Sparse Days 2016

Coarse Correction for Generalized Abstract Schwarz Solvers

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joint work with

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Context

Goal

- Solve $\mathcal{A}x = b$, where $\mathcal{A}$ is a large sparse SPD matrix, on a distributed platform

How?

- Use Domain Decomposition :
  - Geometric partition of a domain & Mathematical transformation of the PDE

Focus of the talk

- Coarse Correction for the MaPHyS solver
- Coarse Space for Generalized Abstract Schwarz methods
  - Need access to local matrices

Installing MaPHyS

- MaPHyS and its dependencies can be installed through `spack` in $\leq$ 15 minutes + coffee break

  morse.gforge.inria.fr/spack/spack.html

- From a laptop to an heterogeneous supercomputer

Using MaPHyS

- Documented test cases
- Centralized/Distributed input
- `CeCILL-C` license

Motivation: Coarse Correction for MaPHyS

MaPHyS: a Massively Parallel Hybrid Solver

- Nested Dissection to distribute the unknowns (Scotch/Metis)
- Local direct solve in each subdomain (PaStiX/MUMPS)
- Iterative solve on the global interface

A Domain Decomposition Method

- Additive Schwarz on the Schur (AS/S)

Need for Coarse Correction

- Good scalability of the direct part ☺
- The size and condition number of the iterative problem increases with the number of subdomains ☹

Example: 2D Test problem

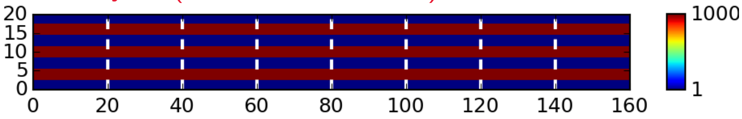
Heterogeneous diffusion

- $\nabla(K\nabla u) = q$
- 7 alternating conductivity layers
- Subdomain: 20×20 elements

Boundary conditions

- Dirichlet on the left
- Neumann elsewhere
- Source: $q = 1$

Conductivity K ($N = 8$ subdomains)



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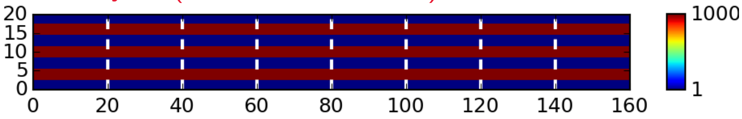
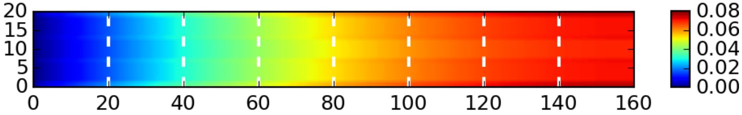
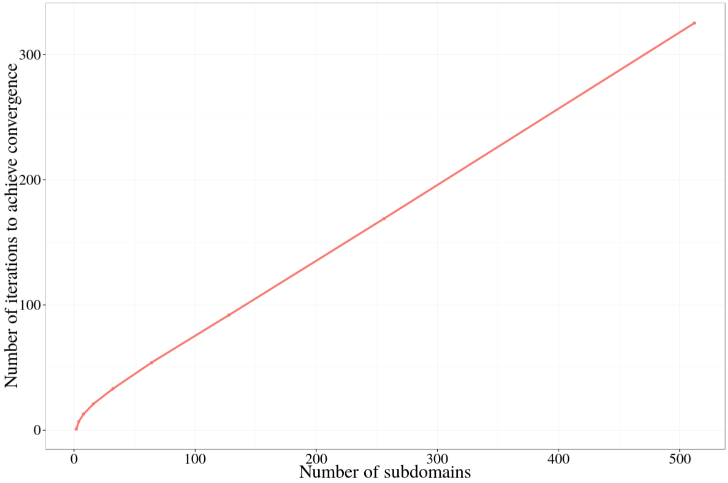


Solution x^* ($N = 8$ subdomains)



