Efficient QR Sequential Least Square algorithm for high frequency GNSS Precise Point Positioning

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AVENUE project

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Geodetic tool Geodesy by Simultaneous Numerical Integration

The GINS geodetic software is being developed for more than 50 years by the French Space Agency (CNES). It can compute:

- gravity of the Earth or other planet,
- orientation parameters of the Earth or any solid body,
- precise position of a station,

by interpreting observations (GNSS, Doppler, GRACE, GOCE) as adjustment of model parameters.
The parameter resolution is based on an iterative inverse Least Squares (LS) method.
In the post-processing of geodetic observations, the unknowns are considered as:

- global as they require the entire observation interval (daily station coordinates)
- local (GNSS satellite clock offsets estimated every data epoch)

Global Navigation Satellite System

GNSS measurements are made on radio signals sent by GNSS satellites to GNSS receivers.
GINS is able to process multi-GNSS (GPS, GLONASS, Galileo) data in a multi-mode (static, kinematic and dynamic) positioning.
GINS is routinely used to compute precise GNSS products delivered to the International GNSS Service (IGS) in the frame of the CNES-CLS Analysis Center activities.
GINS can be used for **PPP** (Precise Point Positioning) method to calculate very precise positions up to few centimeter level and to solve integer phase ambiguities (Integer PPP) using a single (GNSS) receiver.

- Input - the clocks and the orbits of the GNSS satellites
- Output - the clock parameters, the ambiguities, the positions and the tropospheric parameters associated with the receiver.

PPP/IPPP approach

- In PPP/IPPP approach, the mathematical problem to be resolved becomes “lighter” as orbits and clocks are fixed.
- The number of global parameters becomes minority compared to local parameters, especially for high frequency (e.g. 1 Hz) positioning applications (3.e5 unknowns per day).
- This temporal behaviour allows a partition of the parameters involved in the adjustment procedure into active or inactive.
- We implement an algorithm based on a Square Root Information Filter (SRIF) relying on a QR factorization in order to discard from one epoch to another the elements that are not active any more.

Least Square (LS) criterion

The observation vector y is written as:

$$y = Hx + v + y_0,$$

where H is the Jacobian matrix that relates the model variables x to the observations y. $v \sim N(0, I)$. Define $m = y - y_0$ measurement residuals for the parameters.
Define the LS criterion as:

$$J(x) = (m - Hx)^\top (m - Hx) + (x_0 - x)^\top P_0^{-1} (x_0 - x)$$

where x_0 is the a priori state and P_0 the covariance matrix of this a priori.
Consider $P_0^{-1} = H_0^\top H_0$ and define $m_0 = H_0 x_0$. Then:

$$J(x) = \left\| \begin{bmatrix} H_0 \\ H \end{bmatrix} x - \begin{bmatrix} m_0 \\ m \end{bmatrix} \right\|^2$$

Consider an orthogonal transformation:

$$\begin{bmatrix} H & m \end{bmatrix} = Q \begin{bmatrix} R & d \\ 0 & \rho \end{bmatrix}.$$

Q is an orthogonal matrix (size $n+1, n+1$), R an upper triangular matrix (n, n), d a column vector and ρ a scalar (n is the size of the model parameters vector x).

The LS criterion becomes:

$$J(x) = \|Rx - d\|^2 + \rho^2$$

The solution x^* which minimizes the LS criterion is :

$$x^* = R^{-1} d$$

Its covariance matrix has the following form:

$$P = cov(x^*) = R^{-1} (R^{-1})^\top$$

Suppose $\begin{bmatrix} H & m \end{bmatrix} = \begin{bmatrix} H_{1,\alpha} & H_{1,\beta} & m_1 \\ 0 & H_{2,\beta} & m_2 \end{bmatrix}$ corresponding to the state vector divided into two parts $\begin{bmatrix} x_\alpha \\ x_\beta \end{bmatrix}$, and with two measurement subsets 1 and 2 (rows of H).

Applying the orthogonal transformation to subset 1 leads to :

$$\begin{bmatrix} R_{\alpha\alpha} & R_{\alpha\beta} & d_\alpha \\ 0 & R_{\beta\beta} & d_\beta \\ 0 & 0 & \rho \\ 0 & H_{2,\beta} & m_2 \end{bmatrix}$$

The LS criterion can be written as :

$$J(x) = \left\| R_{\alpha\alpha} x_\alpha + R_{\alpha\beta} x_\beta - d_\alpha \right\|^2 + \left\| \begin{bmatrix} R_{\beta\beta} \\ H_{2,\beta} \end{bmatrix} x_\beta - \begin{bmatrix} d_\beta \\ m_2 \end{bmatrix} \right\|^2 + \rho^2$$

If $R_{\alpha\alpha}$ is invertible, the first term is always 0 at the minimum of $J(x)$, and the β subset of the solution x can be found solving only the second term.

