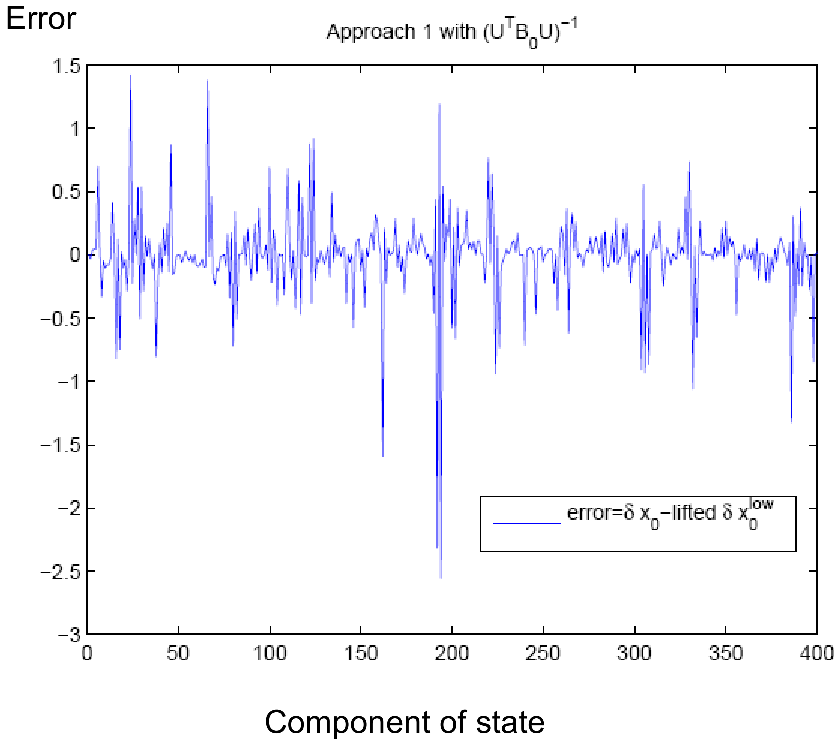
Test matrices:

$M \in \mathbb{R}^{400\times 400}$	from TLM model
$H \in \mathbb{R}^{200\times 400}$	observations at every other point
$B_0^{\frac{1}{2}} \in \mathbb{R}^{400\times 400}$	quite realistic test matrix

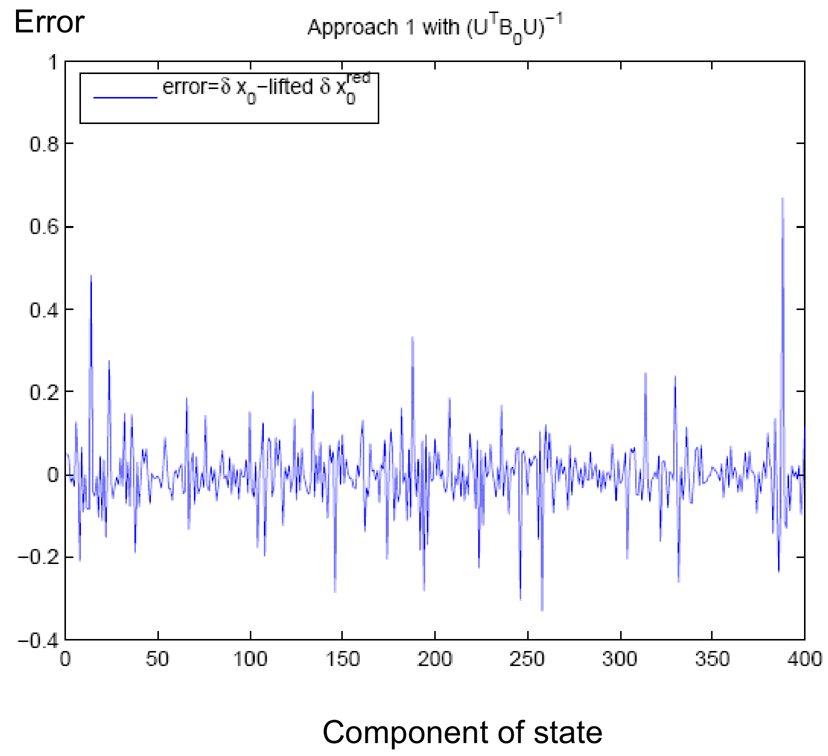
Error norm $nrm = \frac{\| \delta x_0 - \delta x_0^{(lift)} \|_2}{\| \delta x_0 \|_2}$, $\qquad \delta x_0^{(lift)} := V \delta \hat{x}_0$.

