Numerically-aware Nested Dissection Ordering

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Introduction

Solve: $$Ax = b$$

A is:

- Sparse
- Large
- Symmetric
- **Indefinite** - may need pivoting

Sparse factorization phases

Traditionally:

1. Find fill-reducing ordering
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3. Perform numerical factorization

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2. Use matching from Hungarian algorithm to compress matrix
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Aim: replace steps 2-4 with something more intelligent.

	1	2	3	4	5	6
1	0.0	0.9			1.0	0.2
2	0.9	0.0			0.1	1.0
3			1.0	0.2	0.4	0.9
4			0.2	1.0	0.2	0.6
5	1.0	0.1	0.4	0.2	0.5	0.2
6	0.2	1.0	0.9	0.6	0.2	0.8

- Simple nested dissection structure

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$$
\begin{array}{c|cccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & 0.0 & 0.9 & & & \circled{1.0} & 0.2 \\
2 & 0.9 & 0.0 & & & 0.1 & \circled{1.0} \\
3 & & & \circled{1.0} & 0.2 & 0.4 & 0.9 \\
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6	0.2	0.2	1.0	0.8	0.9	0.6
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- Forced pivots
- More numerically stable/Less pivoting

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But wasn't that a "good enough" pivot?

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Let's think about this...

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- Only want to work with entries of A.
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Note: can ignore large a_{ij} if both a_{ii} and a_{jj} are large.

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Problem reduces to:

- For every red diagonal
- Ensure it can be paired with a green off-diagonal
- Without messing up fill-reducing order

WARNING!

"Ensure there is a large entry in column" is insufficient.

$$\begin{pmatrix} \color{red}{\mathbf{x}} & x & \color{green}{\mathbf{x}} \\ x & \color{red}{\mathbf{x}} & \color{green}{\mathbf{x}} \\ \color{green}{\mathbf{x}} & \color{green}{\mathbf{x}} & x \end{pmatrix}$$

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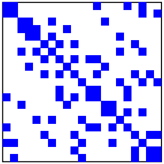
Big entries in row 3 can only be paired with a_{11} **or** a_{22}, not both!

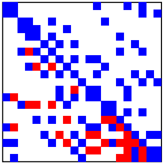
Need a matching!

Nested Dissection Primer



1. Original matrix

Nested Dissection Primer



1. Original matrix (with fill)

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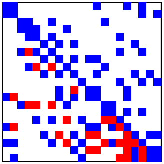

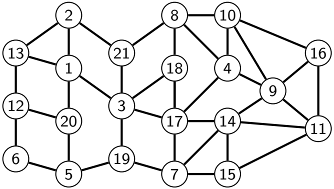

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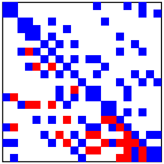

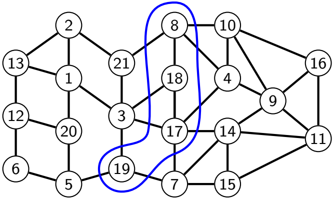

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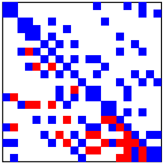
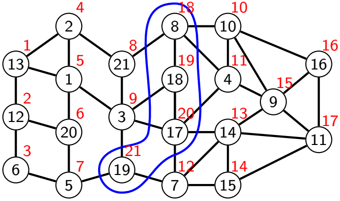
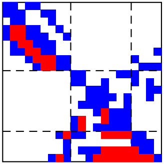




1. Original matrix (with fill)
2. Represent as graph
3. Find separator
4. Reorder to end



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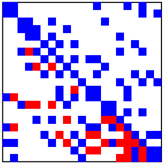

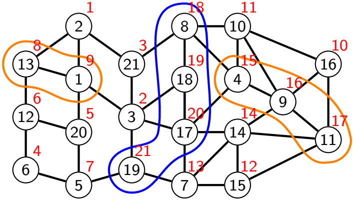

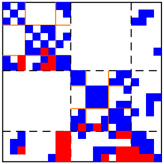

1. Original matrix (with fill)
2. Represent as graph
3. Find separator
4. Reorder to end
5. Recurse

Compatible matchings

Matching $\mathcal{M}$

- Set of pairs (i, j) such that i and j occur in a most one pair.
- Pairs represent 2×2 pivots, i.e. a_{ij} must be **large**.
- Need not have full cardinality.