Recursive Least Square

The recursive method for the least square error LS criterion is used when the observations arrive sequentially in time. Define sequentially the criterion function at time k as:

$$J_k(x) \;=\; \sum_{j=1}^{k} \|H_j x - m_j\|^2$$

H_j and m_j are the parts of H and m corresponding to epoch j.
Sequential estimation usually involves directly the pairs of (P_k,x_k)(Kalman filter) corresponding to the J_k functional minima.
Here we compute the algebraically equivalent pairs (R_k, d_k) (Square Root Information Filter, SRIF), to obtain a recursive representation of $J_k(x)$:

$$\begin{aligned} J_1(x) &= \|H_1 x - m_1\|^2 \\ J_k(x) &= J_{k-1}(x) + \|H_k x - m_k\|^2, \quad \forall k \in [2, n] \end{aligned}$$

The solution at epoch k, (R_k, d_k, ρ_k) can be computed recursively :

$$\begin{aligned} J_k(x) &= \|R_{k-1}x - d_{k-1}\|^2 + \rho_{k-1}^2 + \|H_k x - m_k\|^2 \\ &= \left\| \begin{bmatrix} R_{k-1} \\ H_k \end{bmatrix} x - \begin{bmatrix} d_{k-1} \\ m_k \end{bmatrix} \right\|^2 + \rho_{k-1}^2 \end{aligned}$$

Then, the QR factorization of the matrix $\begin{bmatrix} R_{k-1} & d_{k-1} \\ 0 & \rho_{k-1} \\ H_k & m_k \end{bmatrix}$ produces an upper triangular square matrix $\begin{bmatrix} R_k & d_k \\ 0 & \rho_k \end{bmatrix}$ which leads to:

$$J_k(x) = \|R_k x - d_k\|^2 + \rho_k^2, \quad \forall k \in [1, n]$$

The solution at time k and the corresponding error covariance matrix of the LS problem is given by:

$$\begin{aligned} x_k &= R_k^{-1} d_k \\ P_k &= R_k^{-1} (R_k^{-1})^\top \end{aligned}$$

SRIF Forward-Backward algorithm

The functional can be split into two separated functions, one over the interval $[0, k]$, and another over $[k+1, n]$.

$$J(x) = \sum_{i=1}^{k} \|H_i x - m_i\|^2 + \sum_{i=k+1}^{n} \|H_i x - m_i\|^2$$

$$J(x) = J_k^f(x) + J_k^b(x),$$

Applying the sequential algorithm we have:

$$\begin{aligned} J_k^f(x) &= \left\| R_k^f x - d_k^f \right\|^2 + {\rho_k^f}^2 \\ J_k^b(x) &= \left\| R_{k+1}^b x - d_{k+1}^b \right\|^2 + {\rho_{k+1}^b}^2 \end{aligned}$$

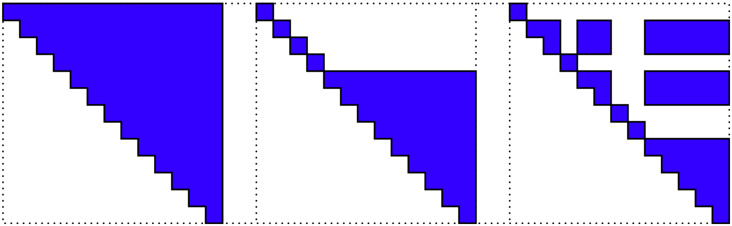
Together these two SRIF forward and backward algorithms work precisely on all the available measurements.

Active parameters

- Proceed with a partition of the state vector in order to reduce the temporal resolution of the estimation problem.
- Define the activity domain of a parameter as the minimal interval of epochs j where the values of H_j are not null.
- Define for each parameter an epoch interval $[b(j), e(j)]$ with $j < b(j)$ or $j > e(j)$ implying that $H_j = 0$ for the column corresponding to this parameter.

To obtain the solution at epoch k, we only need to take into account the **actif parameters**. Then we may reduce the problem and apply the QR factorization.

