Dynamics-based reduction of data assimilation for chaotic models

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1. Premises
   - Motivation
   - Lyapunov vectors and exponents in a nutshell
2. Data assimilation performance in chaotic dynamics
   - Linear perfect dynamics
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   - Out of linearly behaving errors
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Context and Motivation

- The EnKF proves success in high-dimensional system with *only few* members.
- The example below system of dimension $O(10^6)$ and 10^2 members

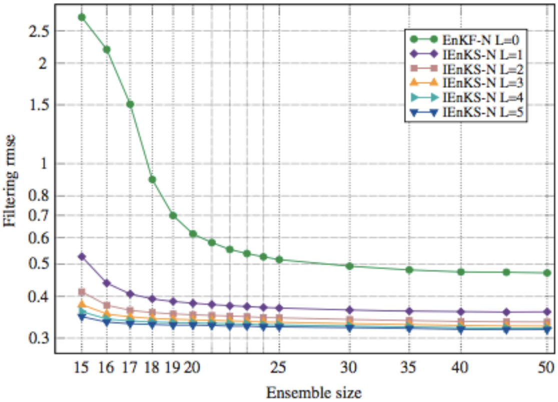
Example of EnKF in high-dimensional system.
TOPAZ ocean-sea ice EnKF integrated system.

In the figure sea ice edge: analysis (black), observations (blue) and 2-days forecast (gray).

Courtesy of Laurent Bertino (NERSC)

Context and Motivation

Example of performance (RMSE) of the EnKF and EnKS (IEnKS in the figure) as a function of the ensemble size in perfect dynamics (L95 model).



Bocquet and Sakov, 2014

Context and Motivation

▶ Several numerical results suggest that the skills of ensemble-based data assimilation methods in chaotic systems are related to the instabilities of the underlying dynamics [Ng et al., 2011].

▶ Numerical evidence that some asymptotic properties of the ensemble-based covariances (rank, range) relate to the unstable modes of the dynamics [Sakov and Oke, 2008; Carrassi et al., 2009].

▶ EnKF-like methods work well in high-dimensional system with only "very few" members (< 100).

Motivation

Some of questions behind this work:

1. (General...) Can DA methods be designed taking into account the dynamical properties of the model?
2. Is the uncertainty in the state estimate driven by the instabilities?
3. Why only a limited number of members is sufficient? (*Caution*: I will not mention localization in this talk!)
4. ...

Lyapunov vectors and exponents

- State and (infinitesimal) error dynamics:

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathscr{M}_t(\mathbf{x}(t)), \qquad \frac{\mathrm{d}\mathbf{e}(t)}{\mathrm{d}t} = \mathbf{M}_{\mathbf{x}(t),t}\mathbf{e}(t). \tag{1}$$

The time integration of the linear error dynamics yields the resolvent:

$$\mathbf{e}(t_1) = \mathbf{M}(t_1, t_0)\mathbf{e}(t_0). \tag{2}$$

- The Oseledec theorem tells that the limiting matrix (far future)

$$\mathbf{S}(t_0) = \lim_{t_1\to\infty} \left\{\mathbf{M}(t_1, t_0)^{\mathrm{T}}\mathbf{M}(t_1, t_0)\right\}^{\frac{1}{2(t_1-t_0)}}. \tag{3}$$

exists, has eigenvalues $e^{\lambda_1} \geq e^{\lambda_2} \geq \ldots \geq e^{\lambda_n}$ where the λ_i are called the Lyapunov exponents that do not depend on t_0, and has eigenvectors that are called the forward Lyapunov vectors which depend on t_0. Symmetrically (far past)

$$\mathbf{S}(t_1) = \lim_{t_0\to-\infty} \left\{\mathbf{M}(t_1, t_0)\mathbf{M}(t_1, t_0)^{\mathrm{T}}\right\}^{\frac{1}{2(t_1-t_0)}}. \tag{4}$$

exists, has the same eigenvalues that do not depend on t_1, and eigenvectors that are called the backward Lyapunov vectors (which depend on t_1).

Error dynamics and Lyapunov vectors/exponents

- The forward and backward Lyapunov vectors are orthonormal. They are norm-dependent and are not covariant under the effect of the dynamics (their subspaces do).

- The positive Lyapunov exponents correspond to exponentially growing error/unstable modes. The negative Lyapunov exponents correspond to exponentially decaying error/stable modes. The zero Lyapunov exponents correspond to neutral modes.

- The Lyapunov vectors generate a sequence of embedded subspaces of $\mathbb{R}^n$ for each t_1 such that

$$F_1^-(t_1) \subset F_2^-(t_1) \subset \cdots \subset F_n^-(t_1) = \mathbb{R}^n \tag{5}$$

where for $\mathbf{e} \in F_i^-(t_1) \backslash F_{i-1}^-(t_1)$, $\|\mathbf{M}^{-1}(t_1, t_0)\mathbf{e}\| \underset{t_0 \to -\infty}{\sim} e^{-\lambda_i(t_1 - t_0)}\|\mathbf{e}\|$; $\mathbf{e}$ are the covariant Lyapunov vectors.

