Interpolation spaces and their applications from a numerical linear algebra perspective

Lecture I: Theory and Algorithms

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CERFACS

June 11, 2017

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Overview of talk

Lecture I

- Norms and duality in finite dimensional Hilbert spaces

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- Discrete Interpolation Norms

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- Norms and duality in finite dimensional Hilbert spaces
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- Finite-element approximation
- Real Interpolation: K-Method
- Fractional Laplacian operator: $(-\Delta)^s$

Finite dimensional Hilbert spaces and R^N

- $(\cdot,\cdot) : \mathbb{H} \times \mathbb{H} \to \mathsf{R}$ scalar product and
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  $\forall u \in \mathbb{H} \qquad u = \sum_{i=1}^{N} u_i \psi_i \qquad u_i \in \mathbf{R} \quad i = 1, \ldots, N$

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- Representation of scalar product in R^N.
  Let $u = \sum_{i=1}^{N} u_i \psi_i$ and $v = \sum_{i=1}^{N} v_i \psi_i$.
  Then

$$(u,v) = \sum_{i=1}^{N} \sum_{j=1}^{N} u_i v_j (\psi_i, \psi_j) = \mathbf{v}^T \mathbf{H} \mathbf{u}$$

  where $\mathbf{H}_{ij} = \mathbf{H}_{ji} = (\psi_i, \psi_j)$ and $\mathbf{u}, \mathbf{v} \in \mathrm{R}^N$.
  Moreover, $\mathbf{u}^T \mathbf{H} \mathbf{u} > 0 \qquad \text{iff} \qquad \mathbf{u} \neq 0$ and, thus $\mathbf{H}$ SPD.

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- Dual vector
  Let $u \in \mathbb{H},\ u \neq 0$, then $\exists f_u \in \mathbb{H}^\star$ such that

$$f_u(u) = \|u\|_{\mathbb{H}}$$

(Riesz, Hahn-Banach).

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  $$\mathbf{f} = \frac{\mathbf{H}\mathbf{u}}{\|\mathbf{u}\|_{\mathbf{H}}}$$

  and

  $$\|f_u\|^2_{\mathbb{H}^\star} = \mathbf{u}^T\mathbf{H}\mathbf{u} = \mathbf{f}^T\mathbf{H}^{-1}\mathbf{f}$$

Dual space basis

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- $f(\psi_i) = \gamma_i$ and $f(u) = f(\sum_{i=1}^{N} u_i \psi_i) = \sum_{i=1}^{N} \gamma_i u_i$

$$\phi_i(u) = \phi(\sum_{i=1}^{N} u_i \psi_i) = u_i \Longrightarrow f = \sum_{i=1}^{N} \alpha_i \phi_i$$

Linear operator

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- $$\kappa_{\mathbf{H}}(\mathbf{M}) = \|\mathbf{M}\|_{\mathbf{H},\mathbf{H}^{-1}} \|\mathbf{M}^{-1}\|_{\mathbf{H}^{-1},\mathbf{H}}.$$

The interesting case is $\kappa_{\mathbf{H}}(\mathbf{M})$ independent of N

Interpolation spaces

$$\begin{aligned}\mathbb{H} &= (\mathrm{R}^N, (u,v)_{\mathbb{H}} = \mathbf{u}^T\mathbf{H}\mathbf{v})\\ \mathbb{M} &= (\mathrm{R}^N, (u,v)_{\mathbb{M}} = \mathbf{u}^T\mathbf{M}\mathbf{v})\end{aligned}$$

Then $\exists \mathscr{S}$ self-adjoint such that

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where $\mathbf{S} = \mathbf{M}^{-1}\mathbf{H}$
S (self-adjoint in the good scalar product!)