Illustration of forward reduction



The first upper triangular matrix depicts the optimal solution obtained at the previous epoch $k-1$. The parameters with the index k equal to $e(j)$ lie in the upper part of the state vector.
They can be reduced and replaced by those parameters having an index which verifies the condition $k = b(j)$.
Next, a permutation is applied in order to sort these parameters according to the $e(j)$ index.
Finally, the measurements of epoch k are assimilated and a new full upper triangular matrix is obtained for the next assimilation cycle.

Software and model parameters

Physical models	
Phase wind-up	yes
Relativity effects	yes
Solid earth deformation	IERS 2010 conventions
Ocean tide loading	Finite-Element Solution (FES) 2004
Troposphere (dry a priori)	GPT pressure model
Troposphere (wet a priori)	Saastamoinen (1973)
Mapping functions	Global Maping Functions
Antenna phase center corrections	2008 IGS Antex file
Strategy	Zero difference (PPP)
Orbits and clocks	GRG-IGS
Data used	L_3 (ionospheric-free)
Ambiguity fixing	IPPP
Cut-off angle	10°
σ factor for observation rejection	3.5

Table: The GINS parametrization.

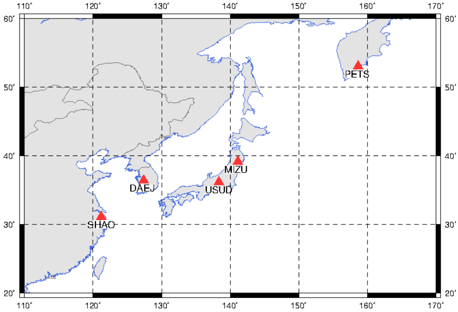
Adjusted parameters	
Receiver clock offset	1 per epoch
Station coordinates	1 per epoch
Integer-valued ambiguity	1 per satellite pass
ZWD correction	1 per 2 hour or per epoch

In GINS the ZWD (tropospheric wet delay) variable was modelled using piecewise linear functions with one constant value every 2 hours, contrary to the position parameters with one value per second or epoch.

Data set

We use 1-Hz GNSS observations collected from two GEONET stations in Japan, on the Pacific tectonic plate, during the 2011, M9 Tohoku-Oki earthquake:

- Mizusawa station (MIZU) located at 140 km from the epicentre
- Usuda station (USUD) at 430 km from the epicentre.



Results

algorithm	cpu	memory	rms
GINS-LS	43min25s	1.541 Gb	0.0062 m
GINS-SRI	01min42s	0.429 Gb	0.0062 m

Table: Comparison of the computational performance between GINS-SRIF and GINS-LS for 1 h of adjustment at the MIZU station.

The IPPP position solutions (East-North-Up)

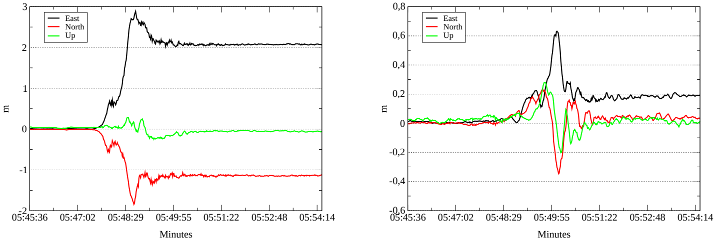
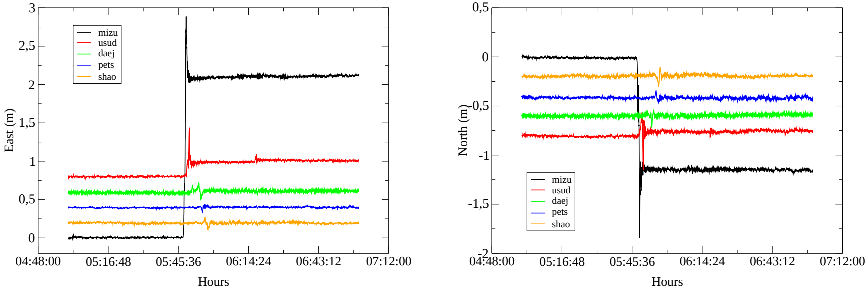


Figure: MIZU (left) and USUD (right) three component solutions, East-North-Up over a short time interval 2011/03/11

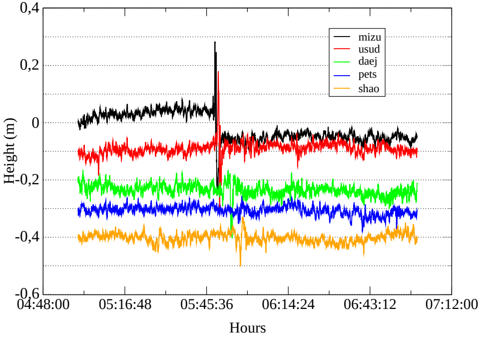
- Horizontal displacements of 2.1 m East and 1.1 m South
- Maximal oscillation amplitudes of 2.8 m (East), 1.8 m (South) and 0.75 m (down).