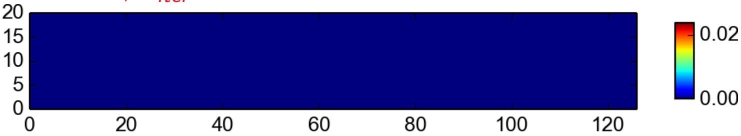
Weak Scalability



N	2	4	8	16	32	64	128	256	512
n_{iter}	1	7	13	21	33	54	92	169	325

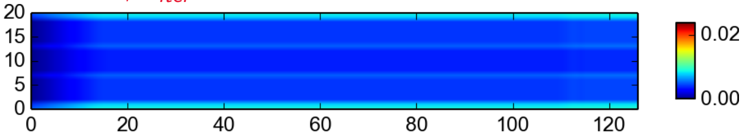
Convergence Behavior

x_Γ, $N = 128$, $n_{iter} = 0$



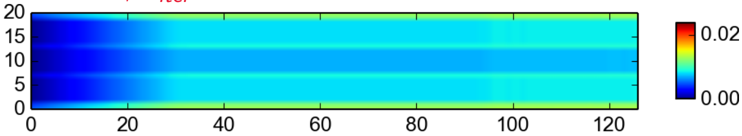
Convergence Behavior

x_Γ, $N = 128$, $n_{iter} = 10$



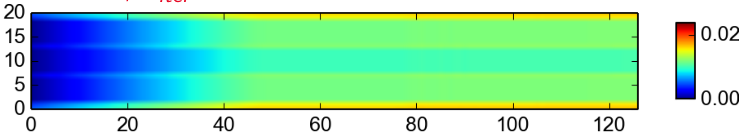
Convergence Behavior

x_Γ, $N = 128$, $n_{iter} = 20$



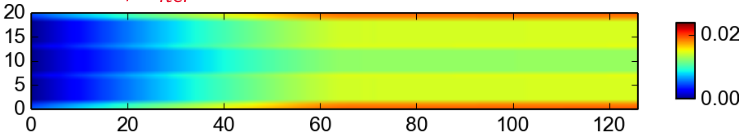
Convergence Behavior

x_Γ, $N = 128$, $n_{iter} = 30$



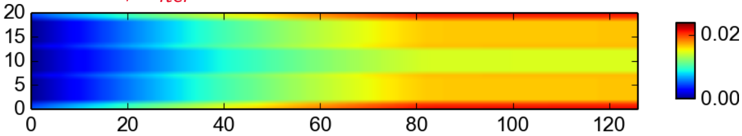
Convergence Behavior

x_Γ, $N = 128$, $n_{iter} = 40$



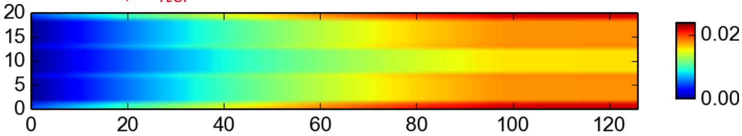
Convergence Behavior

x_Γ, $N = 128$, $n_{iter} = 50$



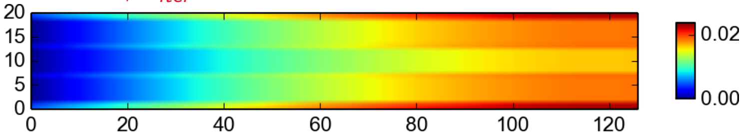
Convergence Behavior

x_Γ, $N = 128$, $n_{iter} = 60$



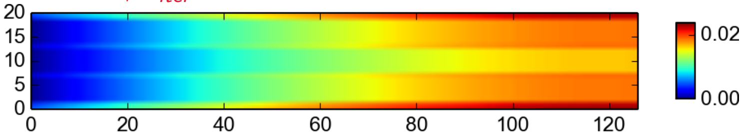
Convergence Behavior

x_Γ, $N = 128$, $n_{iter} = 70$



Convergence Behavior

x_Γ, $N = 128$, $n_{iter} = 70$

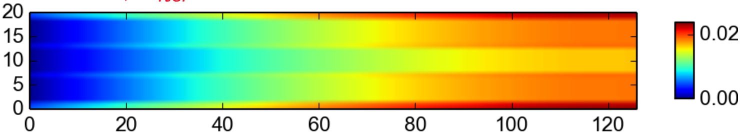


Problem

- No global exchange of information

Convergence Behavior

x_Γ, $N = 128$, $n_{iter} = 70$



Problem

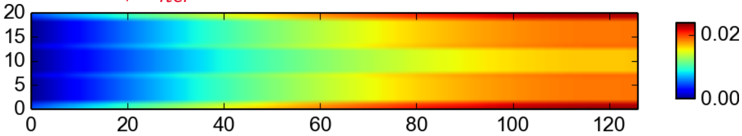
- No global exchange of information

Solution

- Use an exact direct solve on a coarse space V_0

Convergence Behavior

x_Γ, $N = 128$, $n_{iter} = 70$



Problem

- No global exchange of information

Solution

- Use an exact direct solve on a coarse space V_0

Contribution

- Coarse space for MaPHyS
  - but also for a wider class of methods
  - only in the SPD case

Outline

Outline

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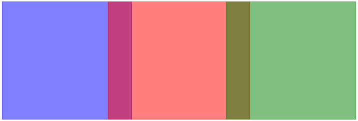
Geometric domain decomposition

Domain decomposition methods on boundary value problems

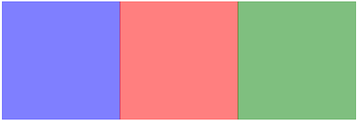
- Geometric partition of a domain
- Mathematical transformation of a PDE

With overlap



- Additive Schwarz method
- Multiplicative Schwarz method

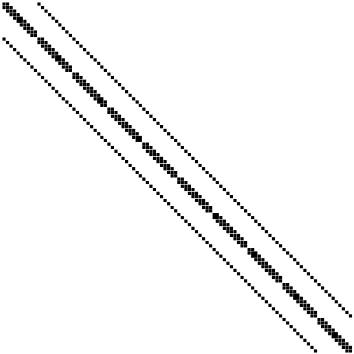
Without overlap



- Substructuring methods (Schur complement)
- Mortar methods (Lagrange multipliers)

Algebraic domain decomposition

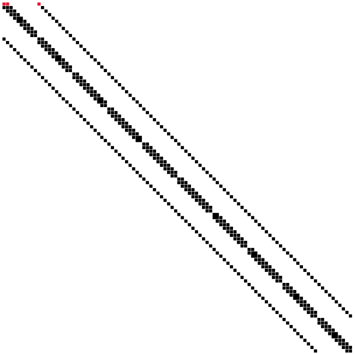
Global Matrix $\mathcal{A}$



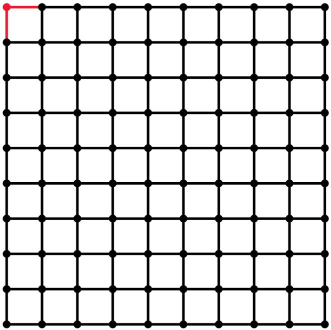
- $\mathcal{A}$ is a sparse matrix. We want to solve $\mathcal{A}x = b$.

Algebraic domain decomposition

Global Matrix $\mathcal{A}$



Adjacency graph G



- The adjacency graph of $\mathcal{A}$ $(n \times n)$ is used as an algebraic mesh:

$$G = (\{1, \ldots, n\}, \{(i,j), \ a_{ij} \neq 0 \mid a_{ji} \neq 0\})$$

- On the first row of $\mathcal{A}$, $a_{1,1}$, $a_{1,2}$ and $a_{1,11} \neq 0$
  $\Rightarrow (1,1)$, $(1,2)$ and $(1,11) \in G$

Algebraic domain decomposition

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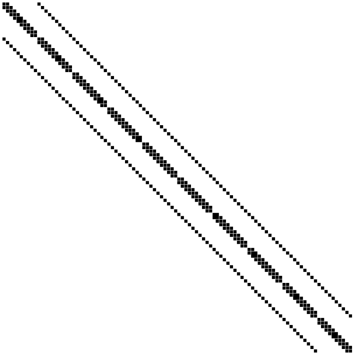

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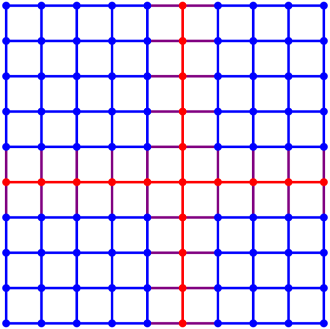
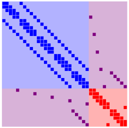