Error between exact and approximate analysis for 1-D SWE model

Low Resolution Model



Reduced Rank Model



Comparison of error norms and conditions numbers - Low resolution vs Reduced order models

	Error norm	Condition number
	low resolution	low resolution
l=200	0.6839	136.4230
	reduced order	reduced order
l=200	0.1728	9.6946
l=150	0.2713	7.5097
l=100	0.4281	6.3893
l=90	0.4638	6.3725
l=80	0.5342	6.3298
l=70	0.6357	6.2807

Comparison of error norms and conditions numbers - Low resolution vs Reduced order models

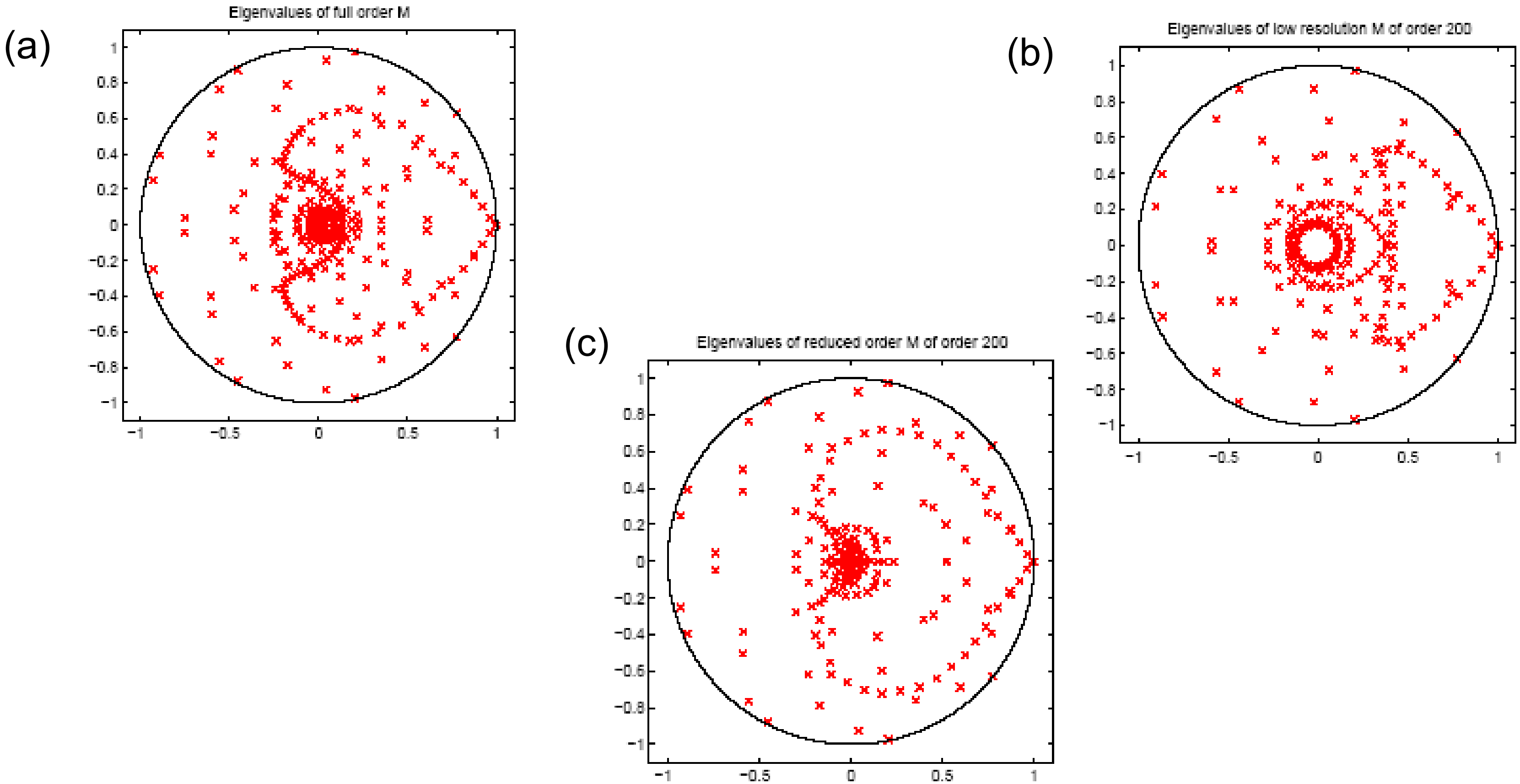
	Error norm	Condition number
	low resolution	low resolution
l=200	0.6839	136.4230
	reduced order	reduced order
l=200	0.1728	9.6946
l=150	0.2713	7.5097
l=100	0.4281	6.3893
l=90	0.4638	6.3725
l=80	0.5342	6.3298
l=70	0.6357	6.2807

Comparison of error norms and conditions numbers - Low resolution vs Reduced order models

	Error norm	Condition number
	low resolution	low resolution
l=200	0.6839	136.4230
	reduced order	reduced order
l=200	0.1728	9.6946
l=150	0.2713	7.5097
l=100	0.4281	6.3893
l=90	0.4638	6.3725
l=80	0.5342	6.3298
l=70	0.6357	6.2807

Comparison of error norms and conditions numbers - Low resolution vs Reduced order models

	Error norm	Condition number
	low resolution	low resolution
l=200	0.6839	136.4230
	reduced order	reduced order
l=200	0.1728	9.6946
l=150	0.2713	7.5097
l=100	0.4281	6.3893
l=90	0.4638	6.3725
l=80	0.5342	6.3298
l=70	0.6357	6.2807



Eigenvalues of (a) full, (b) low resolution (c) reduced rank system matrices

Conclusion

Main conclusion: reduced rank linear models obtained by optimal reduction techniques give more accurate analyses than low resolution linear models that are currently used in practice.