- We define the unstable-neutral subspace $\mathscr{U}_{t_1} \equiv F_{n_0}^-(t_1)$ as the space generated by the n_0 backward Lyapunov vectors that are related to positive and zero Lyapunov exponents. Here, the stable subspace is defined as the orthogonal $\mathscr{U}_{t_1}^\perp$ of $\mathscr{U}_{t_1}$ in $\mathbb{R}^n$.

- See [Legras and Vautard, 1996; Kuptsov and Parlitz, 2012] for a topical introduction.

Laypunov exponents and Attractor dimension

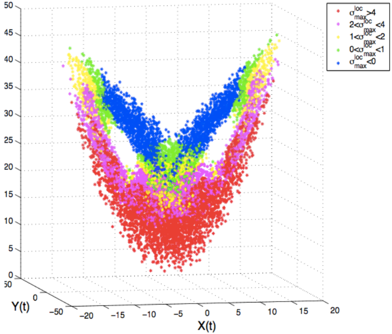
- The Lyapunov exponents characterize and quantify chaos.
- A chaotic system must have at least 1 positive exponent.
- For continuous dynamics at least 1 exponent must be zero and corresponds to the tangent to the flow.
- $\sum_{i=1}^{m} \lambda_i = \overline{\nabla \cdot \mathbf{f}}$ - The sum of the Lyapunov exponents is equal to the averaged divergence of the flow.
- For dissipative systems the above means that $\sum_{i=1}^{m} \lambda_i < 0$.
- The Lyapunov exponents measure the average rate of expansion/contraction of the volumes in the phase-space.
- They are natural indicators of the uncertainty evolution.
- Thus the Lyapunov exponents allow to compute the attractor dimension - Kaplan-Yorke dimension

$$d = d^{-} + \frac{\sum_{i=1}^{d^{-}} \lambda_i}{|\lambda_{d^{+}}|} \tag{6}$$

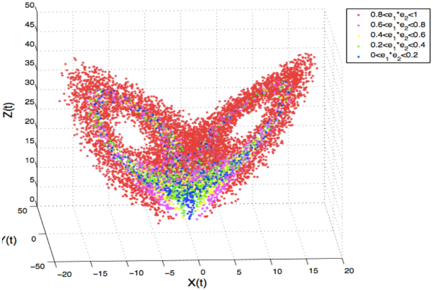
  with d^{-} and d^{+} such that $\sum_{i=1}^{d^{-}} \lambda_i > 0 \qquad \sum_{i=1}^{d^{+}} \lambda_i < 0$
- The attractor dimension can be intended as a measure of the *irreducible degrees of freedom*

Local variability - An example from the L63 model

Leading local Lyapunov exponent



Cosine of angle between $1^{st}/2^{nd}$ CLVs.



Linear case: Kalman filter convergence on the unstable-neutral subspace

- Model dynamics and observation model:

$$\mathbf{x}_k = \mathbf{M}_k \mathbf{x}_{k-1}, \tag{7}$$
$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k. \tag{8}$$

The observation noise, $\mathbf{v}_k$, is assumed an unbiased Gaussian white sequence with statistics

$$\mathrm{E}[\mathbf{v}_k \mathbf{v}_l^{\mathrm{T}}] = \delta_{k,l} \mathbf{R}_k. \tag{9}$$

- We shall show now several analytic convergence results:
  - Linear/Gaussian filters, full rank initial covariance [Gurumoorthy et al, 2017]
  - Linear/Gaussian filters, general initial covariance [Bocquet et al, 2017]
  - Linear/Gaussian smoothers, general initial covariance [Bocquet and Carrassi, 2017]

Key analytic results - Filter case

- **Asymptotic rank**

If n_0 is the dimension of the unstable-neutral subspace, it can be shown that

$$\lim_{k\to\infty} \mathrm{rank}(\mathbf{P}_k) \leq \min\left\{\mathrm{rank}(\mathbf{P}_0), n_0\right\}. \tag{10}$$

- **Convergence rate**

Let σ_i^k, for $i = 1, \ldots, n$ denote the eigenvalues of $\mathbf{P}_k$. We have

$$\sigma_i^k \leq \alpha_i \exp\left(2k\lambda_i^k\right) \tag{11}$$

where $k\lambda_i^k$ is a log-singular value of $\mathbf{M}_{k:0}$ and $\lim_{k\to\infty}\lambda_i^k = \lambda_i$. This gives an upper bound for all eigenvalues of $\mathbf{P}_k$ and a rate of convergence for the $n - n_0$ smallest ones.

- **Projection on the unstable subspace**

If $\mathbf{P}_k$ is uniformly bounded, the stable subspace of the dynamics is asymptotically in the null space of $\mathbf{P}_k$, *i.e.* for any vector $\mathbf{u}_{k:0}$ in the stable subspace

$$\lim_{k\to\infty} \|\mathbf{P}_k \mathbf{u}_{k:0}\| = 0. \tag{12}$$

Key analytic results - Filter case

▶ Explicit dependence of $\mathbf{P}_k$ on $\mathbf{P}_0$

Using either analytic continuation or the symplectic symmetry of the linear representation of covariances, we have proven that

$$\mathbf{P}_k = \mathbf{M}_{k:0}\mathbf{P}_0\left[\mathbf{I} + \mathbf{\Theta}_k\mathbf{P}_0\right]^{-1}\mathbf{M}_{k:0}^{\mathrm{T}} \tag{13}$$

where

$$\mathbf{\Theta}_k \triangleq \mathbf{M}_{k:0}^{\mathrm{T}}\mathbf{\Gamma}_k\mathbf{M}_{k:0} = \sum_{l=0}^{k-1}\mathbf{M}_{l:0}^{\mathrm{T}}\mathbf{\Omega}_l\mathbf{M}_{l:0}. \tag{14}$$

is the *information* matrix, directly related to the *observability*, $\mathbf{\Gamma}_k$. $\mathbf{\Omega}_k \triangleq \mathbf{H}_k^{\mathrm{T}}\mathbf{R}_k^{-1}\mathbf{H}_k$ is the precision of the observations transferred to state space.