- A graph partitioner is used to split the graph

Algebraic domain decomposition

Local Matrices $\mathcal{A}_i$



$$\mathcal{A}_i = \begin{pmatrix} \mathcal{A}_{\mathcal{I}_i\mathcal{I}_i} & \mathcal{A}_{\mathcal{I}_i\Gamma_i} \\ \mathcal{A}_{\Gamma_i\mathcal{I}_i} & \mathcal{A}_{\Gamma_i\Gamma_i} \end{pmatrix}$$

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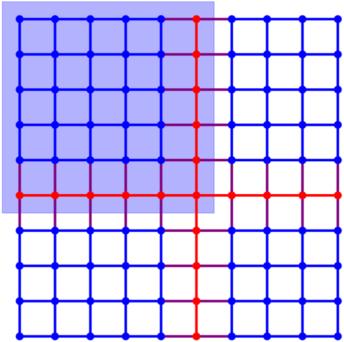

$$\mathcal{A} = \sum_{i=1}^{N} \mathcal{R}_i^T \mathcal{A}_i \mathcal{R}_i$$

Algebraic domain decomposition

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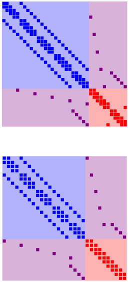

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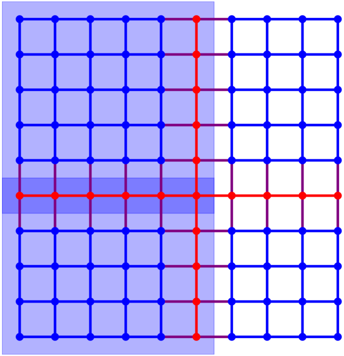

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- We have to split the interface non-zeros

Algebraic domain decomposition

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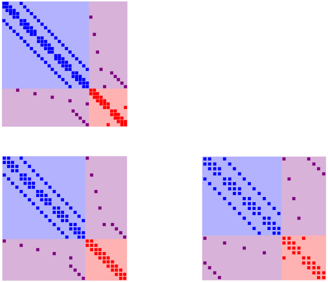

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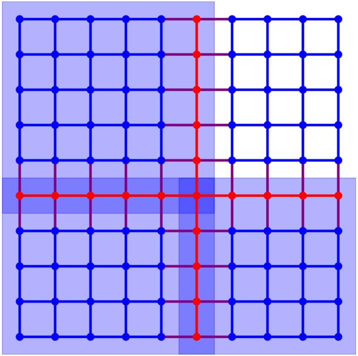

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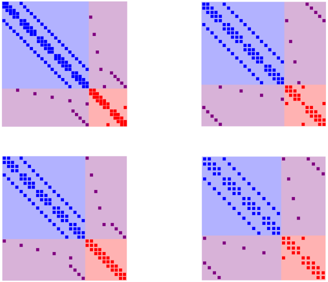

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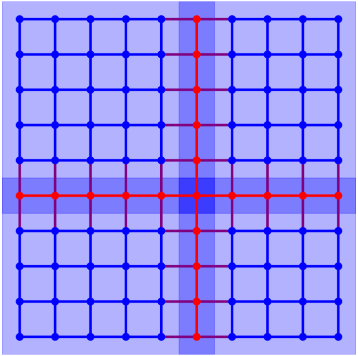

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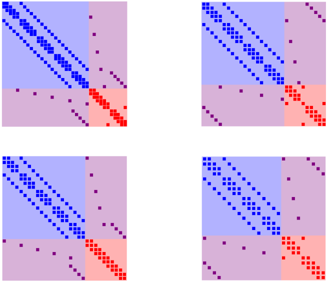
- We have to split the interface non-zeros

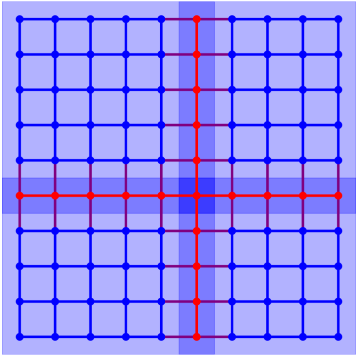
Outline

Abstract Schwarz Preconditioners

Local Preconditioner $\hat{\mathcal{A}}_i$



Adjacency graph G

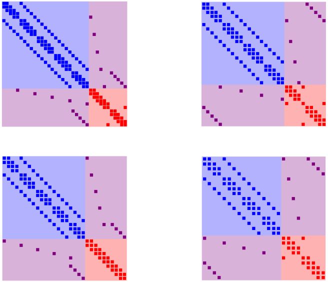


$$\mathcal{A} = \sum_{i=1}^{N} \mathcal{R}_i^T \mathcal{A}_i \mathcal{R}_i \qquad \mathcal{A}^{-1} \approx \mathcal{M} = \sum_{i=1}^{N} \mathcal{R}_i^T \hat{\mathcal{A}}_i^{\dagger} \mathcal{R}_i$$

$\hat{\mathcal{A}}_i$ is a local SPSD matrix *close to* $\mathcal{A}$ inside the subdomain.

Substructuring Methods

Local Matrix $\mathcal{A}_i$



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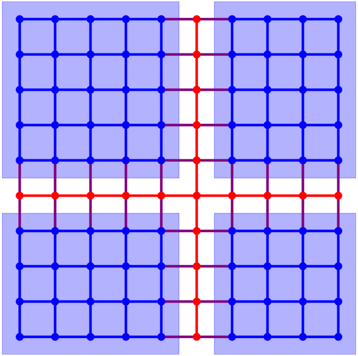
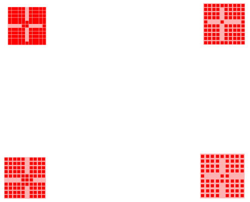


- We factorize $\mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}$ and compute $\mathcal{S}_i = \mathcal{A}_{\Gamma_i\Gamma_i} - \mathcal{A}_{\Gamma_i\mathcal{I}_i}\mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}^{-1}\mathcal{A}_{\mathcal{I}_i\Gamma_i}$