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$$\left\{\mathbf{S}\mathbf{x} = \mu\mathbf{x} \quad \Leftrightarrow \quad \mathbf{H}\mathbf{x} = \mu\mathbf{M}\mathbf{x}\right\} \Rightarrow \mu = \delta^2 > 0$$

$\exists \mathbf{W}$ s.t. $\quad \mathbf{M} = \mathbf{W}^T\mathbf{W}, \quad \mathbf{H} = \mathbf{W}^T\boldsymbol{\Delta}^2\mathbf{W}, \quad \boldsymbol{\Delta}$ diagonal $\quad \boldsymbol{\Delta} \geq 0$

Interpolation spaces

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$$\mathbf{M\Lambda} = \mathbf{W}^{T}\mathbf{W}\mathbf{W}^{-1}\mathbf{\Delta}\mathbf{W}^{-T}\mathbf{W}^{T}\mathbf{W} = \mathbf{\Lambda}^{T}\mathbf{M} \Longrightarrow (\mathbf{u}, \mathbf{\Lambda v})_{\mathbf{M}} = (\mathbf{\Lambda u}, \mathbf{v})_{\mathbf{M}}$$

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$$\mathbf{M}\mathbf{\Lambda} = \mathbf{W}^{T}\mathbf{W}\mathbf{W}^{-1}\mathbf{\Delta}\mathbf{W}^{-T}\mathbf{W}^{T}\mathbf{W} = \mathbf{\Lambda}^{T}\mathbf{M} \Longrightarrow (\mathbf{u}, \mathbf{\Lambda}\mathbf{v})_{\mathbf{M}} = (\mathbf{\Lambda}\mathbf{u}, \mathbf{v})_{\mathbf{M}}$$

and

$$(\mathbf{\Lambda}^{1/2}\mathbf{u}, \mathbf{\Lambda}^{1/2}\mathbf{u})_{\mathbf{M}} = (\mathbf{u}, \mathbf{\Lambda}\mathbf{u})_{\mathbf{M}}$$

Interpolation spaces

$$\left[\mathbb{H}, \mathbb{M}\right]_{\vartheta} = \left\{\mathbf{u} \in \mathsf{R}^N; \left((\mathbf{u}, \mathbf{u})_{\mathbb{M}} + (\mathbf{u}, \mathbf{S}^{1-\vartheta}\mathbf{u})_{\mathbb{M}}\right)^{1/2}\right\}$$

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$$\left[\mathbb{H}, \mathbb{M}\right]_{1/2} = \left\{\mathbf{u} \in \mathsf{R}^N; \left((\mathbf{u}, \mathbf{u})_{\mathbb{M}} + (\mathbf{u}, \mathbf{\Lambda}\mathbf{u})_{\mathbb{M}}\right)^{1/2}\right\}$$

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$$||\mathbf{v}||^2_{\vartheta,h} = ||\mathbf{v}||^2_{\mathbf{H}_{\vartheta,h}} = \mathbf{v}^T\left(\mathbf{M} + \mathbf{M}\mathbf{S}^{1-\vartheta}\right)\mathbf{v}$$

$$\mathbf{H}_{\vartheta,h} = \mathbf{M}\left(\mathbf{I} + \mathbf{S}^{1-\vartheta}\right) = \mathbf{W}^T\left(\mathbf{I} + \boldsymbol{\Delta}^{2(1-\vartheta)}\right)\mathbf{W} \quad \text{(Bessel)}$$

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Let us drop one of the **M**

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$$\mathbf{H}_{\vartheta} = \mathbf{M}\left(\mathbf{S}^{1-\vartheta} \right) = \mathbf{W}^T \left(\mathbf{\Delta}^{2(1-\vartheta)} \right) \mathbf{W} \quad \text{(Riesz)}$$

$$\mathbf{H}_{\vartheta} \sim \mathbf{H}_{\vartheta,h}$$

Interpolation spaces (duality)

$\mathbb{M}^\star$ and $\mathbb{H}^\star$ dual spaces of $\mathbb{M}$ and $\mathbb{H}$

$$\left[\mathbb{H}, \mathbb{M}\right]_\vartheta^\star = \left[\mathbb{M}^\star, \mathbb{H}^\star\right]_{1-\vartheta}$$

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where

$$\mathbf{H}_{1-\vartheta}^\star = \mathbf{H}^{-1}(\mathbf{H}\mathbf{M}^{-1})^\vartheta = \mathbf{W}^{-1}\boldsymbol{\Delta}^{2(\vartheta-1)}\mathbf{W}^{-T} = \mathbf{H}_\vartheta^{-1}$$