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- $(i, j) \in \mathcal{M} \Rightarrow i$ and j in same subset.
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$\Rightarrow$ **Compatibility is exactly what we want**

Outline Algorithm

2. Find a partition $(\mathcal{B}, \mathcal{W}, \mathcal{S})$.

4. Order $\mathcal{S}$ to end.
5. Restrict matrix to $\mathcal{B}$ and recurse.
6. Restrict matrix to $\mathcal{W}$ and recurse.

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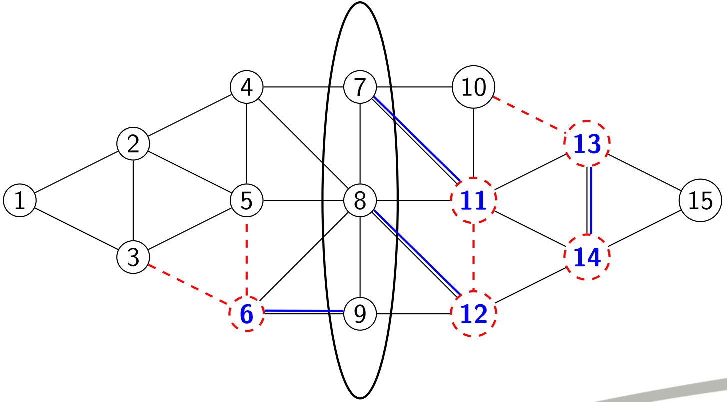
Outline Algorithm

1. Be supplied a matching $\mathcal{M}$.
2. Find a partition $(\mathcal{B}, \mathcal{W}, \mathcal{S})$.
3. Adjust $\mathcal{M}$ and $(\mathcal{B}, \mathcal{W}, \mathcal{S})$ to be compatible.
4. Order $\mathcal{S}$ to end.
5. Restrict matrix and matching to $\mathcal{B}$ and recurse.
6. Restrict matrix and matching to $\mathcal{W}$ and recurse.

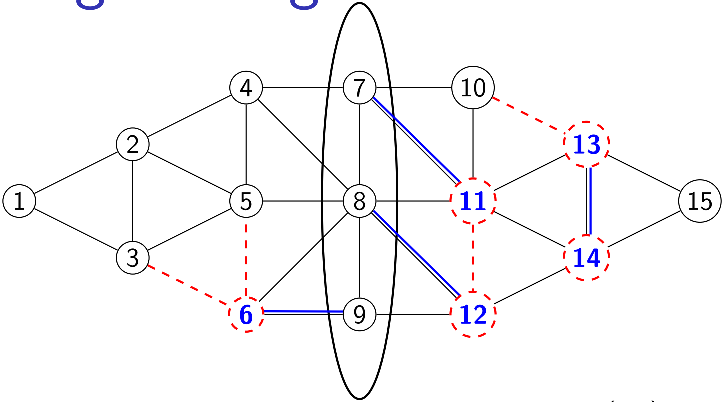
Moving to compatibility

Given incompatible $\mathcal{M}$ and $(\mathcal{B}, \mathcal{W}, \mathcal{S})$:

- Assume $\mathcal{M}$ is compatible on entire matrix.
- There must exist some edge $(i, j) \in \mathcal{M}$ cut by $\mathcal{S}$.



Fixing cut edges



- Find alternating path starting with cut edge (i, j).
- Finish in $\mathcal{S}$ with an edge in $\mathcal{M}$; or
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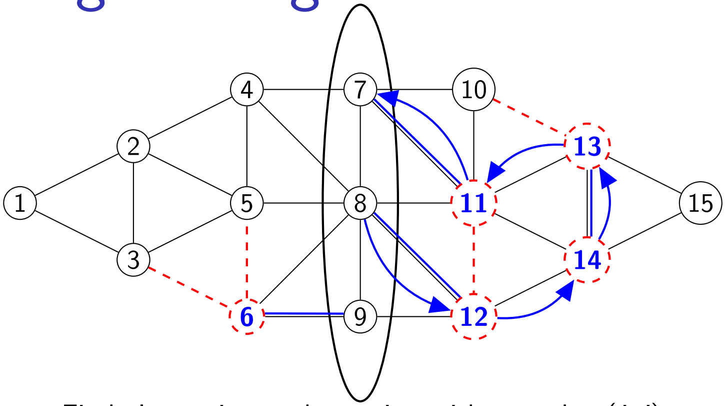