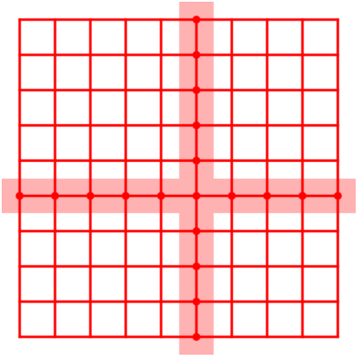
$$\mathcal{A}_i = \begin{pmatrix} \mathcal{A}_{\mathcal{I}_i\mathcal{I}_i} & \mathcal{A}_{\mathcal{I}_i\Gamma_i} \\ \mathcal{A}_{\Gamma_i\mathcal{I}_i} & \mathcal{A}_{\Gamma_i\Gamma_i} \end{pmatrix}$$

Substructuring Methods

Local Schur $\mathcal{S}_i$



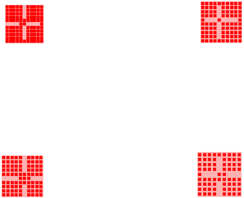
Adjacency graph G



- We factorize $\mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}$ and compute $\mathcal{S}_i = \mathcal{A}_{\Gamma_i\Gamma_i} - \mathcal{A}_{\Gamma_i\mathcal{I}_i}\mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}^{-1}\mathcal{A}_{\mathcal{I}_i\Gamma_i}$
- Now, on each subdomain, the whole local problem is condensed onto the interface (dense matrix)

Substructuring Methods

Local Schur $\mathcal{S}_i$



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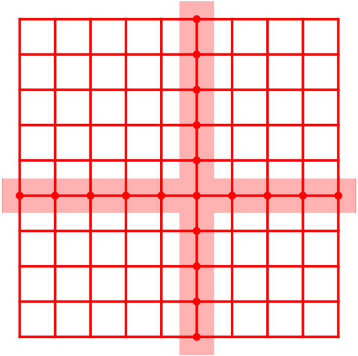

- We solve the interface problem $\mathcal{S}x_\Gamma = f = b_\Gamma - \mathcal{A}_{\Gamma\mathcal{I}}\mathcal{A}_{\mathcal{I}\mathcal{I}}^{-1}b_\mathcal{I}$ with a preconditioned Krylov method
  - $x_{\mathcal{I}_i} = \mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}^{-1}(b_{\mathcal{I}_i} - \mathcal{A}_{\mathcal{I}_i\Gamma_i}x_{\Gamma_i})$

Approximate Schur

Why use the Schur?

- $\mathcal{S}$ is smaller than $\mathcal{A}$ and better conditioned ☺

Problem

- $\mathcal{S}_i = \mathcal{A}_{\Gamma_i\Gamma_i} - \mathcal{A}_{\Gamma_i\mathcal{I}_i}\mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}^{-1}\mathcal{A}_{\mathcal{I}_i\Gamma_i}$ is dense
  - increased memory consumption ☹

Solution

- don't compute $\mathcal{S}_i$ explicitly & use sparsification/compression
- use an approximate Schur $\tilde{\mathcal{S}} = \sum_{i=1}^{N} \mathcal{R}_{\Gamma_i}^T \tilde{\mathcal{S}}_i \mathcal{R}_{\Gamma_i}$ where $\tilde{\mathcal{S}}_i$ is SPSD

$$\text{and} \quad \exists\, \omega_-, \omega_+ > 0, \qquad \forall v \in V \quad \omega_- \leq \frac{v^T \tilde{\mathcal{S}} v}{v^T \mathcal{S} v} \leq \omega_+.$$

Now, we search a preconditioner for $\tilde{\mathcal{S}}$ instead of $\mathcal{S}$.

Generalized Abstract Schwarz Preconditioners

Abstract Schwarz $\rightarrow$ use any local preconditioner $\hat{\mathcal{S}}_i$

- $\mathcal{M}_1 = \sum_{i=1}^{N} \mathcal{R}_{\Gamma_i}^T \hat{\mathcal{S}}_i^{\dagger} \mathcal{R}_{\Gamma_i}$
  - $\hat{\mathcal{S}}_i = \quad \cdots \quad$ should be SPSD
  - $\hat{\mathcal{S}}_i = D_i^{-1} \mathcal{S}_i D_i^{-1}$ for Neumann-Neumann (NN)
  - $\hat{\mathcal{S}}_i = \mathcal{R}_{\Gamma_i} \mathcal{S} \mathcal{R}_{\Gamma_i}^T$ for Additive Schwarz (AS)

Generalized Abstract Schwarz Preconditioners

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2-level Abstract Schwarz

2-level tools	
V_0	Coarse space
$\mathcal{M}_0 = V_0 (V_0^T \mathcal{S} V_0)^\dagger V_0^T$	Coarse solve
$\mathcal{P}_0 = \mathcal{M}_0 \mathcal{S}$	Projector onto the coarse space

- 2-level AS: $\mathcal{M}_D = \mathcal{M}_0 + (\mathcal{I} - \mathcal{P}_0) \mathcal{M}_1 (\mathcal{I} - \mathcal{P}_0)^T$

Generalized Abstract Schwarz Preconditioners

Abstract Schwarz $\rightarrow$ use any local preconditioner $\hat{\mathcal{S}}_i$

- $\mathcal{M}_1 = \sum_{i=1}^{N} \mathcal{R}_{\Gamma_i}^T \hat{\mathcal{S}}_i^{\dagger} \mathcal{R}_{\Gamma_i}$
  - $\hat{\mathcal{S}}_i = \quad \cdots$ should be SPSD
  - $\hat{\mathcal{S}}_i = D_i^{-1} \tilde{\mathcal{S}}_i D_i^{-1}$ for Neumann-Neumann (NN)
  - $\hat{\mathcal{S}}_i = \mathcal{R}_{\Gamma_i} \tilde{\mathcal{S}} \mathcal{R}_{\Gamma_i}^T$ for Additive Schwarz (AS)

Generalized $\rightarrow$ use $\tilde{\mathcal{S}}$ instead of $\mathcal{S}$

2-level tools	
V_0	Coarse space
$\tilde{\mathcal{M}}_0 = V_0 (V_0^T \tilde{\mathcal{S}} V_0)^{\dagger} V_0^T$	Coarse solve
$\tilde{\mathcal{P}}_0 = \tilde{\mathcal{M}}_0 \tilde{\mathcal{S}}$	Projector onto the coarse space

- 2-level GAS: $\tilde{\mathcal{M}}_D = \tilde{\mathcal{M}}_0 + (\mathcal{I} - \tilde{\mathcal{P}}_0) \mathcal{M}_1 (\mathcal{I} - \tilde{\mathcal{P}}_0)^T$