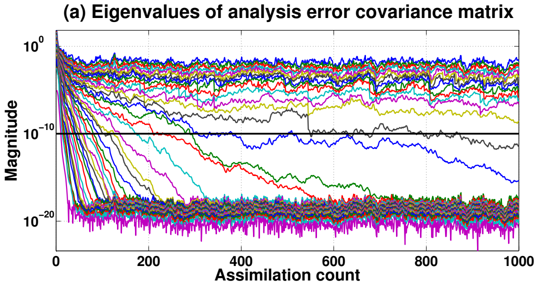
▶ Control of the unstable-neutral modes - Universal limiting covariance

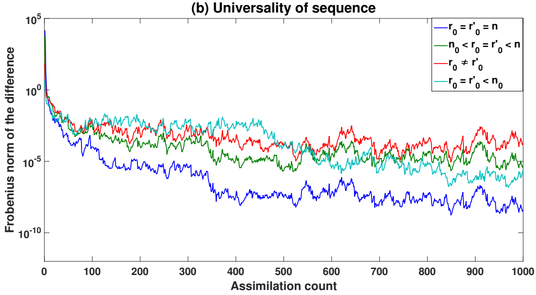
Let $\mathbf{U}_{+,k}$ be a matrix whose columns are the unstable and neutral backward Lyapunov vectors. If they are sufficiently observed we have:

$$\lim_{k\to\infty}\left\{\mathbf{P}_k - \mathbf{U}_{+,k}\left[\mathbf{U}_{+,k}^{\mathrm{T}}\mathbf{\Gamma}_k\mathbf{U}_{+,k}\right]^{-1}\mathbf{U}_{+,k}^{\mathrm{T}}\right\} = \mathbf{0}. \tag{15}$$

The asymptotic sequence does not depend on $\mathbf{P}_0$, only $\mathbf{\Gamma}_k$!

Numerical illustration and verification



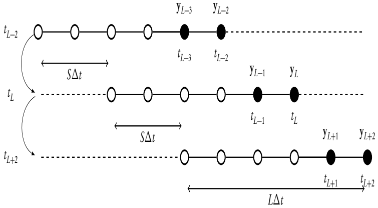


Linearized Lorenz-95 model around a Lorenz-95 trajectory.

(a) Evolution of the eigenvalues of $\mathbf{P}_k$.

(b) Frobenius norm of the difference between two different $\mathbf{P}_0$ when the conditions are satisfied, *i.e.* $\|\mathbf{P}^{\mathrm{a}}_k - \mathbf{P}'^{\mathrm{a}}_k\|$.

Key analytic results - The smoother case



▶ Explicit dependence of X_k on X_0

Left-transform update; if $k = pS$, $p = 0, 1, \dots$:

$$\mathbf{X}_k = \mathbf{M}_{k:0}\mathbf{X}_0 \left[\mathbf{I}_N + \mathbf{X}_0^{\mathrm{T}}\widehat{\boldsymbol{\Theta}}_k\mathbf{X}_0\right]^{-\frac{1}{2}} \boldsymbol{\Psi}_k \tag{16}$$

where

$$\widehat{\boldsymbol{\Theta}}_k \triangleq \sum_{q=0}^{p-1} \mathbf{M}_{qS:0}^{\mathrm{T}}\widehat{\boldsymbol{\Omega}}_{qS}\mathbf{M}_{qS:0}\,. \tag{17}$$

and $\boldsymbol{\Psi}_k$ is an orthogonal matrix. It is also possible to derive the right-transform update [see Bocquet and Carrassi, 2017 for the full derivation].

Analytic Results - The smoother case

▶ Convergence rate

The convergence rate of the collapse of $\mathbf{P}_k$ of the smoother is not faster than the filter's: the bounding rate is the same.