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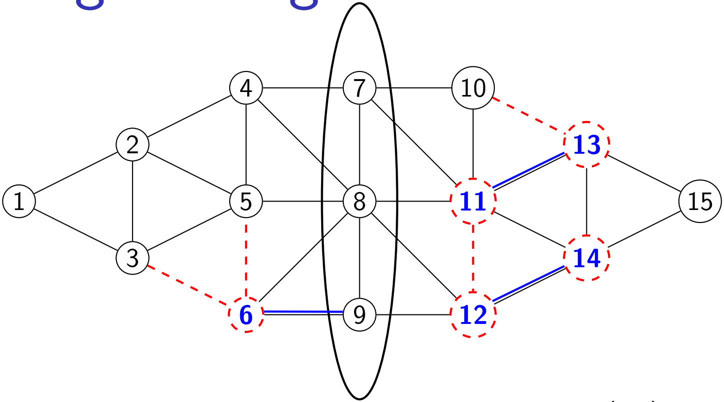

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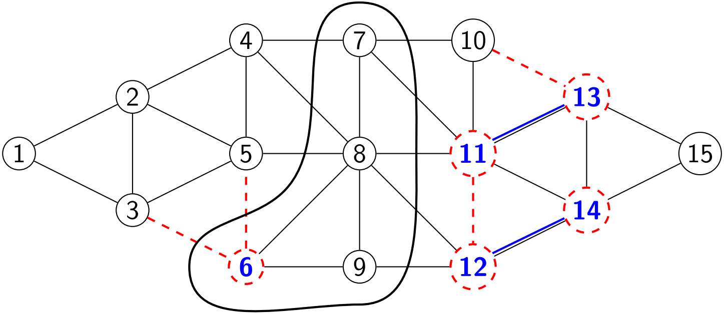

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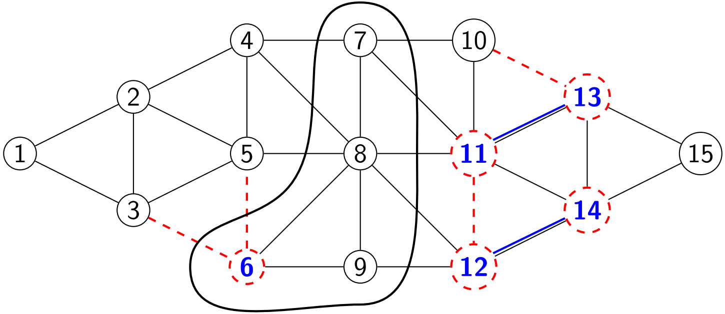
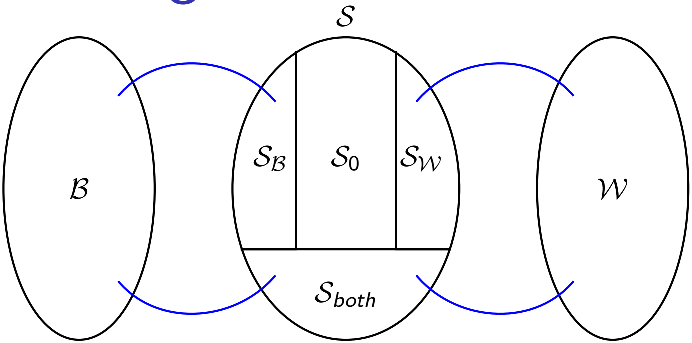


But now we don't need $9 \in \mathcal{S}$?

Trimming



- Try pulling nodes out of $\mathcal{S}_\mathcal{B}$ into $\mathcal{B}$.
- If they have a small diagonal, find a partner from $\mathcal{S}_\mathcal{B} \cup \mathcal{S}_0$.
- Same for $\mathcal{S}_\mathcal{W}$.
- Update sets and repeat until nothing moves.
- Can allow alternating paths in $\mathcal{B}$ and $\mathcal{W}$ too: complicated