Common GAS Preconditioners

Generalized Neumann-Neumann Preconditioner

[Le Tallec & Vidrascu 98]

$$\tilde{\mathcal{M}}_{NN} = \tilde{\mathcal{M}}_0 + (\mathcal{I} - \tilde{\mathcal{P}}_0) \sum_{i=1}^{N} \mathcal{R}_{\Gamma_i}^T \left(D_i^{-1} \tilde{\mathcal{S}}_i D_i^{-1}\right)^{\dagger} \mathcal{R}_{\Gamma_i} (\mathcal{I} - \tilde{\mathcal{P}}_0)^T$$

$$\lambda_{\min}(\tilde{\mathcal{M}}_{NN}\mathcal{S}) \geq \frac{1}{\omega_+}$$

Generalized Additive Schwarz Preconditioner [MaPHyS]

$$\tilde{\mathcal{M}}_{AS} = \tilde{\mathcal{M}}_0 + (\mathcal{I} - \tilde{\mathcal{P}}_0) \sum_{i=1}^{N} \mathcal{R}_{\Gamma_i}^T \left(\mathcal{R}_{\Gamma_i} \tilde{\mathcal{S}} \mathcal{R}_{\Gamma_i}^T\right)^{-1} \mathcal{R}_{\Gamma_i} (\mathcal{I} - \tilde{\mathcal{P}}_0)^T$$

$$\lambda_{\max}(\tilde{\mathcal{M}}_{AS}\mathcal{S}) \leq \frac{\max_{i \leq N} (N_i + 1)}{\omega_-} = \frac{N_c}{\omega_-}$$

$N_i = \#\{j \neq i,\ \mathcal{R}_{\Gamma_i} \tilde{\mathcal{S}} \mathcal{R}_{\Gamma_j}^T \neq 0\}$ is the number of neighbors of subdomain i.

Outline

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Convergence Theorem for GAS

Theorem

$$\kappa(\tilde{\mathcal{M}}_D \mathcal{S}) \leq \frac{\omega_+}{\omega_-} \left(1 + \max_{i \leq N} \sup_{v \in \tilde{V}_i^\perp} \frac{|v|^2_{\hat{\mathcal{S}}_i}}{|v|^2_{D_i^{-1} \tilde{\mathcal{S}}_i D_i^{-1}}} \right) \max_{i \leq N} (N_i + 1) \sup_{v \in \hat{V}_i^\perp} \frac{|v|^2_{\mathcal{R}_{\Gamma_i} \tilde{\mathcal{S}} \mathcal{R}_{\Gamma_i}^T}}{|v|^2_{\hat{\mathcal{S}}_i}}$$

where $\tilde{V}_i^\perp = \left(D_i^{-1} \tilde{\mathcal{S}}_i D_i^{-1} V_i^0 \right)^\perp$, $\hat{V}_i^\perp = \left(\hat{\mathcal{S}}_i V_i^0 \right)^\perp$, $V_0 = \sum_{i=1}^N \mathcal{R}_{\Gamma_i}^T V_i^0$ and

$$\tilde{\mathcal{M}}_D = \tilde{\mathcal{M}}_0 + (\mathcal{I} - \tilde{\mathcal{P}}_0) \sum_{i=1}^N \mathcal{R}_{\Gamma_i}^T \hat{\mathcal{S}}_i^\dagger \mathcal{R}_{\Gamma_i} (\mathcal{I} - \tilde{\mathcal{P}}_0)^T$$

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Convergence Theorem for GAS

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- and to the Additive Schwarz preconditioner.

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Outline of the proof

$$\kappa(\tilde{\mathcal{M}}_D\mathcal{S}) = \frac{\lambda_{\mathsf{max}(\tilde{\mathcal{M}}_D\mathcal{S})}}{\lambda_{\mathsf{min}(\tilde{\mathcal{M}}_D\mathcal{S})}}$$

Outline of the proof

$$\kappa(\tilde{\mathcal{M}}_D \mathcal{S}) = \frac{\lambda_{\max(\tilde{\mathcal{M}}_D \mathcal{S})}}{\lambda_{\min(\tilde{\mathcal{M}}_D \mathcal{S})}}$$

$$\lambda_{\min}(\tilde{\mathcal{M}}_{NN} \mathcal{S}) \geq \frac{1}{\omega_+}$$

$$\hat{\mathcal{S}}_i = D_i^{-1} \tilde{\mathcal{S}}_i D_i^{-1}$$

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Outline of the proof

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The (few) vectors where $\hat{\mathcal{S}}_i$ is far worse than AS or NN should be in the coarse space V_0.

Outline

Extending GenEO [Spillane 2012] for GAS

$$\kappa(\tilde{\mathcal{M}}_D\mathcal{S}) \leq \frac{\omega_+}{\omega_-}\left(1 + \max_{i\leq N} \sup_{v\in\tilde{V}_i^\perp} \frac{|v|^2_{\hat{\mathcal{S}}_i}}{|v|^2_{D_i^{-1}\tilde{\mathcal{S}}_i D_i^{-1}}}\right)\max_{i\leq N}(N_i+1)\sup_{v\in\hat{V}_i^\perp}\frac{|v|^2_{\mathcal{R}_{\Gamma_i}\tilde{\mathcal{S}}\mathcal{R}_{\Gamma_i}^T}}{|v|^2_{\hat{\mathcal{S}}_i}}$$

1. Choose two thresholds $\alpha > 0$ and $\beta \geq 1$.

Extending GenEO [Spillane 2012] for GAS

$$\kappa(\tilde{\mathcal{M}}_D\mathcal{S}) \leq \frac{\omega_+}{\omega_-}\left(1 + \max_{i\leq N} \sup_{v\in\tilde{V}_i^\perp} \frac{|v|^2_{\hat{\mathcal{S}}_i}}{|v|^2_{D_i^{-1}\tilde{\mathcal{S}}_i D_i^{-1}}}\right) \max_{i\leq N} (N_i+1) \sup_{v\in\hat{V}_i^\perp} \frac{|v|^2_{\mathcal{R}_{\Gamma_i}\tilde{\mathcal{S}}\mathcal{R}_{\Gamma_i}^T}}{|v|^2_{\hat{\mathcal{S}}_i}}$$

1. Choose two thresholds $\alpha > 0$ and $\beta \geq 1$.
2. Solve locally the generalized eigenproblems

$$D_i^{-1}\tilde{\mathcal{S}}_i D_i^{-1} p = \lambda\, \hat{\mathcal{S}}_i p \qquad \text{and} \qquad \hat{\mathcal{S}}_i p = \eta\, \mathcal{R}_{\Gamma_i}\tilde{\mathcal{S}}\mathcal{R}_{\Gamma_i}^T p$$

for eigenvalues $\lambda \leq \alpha^{-1}$ and $\eta \leq (N_i+1)\beta^{-1}$.