IPPP GPS horizontal solution series



SRIF is clearly able to detect the displacement waveforms at both Japanese stations and to detect less stronger wave effects propagated to the other tectonic plates.

IPPP GPS up solution series



Generally, the vertical component is more poorly determined by GNSS positioning than the horizontal components, partly due to the ZWD effect.

Stochastic parameter

ZWD is modelled as a high rate sampled parameter at 1 sec time interval by using a random walk process.

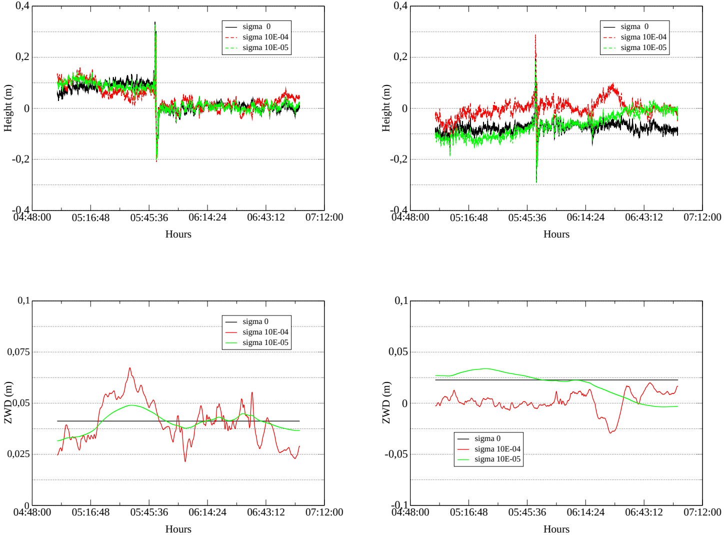
$$ZWD_{current} = ZWD_{past} + \delta.$$

station	$\sigma_\delta = 0$	$\sigma_\delta = 10E-04$	$\sigma_\delta = 10E-05$
MIZU	100%	100%	100%
USUD	92.86%	71.43%	100%

Table: Table showing the fixing ambiguity rate corresponding to the constant ZWD ($\sigma_\delta = 0$) and to two σ_δ of 10^{-4} m and 10^{-5} m associated with the random walk process.

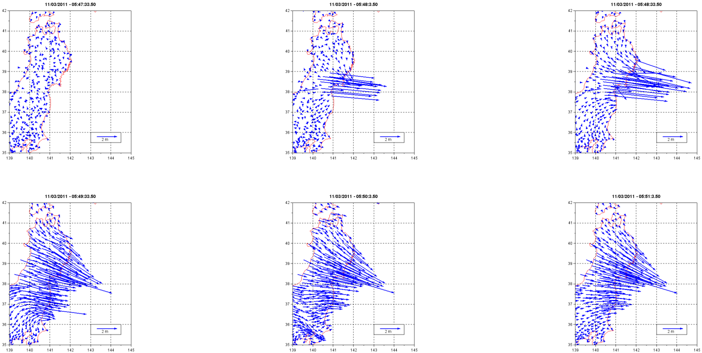
A major limiting factor for the achievement of centimetre accuracy in kinematic positioning is the occurrence of cycle slips.
A rate of fixing ambiguities of less than 100% may indicate that the cycle slips were not correctly detected and repaired.

Displacements in the up component corresponding to the ZWD parameter



Horizontal seismic displacement based on 30-s sampling ; plate deformation and wave propagation

GNSS data with 30 sec sampling provided by Geospatial Information Authority of Japan (GSI)(http://www.gsi.go.jp/) that operates GNSS-based control stations of the GPS Earth Observation Network system (GEONET)



Perspectives

- In the current GINS configuration the tropospheric parameters are adjusted every 2 h. This may introduce artefacts. The SRIF helps to better take into account the **tropospheric effects** by adjusting the parameters at the epoch resolution (every sec.)
- A better correction of the tropospheric parameters may help to better synchronize the atomic clocks by using a satellite constellation clocks. This is called **time transfer**.
- In the current GINS configuration the gravity field is adjusted once per day. Using the **GRACE** measurements and SRIF it will be possible to adjust it at a better time resolution.
- No dynamic filter is used in the GINS, but it is envisaged to adapt GINS in order to take it into consideration.

Thank you