▶ Asymptotic of $\mathbf{X}_k$

The accuracy of the smoother for re-analysis is better and this impacts the asymptotic sequences. Indeed we have, for $k = pS$, $p = 0, 1, \ldots$:

$$\lim_{k \to \infty} \left\{ \mathbf{X}_k - \mathbf{U}_{+,k} \left[\mathbf{U}_{+,k}^{\mathrm{T}} \widehat{\mathbf{\Gamma}}_k \mathbf{U}_{+,k} \right]^{-\frac{1}{2}} \mathbf{\Psi}_k \right\} = \mathbf{0}. \tag{18}$$

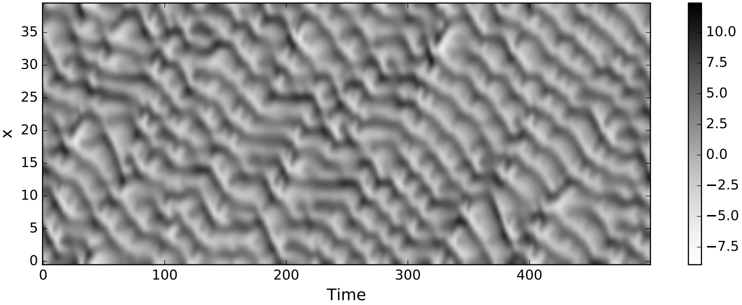
The only difference with respect to filter's is in the observability matrix $\widehat{\mathbf{\Gamma}}_k$, for $k = pS$, $p = 0, 1, \ldots$:

$$\widehat{\mathbf{\Gamma}}_k = \mathbf{\Gamma}_k + \sum_{l=k}^{k+L-S} \mathbf{M}_{k:l}^{-\mathrm{T}} \mathbf{\Omega}_l \mathbf{M}_{k:l}^{-1}. \tag{19}$$

which guarantees that the limiting $\mathbf{X}_k$ for smoother is smaller than in filter

$$\mathbf{U}_{+,k} \left[\mathbf{U}_{+,k}^{\mathrm{T}} \widehat{\mathbf{\Gamma}}_k \mathbf{U}_{+,k} \right]^{-1} \mathbf{U}_{+,k}^{\mathrm{T}} \leq \mathbf{U}_{+,k} \left[\mathbf{U}_{+,k}^{\mathrm{T}} \mathbf{\Gamma}_k \mathbf{U}_{+,k} \right]^{-1} \mathbf{U}_{+,k}^{\mathrm{T}}. \tag{20}$$

Nonlinear chaotic models: the Lorenz-95 low-order model



- It represents a mid-latitude zonal circle of the global atmosphere.
- Set of $M = 40$ ordinary differential equations [Lorenz and Emmanuel 1998]:

$$\frac{\mathrm{d}x_m}{\mathrm{d}t} = (x_{m+1} - x_{m-2})x_{m-1} - x_m + F\,, \tag{21}$$

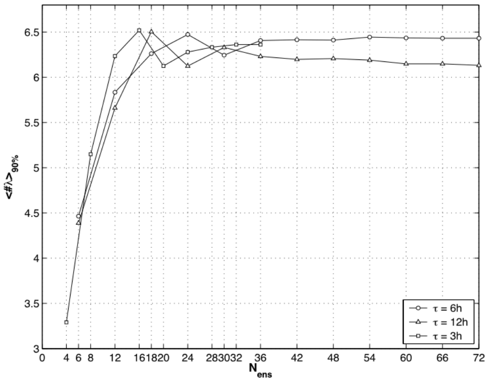
where $F = 8$, and the boundary is cyclic.

- Chaotic dynamics, for $M = 40$, it possesses 13 positive and 1 neutral Lyapunov exponents ($n_0 = 14$).

Rank deficiency of the EnKF covariance

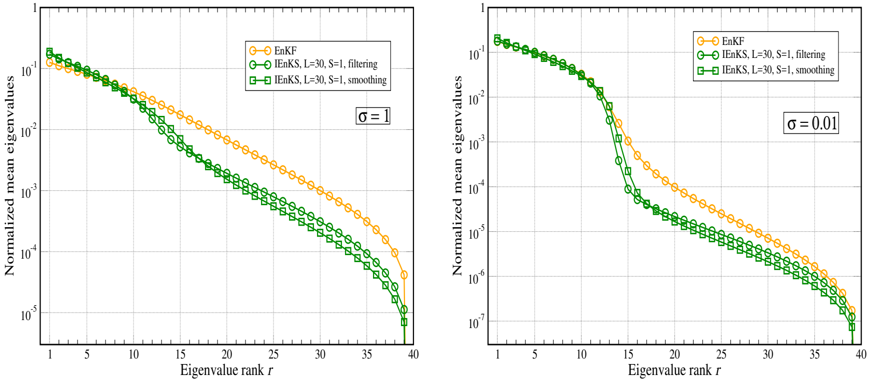
- $M = 36,\ n_0 = 12$
- EnKF-like method: Maximum Likelihood Ensemble filter [Zupanski, 2005] - Hybrid ensemble/variational

Mean number of eigenvalues accounting for 90% of the variance vs N_{ens} for $\tau = 12$, 6 and 3 hours



Carrassi *et al.*, 2009

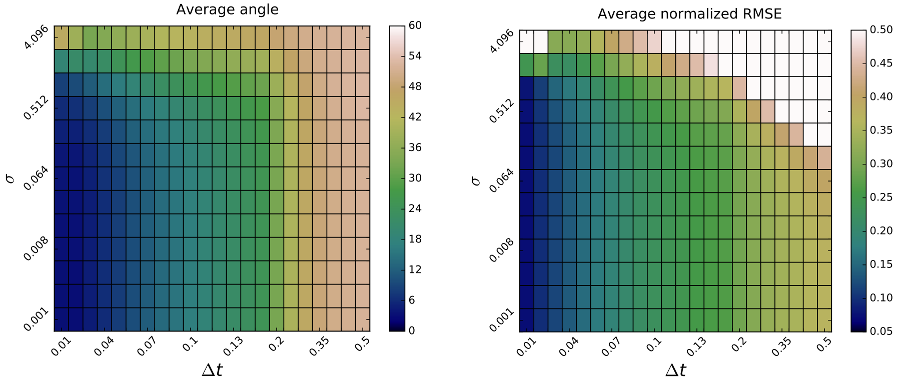
Spectrum of the analysis error covariance matrix



- $M = 40$, it possesses 13 positive and 1 neutral Lyapunov exponents ($n_0 = 14$).
- Time-average spectra of $\mathbf{P}_k^{\mathrm{a}}$: A visible transition at $r = 15$.