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$$\kappa(\tilde{\mathcal{M}}_D \mathcal{S}) \leq \frac{\omega_+}{\omega_-} \left(1 + \max_{i \leq N} \sup_{v \in \tilde{V}_i^\perp} \frac{|v|^2_{\hat{\mathcal{S}}_i}}{|v|^2_{D_i^{-1} \tilde{\mathcal{S}}_i D_i^{-1}}} \right) \max_{i \leq N} (N_i + 1) \sup_{v \in \hat{V}_i^\perp} \frac{|v|^2_{\mathcal{R}_{\Gamma_i} \tilde{\mathcal{S}} \mathcal{R}_{\Gamma_i}^T}}{|v|^2_{\hat{\mathcal{S}}_i}}$$

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3. Assemble the resulting coarse space $V_0 = \sum_{i=1}^N \mathcal{R}_{\Gamma_i}^T V_i^0$.

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3. Assemble the resulting coarse space $V_0 = \sum_{i=1}^N \mathcal{R}_{\Gamma_i}^T V_i^0$.

Then:

$$\kappa(\tilde{\mathcal{M}}_D\mathcal{S}) \leq \frac{\omega_+}{\omega_-}(1+\alpha)\,\beta.$$

Outline

Convergence Theorem for AS,2

- In the Additive Schwarz case, we don't always need the projection step:

$$\tilde{\mathcal{M}}_{AS,2} = \tilde{\mathcal{M}}_0 + \sum_{i=1}^{N} \mathcal{R}_{\Gamma_i}^T \left(\mathcal{R}_{\Gamma_i} \tilde{\mathcal{S}} \mathcal{R}_{\Gamma_i}^T\right)^{-1} \mathcal{R}_{\Gamma_i}$$

$$\kappa(\tilde{\mathcal{M}}_{AS,2}\mathcal{S}) \leq \frac{\omega_+}{\omega_-} \left[N_c + 1 + \alpha(N_c + 2)\right] (N_c + 1)$$

where $N_c = \max N_i + 1$.

- The bound on the condition number is bigger (N_c^2 instead of N_c) ☹
- Each iteration is cheaper (one less matrix-vector product) ☺
- Possibility to overlap some computation and communications ☺

Step 1: Domain Decomposition

- $\mathcal{A} = \sum_{i=1}^{N} \mathcal{R}_i^T \mathcal{A}_i \mathcal{R}_i$

Step 2: Factorization

- Computation of $\mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}^{-1}$ and $\mathcal{S}_i = \mathcal{A}_{\Gamma_i\Gamma_i} - \mathcal{A}_{\Gamma_i\mathcal{I}_i}\mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}^{-1}\mathcal{A}_{\mathcal{I}_i\Gamma_i}$

Step 3: Preconditioner Setup

- $\mathcal{M}_{AS} = \sum_{i=1}^{N} \mathcal{R}_{\Gamma_i}^T \left(\mathcal{R}_{\Gamma_i}\mathcal{S}\mathcal{R}_{\Gamma_i}^T\right)^{-1} \mathcal{R}_{\Gamma_i}$

Step 4: Solve

- on Γ: *Krylov method* $\mathcal{S}\, x_\Gamma = f$ preconditioned with $\mathcal{M}_{AS}$
- on $\mathcal{I}$: *Direct method* $x_{\mathcal{I}_i} = \mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}^{-1}\left(b_{\mathcal{I}_i} - \mathcal{A}_{\mathcal{I}_i\Gamma_i}x_{\Gamma_i}\right)$

AS, 2 Step by step

Step 1: Domain Decomposition (Application level)

- $\mathcal{A} = \sum_{i=1}^{N} \mathcal{R}_i^T \mathcal{A}_i \mathcal{R}_i$

Step 2: Factorization

- Computation of $\mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}^{-1}$ and $\mathcal{S}_i = \mathcal{A}_{\Gamma_i\Gamma_i} - \mathcal{A}_{\Gamma_i\mathcal{I}_i}\mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}^{-1}\mathcal{A}_{\mathcal{I}_i\Gamma_i}$

Step 3: Preconditioner Setup

- $\mathcal{M}_{AS,2} = \mathcal{M}_0 + \sum_{i=1}^{N} \mathcal{R}_{\Gamma_i}^T \left(\mathcal{R}_{\Gamma_i}\mathcal{S}\mathcal{R}_{\Gamma_i}^T\right)^{-1} \mathcal{R}_{\Gamma_i}$

Step 4: Solve

- on Γ: *Krylov method* $\mathcal{S}\, x_\Gamma = f$ preconditioned with $\mathcal{M}_{AS,2}$
- on $\mathcal{I}$: *Direct method* $x_{\mathcal{I}_i} = \mathcal{A}_{\mathcal{I}_i\mathcal{I}_i}^{-1}\left(b_{\mathcal{I}_i} - \mathcal{A}_{\mathcal{I}_i\Gamma_i} x_{\Gamma_i}\right)$