Interpolation spaces (∞ dimensional case)

- $\mathbb{X}, \mathbb{Y}$ two Hilbert spaces with $\mathbb{X} \subset \mathbb{Y}$, $\mathbb{X}$ **dense and continuously embedded** in $\mathbb{Y}$. $\langle \cdot, \cdot \rangle_{\mathbb{X}}, \langle \cdot, \cdot \rangle_{\mathbb{Y}}$ scalar product and $\| \cdot \|_{\mathbb{X}}, \| \cdot \|_{\mathbb{Y}}$ the respective norms.
- (Riesz representation theory) $\exists \mathscr{S} : \mathbb{X} \to \mathbb{Y}$ positive and self-adjoint with respect to $\langle \cdot, \cdot \rangle_{\mathbb{Y}}$ such that $\langle u, v \rangle_{\mathbb{X}} = \langle u, \mathscr{S} v \rangle_{\mathbb{Y}}$. $\mathscr{E} = \mathscr{S}^{1/2} : \mathbb{X} \to \mathbb{Y}$,
- $\mathbb{X} = D(\mathscr{E})$ with $\|u\|_{\mathbb{X}} \sim \|u\|_{\mathscr{E}} := \left(\|u\|_{\mathbb{Y}}^2 + \|\mathscr{E} u\|_{\mathbb{Y}}^2 \right)^{1/2}$.
- $\|u\|_{\theta} := \left(\|u\|_{\mathbb{Y}}^2 + \|\mathscr{E}^{1-\theta} u\|_{\mathbb{Y}}^2 \right)^{1/2}$.
- The *interpolation space of index* θ $[\mathbb{X}, \mathbb{Y}]_{\theta} := D(\mathscr{E}^{1-\theta}), \quad 0 \leq \theta \leq 1$, with the inner-product $\langle u, v \rangle_{\theta} = \langle u, v \rangle_{\mathbb{Y}} + \left\langle u, \mathscr{E}^{1-\theta} v \right\rangle_{\mathbb{Y}}$ is a Hilbert space (Lions Magenes 1968).
- $[\mathbb{X}, \mathbb{Y}]_0 = \mathbb{X}$ and $[\mathbb{X}, \mathbb{Y}]_1 = \mathbb{Y}$. If $0 < \theta_1 < \theta_2 < 1$ then

$$\mathbb{X} \subset [\mathbb{X}, \mathbb{Y}]_{\theta_1} \subset [\mathbb{X}, \mathbb{Y}]_{\theta_2} \subset \mathbb{Y}.$$

Interpolation Theorem

Let $\mathfrak{X}, \mathfrak{Y}$ *Hilbert spaces* $\mathfrak{X} \subset \mathfrak{Y}$ *with* $\mathfrak{X}$ *dense in* $\mathfrak{Y}$*, and with inclusion compact and continuous. Let* $\mathbb{X}, \mathbb{Y}$ *satisfy similar properties. Let* $\pi \in \mathcal{L}(\mathfrak{X}; \mathbb{X}) \cap \mathcal{L}(\mathfrak{Y}; \mathbb{Y})$*. Then for all* $\theta \in (0, 1)$*,*

$$\pi \in \mathcal{L}([\mathfrak{X}, \mathfrak{Y}]_\theta; [\mathbb{X}, \mathbb{Y}]_\theta).$$

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Let $\mathfrak{X} \supset \mathbb{X}_h$ and $\mathfrak{Y} \supset \mathbb{Y}_h$ ($\mathbb{X}_h$ and $\mathbb{Y}_h$ finite-dimensional spaces) $i_h : \mathscr{L}(\mathbb{X}_h; \mathfrak{X}) \cap \mathcal{L}(\mathbb{Y}_h; \mathfrak{Y})$ the continuous injection operator

$$i_h \in \mathcal{L}([\mathbb{X}_h, \mathbb{Y}_h]_\theta; [\mathfrak{X}, \mathfrak{Y}]_\theta).$$