Bocquet and Carrassi, 2017

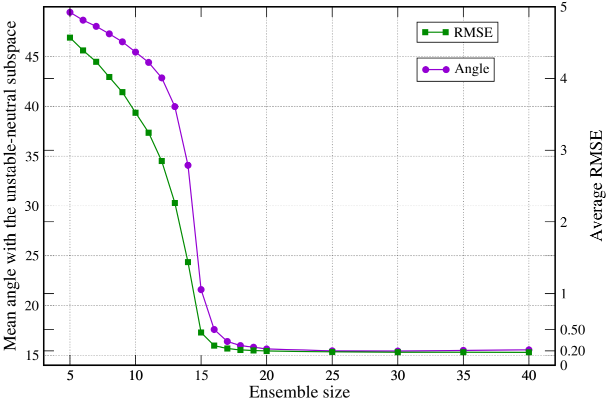
EnKF performance and Lyapunov directions



▶ Time- and ensemble-averaged angle an anomaly of the EnKF and the unstable-neutral subspace (left), and the EnKF RMSE normalized by σ (right panel), on the plane $(x,y) = (\Delta t, \sigma)$. The setup is $\mathbf{H} = \mathbf{I}_d$, $\mathbf{R} = \sigma \mathbf{I}_d$ and $N = 20$.

Bocquet and Carrassi, 2017

Projection on the unstable-neutral subspace and accuracy of EnKF



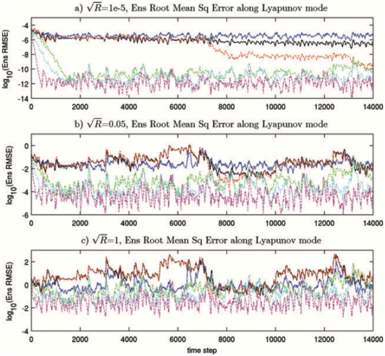
▶ EnKF - Average angle (left y-axis) between an anomaly (from the ensemble) and the unstable-neutral subspace, and RMSE of the analysis (right y-axis), as a function of the ensemble size.

Bocquet and Carrassi, 2017

Deviation from linearity - Does the unstable neutral subspace suffice?

- 6dim - Lorenz 1995 Model
- 2 positive Lyapunov exponents, 1 neutral, and 3 negative

Time series of EnKF RMSE along Lyapunov vectors



Ng *et al.*, 2011

The greatest separation between decaying and growing modes occurs for small observational error ⇒ Suggestion to *include some weakly stable modes* to describe the uncertainty.

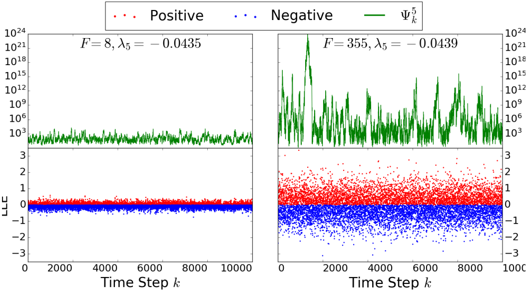
Linear noisy dynamics - The effect of the stable modes

- Model dynamics and observation model:

$$\mathbf{x}_k = \mathbf{M}_k \mathbf{x}_{k-1} + \mathbf{w}_k, \quad (22)$$
$$\mathbf{y}_k = \mathbf{H}_k \mathbf{x}_k + \mathbf{v}_k. \quad (23)$$

The model and observation noises, $\mathbf{w}_k$ and $\mathbf{v}_k$, are assumed mutually independent, unbiased Gaussian white sequences with statistics

$$\mathrm{E}[\mathbf{v}_k \mathbf{v}_l^{\mathrm{T}}] = \delta_{k,l}\mathbf{R}_k, \quad \mathrm{E}[\mathbf{w}_k \mathbf{w}_l^{\mathrm{T}}] = \delta_{k,l}\mathbf{Q}_k, \quad \mathrm{E}[\mathbf{v}_k \mathbf{w}_l^{\mathrm{T}}] = \mathbf{0}. \quad (24)$$



Grudzien et al., 2017

- Discrete, linearized L95 with 10 dimensions and 6 stable modes.
- The error in the stable modes does not converge to zero but it is bounded.
- Variability in the local LE of the stable modes forces transient instabilities.
- Unconstrained uncertainty in the i^{th} mode computable recursively using the QR decomposition $\psi_k^i = \sum_{l=0}^{k} ||(\mathbf{T}_{k:l}^{\mathrm{T}})^i||^2$