Outline

3D Test problem

Heterogeneous diffusion

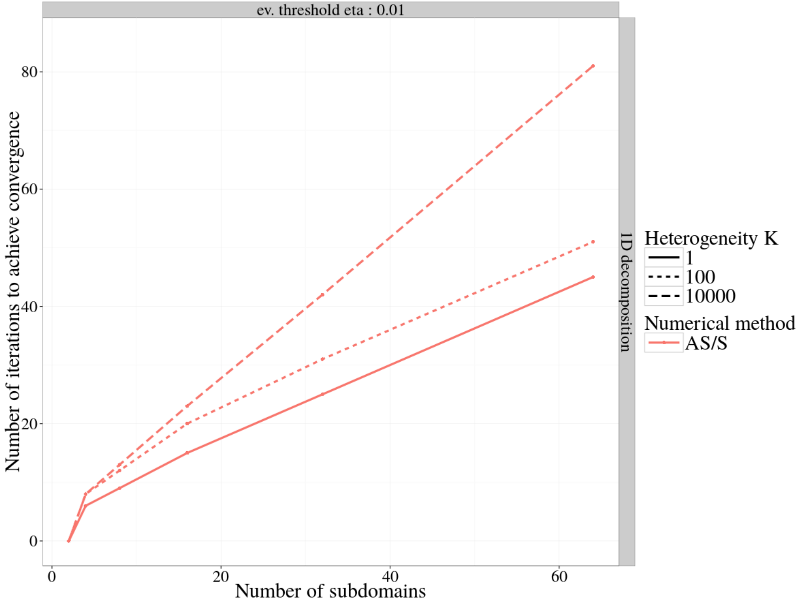
- $\nabla(K\nabla u) = 1$
- Alternating conductivity layers of 3 elements (ratio K between layers)
- Dirichlet on the left, Neumann elsewhere

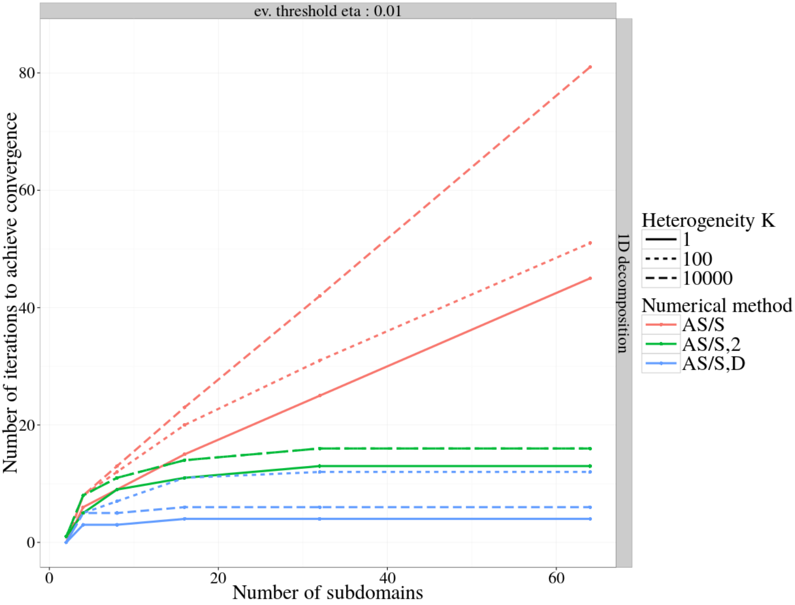
Domain decomposition

- $N \times 1 \times 1$ (1D decomposition)
- $N/2 \times 2 \times 1$ (2D decomposition)
- Constant subdomain size: $10 \times 10 \times 10$ elements

Implementation

- python/MPI


ev. threshold eta : 0.01
1D decomposition
Number of iterations to achieve convergence
Number of subdomains
0
20
40
60
80
Heterogeneity K
1
100
10000
Numerical method
AS/S


ev. threshold eta : 0.01
1D decomposition
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0
20
40
60
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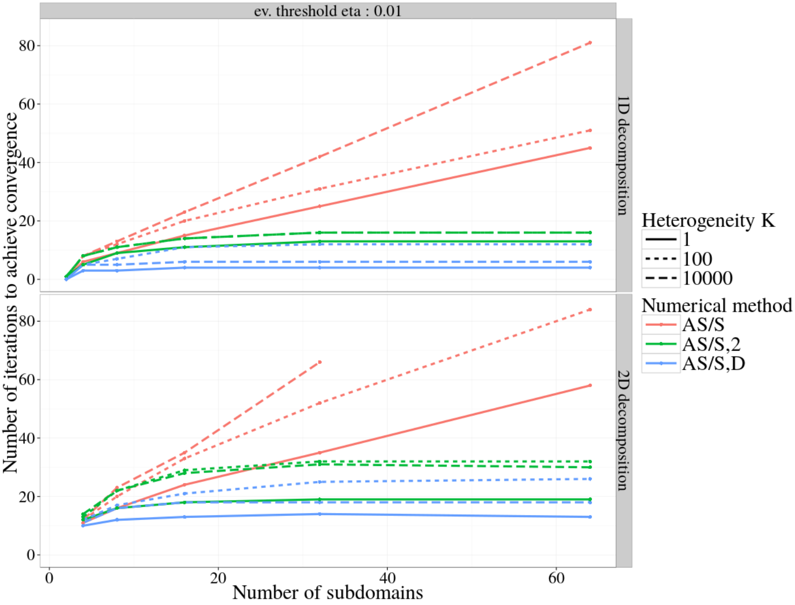
1
100
10000
Numerical method
AS/S
AS/S,2
AS/S,D