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Assume now that there exists an interpolation operator $\exists I_h$ such that $I_h : \mathcal{L}(\mathfrak{X}; \mathbb{X}_h) \cap \mathcal{L}(\mathfrak{Y}; \mathbb{Y}_h)$ and $I_h u = u_h$ for all $u_h \in \mathbb{X}_h$.

$$I_h \in \mathcal{L}([\mathfrak{X}, \mathfrak{Y}]_\theta; [\mathbb{X}_h, \mathbb{Y}_h]_\theta)$$

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Let $\mathfrak{X}, \mathfrak{Y}$ Hilbert spaces $\mathfrak{X} \subset \mathfrak{Y}$ with $\mathfrak{X}$ dense in $\mathfrak{Y}$, and with inclusion compact and continuous. Let $\mathbb{X}, \mathbb{Y}$ satisfy similar properties. Let $\pi \in \mathcal{L}(\mathfrak{X}; \mathbb{X}) \cap \mathcal{L}(\mathfrak{Y}; \mathbb{Y})$. Then for all $\theta \in (0,1)$,

$$\pi \in \mathcal{L}([\mathfrak{X}, \mathfrak{Y}]_\theta; [\mathbb{X}, \mathbb{Y}]_\theta).$$

$$\forall u_h \in [\mathbb{X}_h, \mathbb{Y}_h]_\theta, \ \|i_h u_h\|_\theta = \|u_h\|_\theta \leq C_1 \|u_h\|_{\theta,h}.$$

$$\forall u \in [\mathfrak{X}, \mathfrak{Y}]_\theta, \ \|I_h u\|_{\theta,h} \leq C_2 \|u\|_\theta.$$

Since $[\mathbb{X}_h, \mathbb{Y}_h]_\theta \subset [\mathfrak{X}, \mathfrak{Y}]_\theta$ then $\frac{1}{C_1}\|u_h\|_\theta \leq \|u_h\|_{\theta,h} \leq C_2 \|u_h\|_\theta$.

i.e. $\|u_h\|_\theta \sim \|u_h\|_{\theta,h}$

Interpolation Theorem

Let $\mathfrak{X}, \mathfrak{Y}$ *Hilbert spaces* $\mathfrak{X} \subset \mathfrak{Y}$ *with* $\mathfrak{X}$ *dense in* $\mathfrak{Y}$*, and with inclusion compact and continuous. Let* $\mathbb{X}, \mathbb{Y}$ *satisfy similar properties. Let* $\pi \in \mathcal{L}(\mathfrak{X}; \mathbb{X}) \cap \mathcal{L}(\mathfrak{Y}; \mathbb{Y})$*. Then for all* $\theta \in (0, 1)$*,*

$$\pi \in \mathcal{L}([\mathfrak{X}, \mathfrak{Y}]_\theta; [\mathbb{X}, \mathbb{Y}]_\theta).$$

$$\|\pi\|_{[\mathfrak{X},\mathfrak{Y}]_\theta \to [\mathbb{X},\mathbb{Y}]_\theta} \leq c \|\pi\|^{1-\theta}_{\mathfrak{X} \to \mathbb{X}} \, \|\pi\|^{\theta}_{\mathfrak{Y} \to \mathbb{Y}_h}$$

Interpolation spaces (∞ dimensional case)

For $0 < s < 1$ and given $\Omega \subsetneq \mathrm{R}^n$ with boundary Γ sufficiently smooth (Lipschitz is fine):

$$|v|^2_{H^s(\Omega)} = \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \mathbf{d}x\mathbf{d}y,$$

is a seminorm and

$$H^s(\Omega) = \left\{ v \in L^2(\Omega) : |v|_{H^s(\Omega)} < \infty \right\},$$

with

$$\|v\|^2_{H^s(\Omega)} = \left(\|v\|^2_{L^2(\Omega)} + |v|^2_{H^s(\Omega)} \right),$$

is an Hilbert space (Gagliardo-Slobodeckiĭ)

Interpolation spaces (∞ dimensional case)