Nonlinear System - Stability $\sim$ Observability

- Consider again the deterministic (perfect) nonlinear-chaotic system $\mathbf{x}(t) = \mathscr{M}(\mathbf{x}(t_0))$ (unknown)
- We aim at estimating/approximating it by the sequence $\mathbf{x}^a_k = (\mathbf{I} - \mathbf{K}_k \mathscr{H})\mathscr{M}(\mathbf{x}_{k-1}) + \mathbf{K}_k \mathbf{y}_k$
- Linear perturbation evolution is given by $\boldsymbol{\delta}\mathbf{x}(t) = \mathbf{M}\boldsymbol{\delta}\mathbf{x}(t_0)$
- The linear perturbations about the forecast/analyses sequence follow $\boldsymbol{\delta}\mathbf{x}^a_k = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)\mathbf{M}_{k-1}\boldsymbol{\delta}\mathbf{x}^a_{k-1}$
- The term $(\mathbf{I} - \mathbf{K}_k \mathbf{H}_k)$ reflects the effect induced by the assimilation
- It modifies the stability properties of the perturbative dynamics relative to the free (unconstrained) system.
- We shall use this linear stability framework in nonlinear system $\Leftrightarrow$ we will assume that error will behave (quasi)-linearly.

Nonlinear System - Stability $\sim$ Observability

- **Complete stabilization** by the updating process is **sufficient for the uniqueness of the solution**
- It is **necessary for the convergence** of this solution to the true state of the system.
- Conjecture $\Rightarrow$ The complete stabilization will drive analysis errors to zero in the noiseless case and to the lowest possible values when noise or nonlinear effects are present.

We have two ways to achieve this stabilization:

1. Design of the observational network (types, distribution, frequency) $\Longleftrightarrow$ Acting on the operator $\mathscr{H}$
2. Design of the DA scheme $\Longleftrightarrow$ Acting on $\mathbf{K}$

Assimilation in the Unstable Subspace - AUS

Assimilation in the Unstable Subspace $\Leftrightarrow$ Confine the analysis correction in the unstable subspace

- The growth of the initial uncertainty strongly projects on the unstable manifold of the forecast model.
- The AUS approach consists in confining the analysis update in the subspace spanned by the leading unstable directions **E**:

$$\mathbf{x^a} = \mathbf{x^b} + \mathbf{E}\boldsymbol{\Gamma}\mathbf{E}^T\mathbf{H}^T(\mathbf{R} + \mathbf{H}\mathbf{E}\boldsymbol{\Gamma}\mathbf{E}^T\mathbf{H}^T)^{-1}(\mathbf{y}^o - \mathbf{H}\mathbf{x^b})$$

- While all assimilation methods, more or less implicitly, exert some control on the flow dependent instabilities, AUS exploits the unstable subspace, as key dynamical information in the assimilation process.
- The columns of the matrix **E** are the Lyapunov modes of the forecast-analysis cycle
- They can be approximated by Breeding on the Data Assimilation System **BDAS**

Trevisan and Uboldi, 2004; Palatella et al, 2013 for a review

Illustration: AUS in 1 dimension

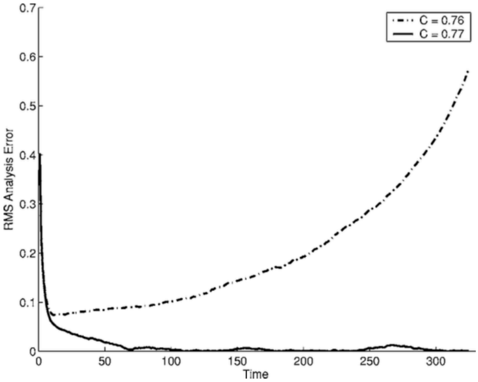
- $\mathbf{M}$ has a single eigenvalue, $\Gamma > 1$ corresponding to a positive Lyapunov exponent
- $\mathbf{e}$ be the associated eigenvector
- 1 scalar observation $\Rightarrow$ $\mathbf{H}$ is a constant row vector
- Kalman gain approximated by $\mathbf{K}_k = c\mathbf{e}_k$ where c is a constant scalar.
- A **sufficient condition** for the complete stabilization of the forecast-analysis cycle is:

$$c\mathbf{He} > 1 - \Gamma^{-1}$$

- The choice $\mathbf{K}_k || \mathbf{e}_k$ makes the confinement on the unstable direction
- The amplitude of the correction, c, must be large enough to counteract the unstable growth.

Illustration: AUS in 1 dimension

- Lorenz-63 at the origin $x = y = z = 0$
- Observation of the y-component
- **M** possesses one eigenvalue larger than 1, $\Gamma = 3.26$.
- **He** converges to 0.91
- Stabilization is achieved if $c > 0.762$



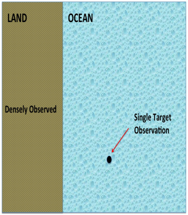
Carrassi et al., 2008

AUS and Target Observations

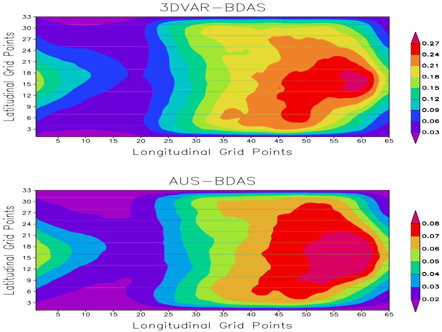
TARGET OBSERVATION STRATEGY: Breeding on the Data Assimilation System BDAS