ev. threshold eta : 0.01
Number of iterations to achieve convergence
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0
20
40
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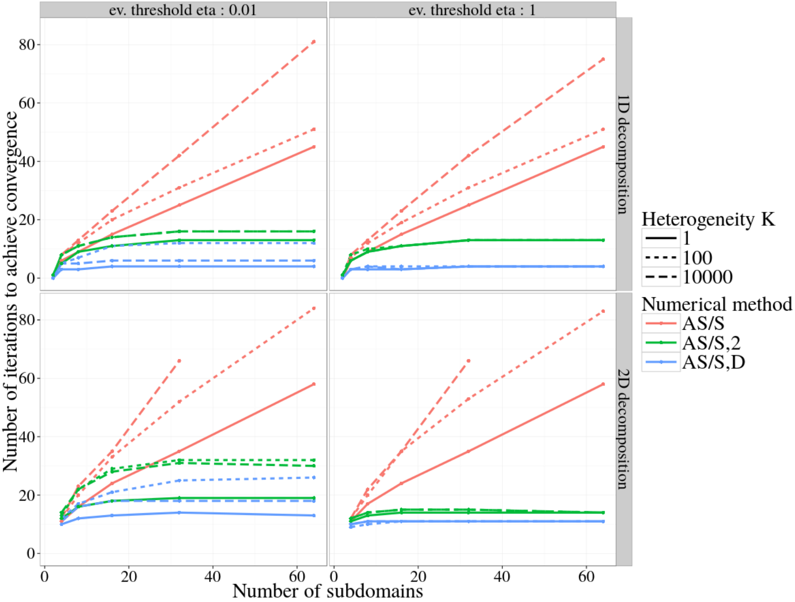
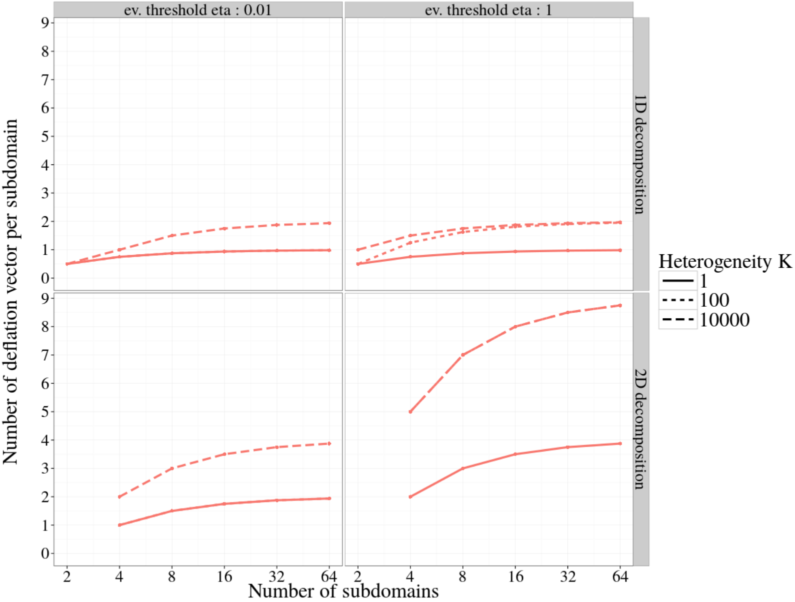
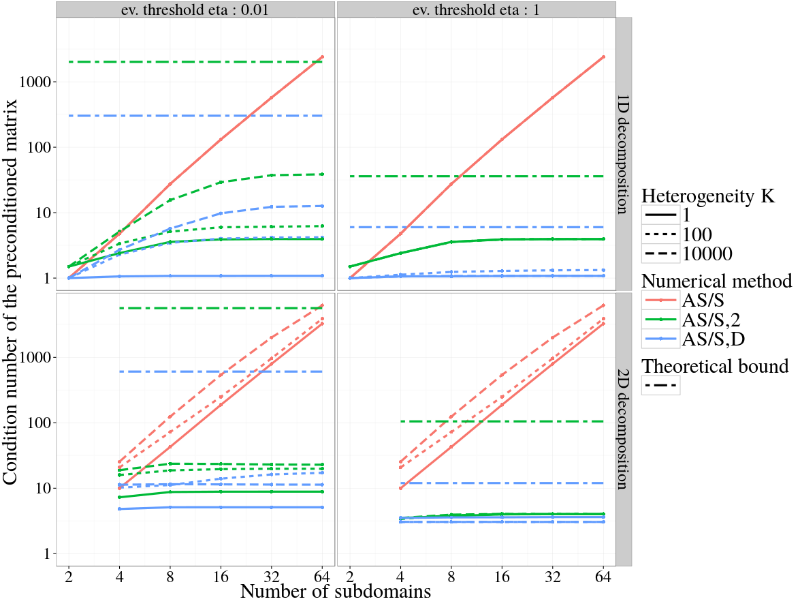
1D decomposition
2D decomposition
Heterogeneity K
1
100
10000
Numerical method
AS/S
AS/S,2
AS/S,D

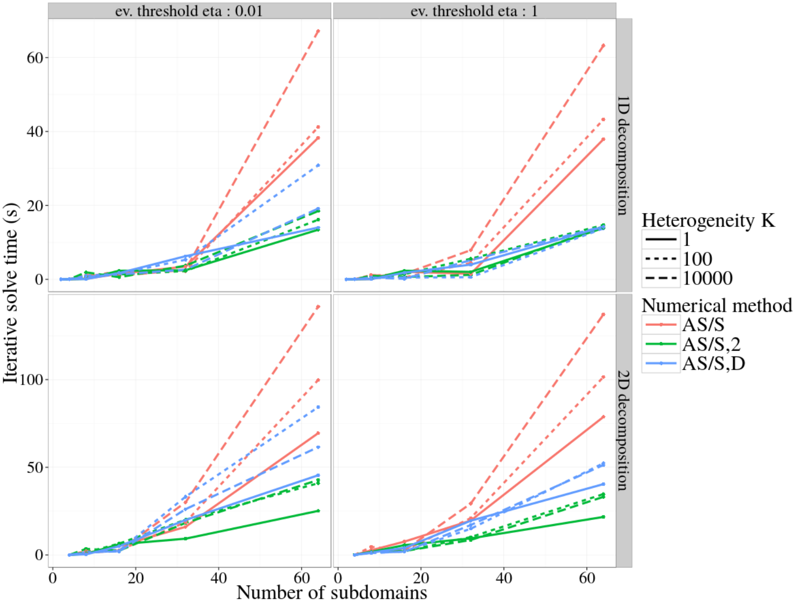


- The number of iterations is stabilized independently of K and N



- More difficult problems require a bigger coarse space


ev. threshold eta : 0.01
ev. threshold eta : 1
1D decomposition
2D decomposition
Condition number of the preconditioned matrix
1000
100
10
1
2
4
8
16
32
64
Number of subdomains
Heterogeneity K
1
100
10000
Numerical method
AS/S
AS/S,2
AS/S,D
Theoretical bound


ev. threshold eta : 0.01
ev. threshold eta : 1
1D decomposition
2D decomposition
Iterative solve time (s)
Number of subdomains
Heterogeneity K
1
100
10000
Numerical method
AS/S
AS/S,2
AS/S,D

Perspectives

GenEO in MaPHyS

- Loosening the assumptions ($\mathcal{A}_i$ SPSD and $\mathcal{A}$ SPD)
- Implementation and test of the 2-level preconditioner on real applications

Other recent/ongoing efforts in MaPHyS

- Partioning/balancing both interface and interior vertices (A. Casadei)
- Parallel analysis and dist. sub. API (M. Kuhn)
- $\mathcal{H}$-arithmetic for local solve ($\mathcal{H}$-PaStiX) and preconditioner (A. Falco, G. Pichon, Y. Harness)
- Numerical resilience policies (M. Zounon)
- Task-based implementation (S. Nakov)

Thanks for your attention!

Questions?

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