The space $H_0^s(\Omega)$ is defined as completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{H^s(\Omega)}$:

$$H_0^s(\Omega) = \overline{C_0^\infty(\Omega)}^{H^s(\Omega)}.$$

Interpolation spaces (∞ dimensional case)

If

$$\boxed{\theta = 1 - s}$$

we have

$$H^s(\Omega) = \left[H^1(\Omega), L^2(\Omega)\right]_\theta.$$

EQUIVALENCE OF the NORMS

Interpolation spaces (∞ dimensional case)

If

$$\boxed{\theta = 1 - s}$$

we have

$$H_0^s(\Omega) \supseteq \left[H_0^1(\Omega), L^2(\Omega)\right]_\theta.$$

Interpolation spaces (∞ dimensional case)

The space $\left[H^1_0(\Omega), L^2(\Omega)\right]_{\frac{1}{2}}$ (*Lions-Magenes space*)

$$H^{\frac{1}{2}}_{00}(\Omega) = \left[H^1_0(\Omega), L^2(\Omega)\right]_{\frac{1}{2}}$$

can be characterised as

$$H^{\frac{1}{2}}_{00}(\Omega) = \left\{ v \in H^{\frac{1}{2}} : \int_\Omega \frac{v^2(y)}{dist(y,\Gamma)} \mathbf{d}y \ \leq \infty \right\}$$

see Lions Magenes Chapter 11

Interpolation spaces (∞ dimensional case)

Let u a constant function on Ω and zero outside:

$$u \in H_0^{\frac{1}{2}}(\Omega) \qquad u \notin H_{00}^{\frac{1}{2}}(\Omega)$$

$$\boxed{H_{00}^{\frac{1}{2}}(\Omega) \underset{\neq}{\subseteq} H^{\frac{1}{2}}(\Omega)}$$

Interpolation spaces (∞ dimensional case)

$\Omega \subset \mathbb{R}^n$ open bounded with smooth boundary Γ and let $\boldsymbol{\alpha}$ denote a multi-index of order m where m is a positive integer

$$H^m(\Omega) = \left\{ u : D^{\boldsymbol{\alpha}} u \in L^2(\Omega), \quad |\boldsymbol{\alpha}| \leq m \right\} \qquad (H^0(\Omega) = L^2(\Omega))$$

$$H^s(\Omega) := [H^m(\Omega), H^0(\Omega)]_{1-s/m}$$

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$H_0^s(\Omega)$ completion of $C_0^\infty(\Omega)$ in $H^m(\Omega)$, where $s > 0$.

Interpolation spaces (∞ dimensional case)

$\Omega \subset \mathbb{R}^n$ open bounded with smooth boundary Γ and let $\boldsymbol{\alpha}$ denote a multi-index of order m where m is a positive integer

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$$H^s(\Omega) := [H^m(\Omega), H^0(\Omega)]_{1-s/m}$$

For $0 \leq s_2 < s_1$ and k integer

$$\begin{array}{rcl} [H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta & \subset & H_0^{(1-\theta)s_1+\theta s_2}(\Omega) \\ & \text{if} & (1-\theta)s_1 + \theta s_2 \neq k + 1/2 \\ & & \\ [H_0^{s_1}(\Omega), H_0^{s_2}(\Omega)]_\theta & = & H_{00}^{k+1/2}(\Omega) \subsetneq H_0^{k+1/2}(\Omega) \\ & \text{if} & (1-\theta)s_1 + \theta s_2 = k + 1/2 \end{array}$$

$$H^{-s}(\Omega) = (H_0^s(\Omega))^\star \quad s > 0$$

If $(1-\theta)s_1 + \theta s_2 = 1/2$

$$[H^{-s_1}(\Omega), H^{-s_2}(\Omega)]_\theta = \left(H_{00}^{1/2}(\Omega)\right)^\star.$$