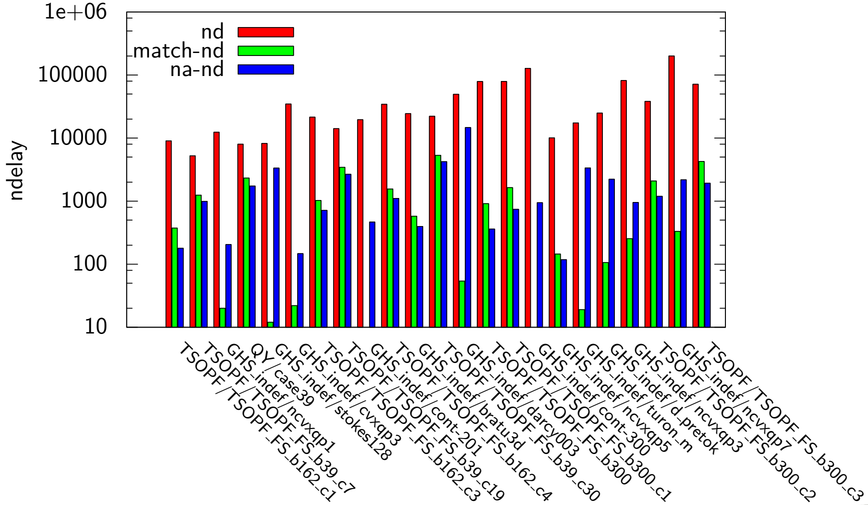
Numerical results

- 23 very difficult problems from UFL (just scaling doesn't work)
- Ordering with in-house Nested Dissection code
- Pre-scaled by symmetrized Hungarian scaling (MC64)
- Factorized with `HSL_MA97`
- Consider:

  ndelay — number of delayed pivots (i.e. how far from predicted order)

  fflop — number of floating point operation to calculate L

ndelay



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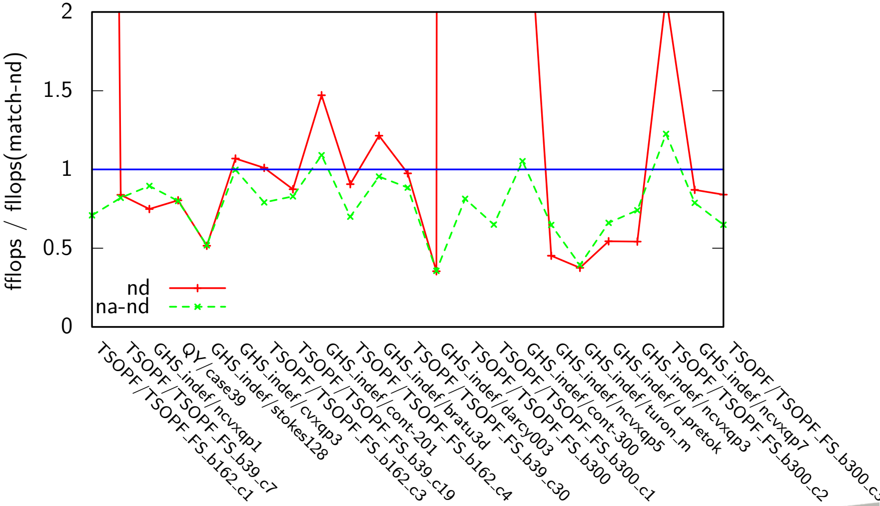
fflops



fflops / fflops(match-nd)
2
1.5
1
0.5
0
nd
na-nd
TSOPF/TSOPF_FS_b162_c1
TSOPF/TSOPF_FS_b39_c7
GHS_indef/ncvxqp1
QY/case39
GHS_indef/stokes128
GHS_indef/cvxqp3
TSOPF/TSOPF_FS_b162_c3
TSOPF/TSOPF_FS_b39_c19
GHS_indef/cont-201
TSOPF/TSOPF_FS_b162_c4
GHS_indef/bratu3d
TSOPF/TSOPF_FS_b39_c30
GHS_indef/darcy003
TSOPF/TSOPF_FS_b300
TSOPF/TSOPF_FS_b300_c1
GHS_indef/cont-300
GHS_indef/ncvxqp5
GHS_indef/turon_m
GHS_indef/d_pretok
GHS_indef/ncvxqp3
TSOPF/TSOPF_FS_b300_c2
GHS_indef/ncvxqp7

For AMD

Modifying Approximate Minimum Degree

- Approach by Duff and Pralet in 2005
- Relaxed version of “compress then order” approach
- Rather than glue pivots together, constrain them
- Only allow small pivots to be ordered after update by **big** off-diagonal.

Reasonable results obtained.
Unclear if used by any solvers in practice.

Summary

- Post processing to a dissection algorithm
- But part of a *nested* dissection algorithm
- Could be put into any ND code with a little work
- Effective at reducing delays without significantly increasing size/work for factors
- More details in technical report: RAL-P-2016-004

Thanks for listening!

Questions?

http://www.numerical.rl.ac.uk/spral

Funded by ESPRC grant EP/M025179/1

Say what now

An appendix?