Experiment	Ocean Obs Type/Positioning/Assimilation	RMS Error
LO	-	0.462
FO	vert.Prof/fixed(in the max(err))/3DVar	0.338
RO	vert.Prof/random/3DVar	0.311
3DVar-BDAS	vert.Prof/BDAS/3DVar	0.184
AUS-BDAS	temp.1-Level/BDAS/AUS	0.060

- Quasi-geostrophic atmospheric model (Rotunno and Bao, 1996 MWR)
- Perfect model setup - Observation Dense area (1-20 Longitude) - Target Area, one obs between 21-64 Longitude

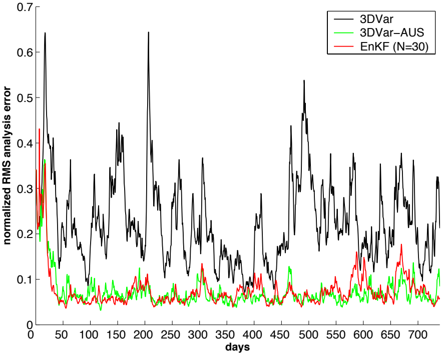


Carrassi et al., 2007



Hybrid 3DVar - AUS

Enhancing the performance of a 3DVar by using AUS
Comparison with EnKF



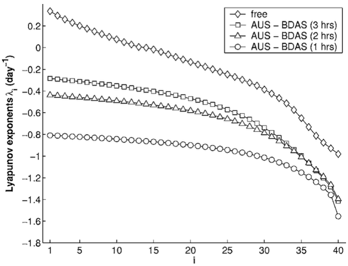
Carrassi et al, 2008

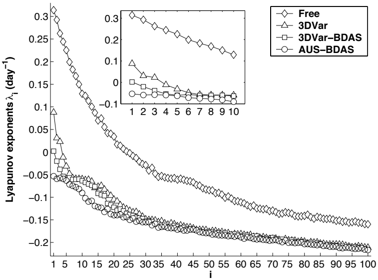
- A network of randomly distributed obs (vertical soundings)
- 3DVar-AUS: (1) AUS assimilate the obs able to control an unstable mode; (2) 3DVar process the remaining obs
- 3DVar-AUS comparable to EnKF with only one BDAS mode ⇒ Reduced computational cost and implementation on a pre-existing 3DVar scheme

Does stabilization improve estimation?

Quasi-Geostrophic Model (Rotunno and Bao, 1996)

Lorenz 1995 Model





Experiment	RMS Error
Free	1
AUS-BDAS - $\tau = 3$ h	0.014
AUS-BDAS - $\tau = 2$ h	0.011
AUS-BDAS - $\tau = 1$ h	0.009

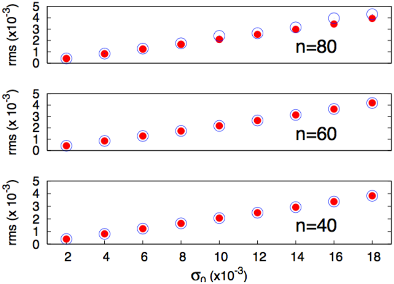
Experiment	RMS Error
Free	1
3DVar	0.321
3DVar-BDAS	0.163
AUS-BDAS	0.058

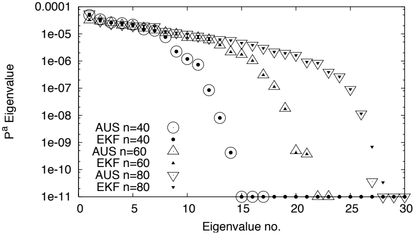
- DA provides a stabilizing effect (compare 3DVar with free system Lyapunov spectrum) but ...
- if the DA is designed to kill the instabilities, the estimation error is efficiently reduced

Carrassi et al., 2008

EKF-AUS

- Lorenz 1995 with dimension $n = 40(14Lyap^+)$, $60(19Lyap^+)$ and $80(25Lyap^+)$
- When errors behave linearly the error covariance projects on the unstable subspace
- EKF and its reduced unstable space counterpart EKF-AUS gives the same performance

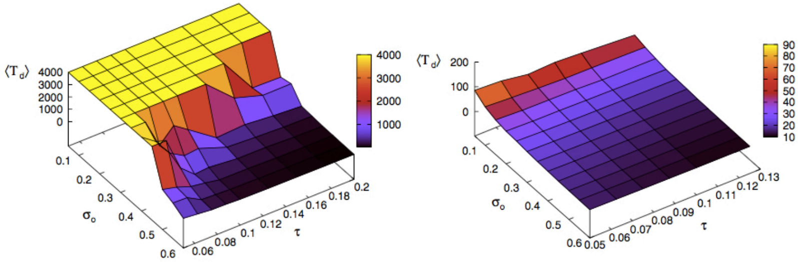




Trevisan and Palatella, 2011

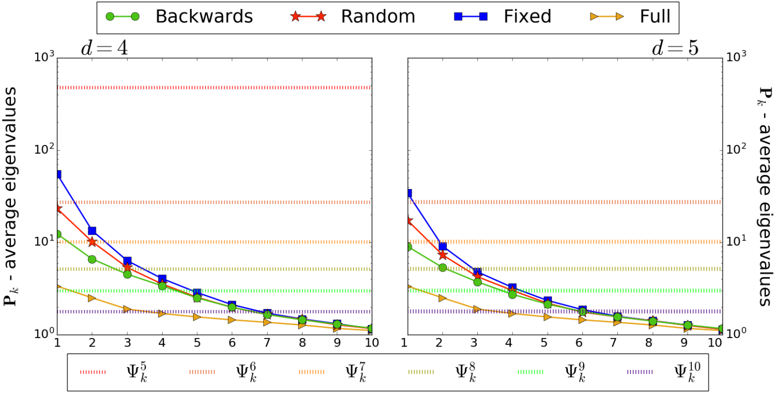
EKF-AUS extension to nonlinear error

- When errors do not behave linearly the unstable-neutral subspace alone does not describe fully the uncertainty.
- An extension to include stable modes is considered in the extension EKF-AUS-NL [Palatella and Trevisan, 2015].
- In the EKF-AUS-NL n_0 unstable-neutral direction + m_l additional stable modes are used.
- The stable modes and their interactions with all m_0 are considered by expanding the error dynamics up to the 2^{nd} order.
- EKF-AUS-NL works well when nonlinearity in the dynamics are primarily of quadratic nature, likewise the advection.
- Below example with L95 ($n_0 = 14$) and $m_l = 4$
- The figure shows the time before divergence (no divergence set with $\tau_d = 4000$ in the figure) as a function of the obs error σ_o and assimilation interval τ.



Linear noisy dynamics - The unconstrained stable forecast

- Spectrum of the $\mathbf{P}^f$ for different observational design
- The horizontal lines depicts the unconstrained forecast error, ψ^i in the stable modes [Grudzien et al, 2017]



Grudzien et al., 2017

Conclusion and directions

- Deterministic ensemble filters and smoothers are sensitive to the existence of a low-dimensional unstable-neutral subspace where a large portion of the uncertainty is confined.
- This seems to support/explain their success in high dimension.
- It is possible to design reduced-order formulation based on the unstable-neutral subspace.
- These formulation will be quasi-optimal when errors behave linearly and/or models are not perfect.
- Sub-optimal, but accurate, reduced-order formulations can be designed by including weakly stable modes.

Directions

- Design of "optimal" reduced order-filter in the presence of noise.
- Bayesian data assimilation using unstable subspace to define the proposal density.

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Upcoming conference on dynamical system & geosciences

Dear friends and colleagues, we would like to draw your attention to the following meeting:

Numerical Modeling, Predictability and Data Assimilation in Weather, Ocean and Climate

A Symposium Honoring the Legacy of Anna Trevisan

The Symposium will be held on October 17-20, 2017, at *the Institute of Atmospheric Sciences and Climate* (ISAC) of the *National Research Council of Italy* (CNR), in Bologna.

The event aims to honor Anna´s scientific contributions and her impact on the atmospheric, oceanic and climate sciences. The sessions are open to all those scientists, students and colleagues who have worked with Anna in person, to those whose past and current research has been influenced by Anna's work and by her personality, as well as to those who expect to gain future insights from the communications presented at this symposium.

Full details on the Symposium website: www.isac.cnr.it/Anna-Trevisan-Memorial-Symposium

The Symposium will be structured along three thematic areas:

1. Numerical modeling of the atmosphere and ocean
   Keynote Speaker: **E. Kalnay** (Un. of Maryland, USA)
2. Predictability
   Keynote Speaker: **M. Ghil** (ENS & UCLA, France & USA)
3. Data Assimilation
   Keynote Speaker: **O. Talagrand** (ENS & CNRS, France)

We welcome abstracts on any of the three areas and for either oral or poster contributions. The contributions will be published in a **Special Issue of Nonlinear Processes in Geophysics** subject to standard peer-review process.

Important dates: **Deadline for submission of abstract July 15th**
Deadline for registration September 15th (no fee).

The invited speakers of the Symposium are:
R. Benzi (Un. La Sapienza, Rome, Italy), **M. Bocquet** (ENPC, France), **M. Bonavita** (ECMWF), **A. Buzzi** (ISAC, CNR), **P. Cessi** (Scripps, USA), **S. Corti** (ISAC, CNR), **V. Lucarini** (Un. of Reading, UK), **C. Nicolis** (RMI, Belgium), **L. Palatella** (Italy), **S. Penny** (Un. of Maryland, USA), **N. Pinardi** (Un. of Bologna, Italy), **R. Rotunno** (NCAR, USA), **S. Tibaldi** (CMCC, Italy), **F. Uboldi** (Italy) and **S. Vannitsem** (RMI, Belgium)

Thank you for bringing the symposium to the attention of your colleagues, students and collaborators.

We look forward to welcome you in Bologna.

Best regards,

Alberto Carrassi
On behalf of the organizing and scientific committee: Michael Ghil, Eugenia Kalnay, Alberto Maurizi, Franco Prodi, Antonio Speranza and Olivier Talagrand.

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