Trace Theorem

Let Ω open, bounded, and connected subset of$\mathbb{R}^n$, $(n \geq 1)$ with boundary Γ Lipschitz. The application

$$u \in H^m(\Omega) : u \rightarrow \left\{ \frac{\partial^j u}{\partial \nu^j} : j = 0, 1, \ldots, m-1 \right\}$$

where $\frac{\partial^j u}{\partial \nu^j} : \mathcal{D}(\overline{\Omega}) \rightarrow \left(\mathcal{D}(\Gamma)\right)^m$ is the normal derivative of order j oriented to the exterior of Ω, can be extended to a linear continuous application of

$$H^m(\Omega) \rightarrow \prod_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\Gamma).$$

This application is surjective and $\exists \mathcal{R}$ linear and continuous

$$\vec{g} = \{g_j\} \rightarrow \mathcal{R}\vec{g} \text{ of } \prod_{j=0}^{m-1} H^{m-j-\frac{1}{2}}(\Gamma) \rightarrow H^m(\Omega)$$

such that $\frac{\partial^j u}{\partial \nu^j} \mathcal{R}\vec{g} = g_j \;\; 0 \leq j \leq m-1$.

Finite-element example

$$H^{1/2}_{00}(\Omega) = [H^1_0(\Omega), L^2(\Omega)]_{1/2}.$$

Let $\mathbb{X}_h \subset H^1_0(\Omega), \mathbb{Y}_h \subset L^2(\Omega)$. Let $\{\phi_i\}_{1\leq i\leq n} \in \mathbb{X}_h$ be a spanning set for $\mathbb{Y}_h$ and let $\mathbf{L}_k \in \mathrm{R}^{n\times n}$ denote the Grammian matrices corresponding to the $\langle\cdot,\cdot\rangle_{H^k_0(\Omega)}$-inner product ($H^0(\Omega) = L^2(\Omega)$):

$$(\mathbf{L}_k)_{ij} = \langle\phi_i, \phi_j\rangle_{H^k_0(\Omega)}.$$

$\mathbf{H} = \mathbf{L}_1, \quad \mathbf{M} = \mathbf{L}_0$ and $\mathbf{H}_{1/2,h} = \mathbf{L}_0\left(\mathbf{I} + (\mathbf{L}_0^{-1}\mathbf{L}_1)^{1/2}\right)$ (Bessel)

Moreover, we have

$\mathbf{H}_{1/2,h} \sim \mathbf{H}_{1/2} = \mathbf{L}_0\left(\mathbf{L}_0^{-1}\mathbf{L}_1\right)^{1/2}$ (Riesz)

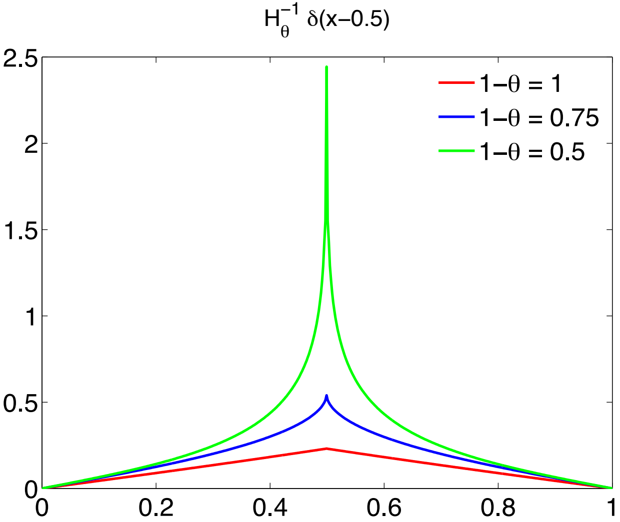
Interpolation theorem for FEM

Let the assumptions of Interpolation Theorem hold with $(\mathbb{X}, Y)$ replaced by $(\mathbb{X}_h, \mathbb{Y}_h)$ defined above. Let $\mathbf{H}_{\theta,h} = \mathbf{L}_0\left(\mathbf{I} + (\mathbf{L}_0^{-1}\mathbf{L}_1)^{1-\theta}\right)$, $\mathbf{H}_\theta = \mathbf{L}_0(\mathbf{L}_0^{-1}\mathbf{L}_1)^{1-\theta}$. Then there exist constants c, C independent of n such that

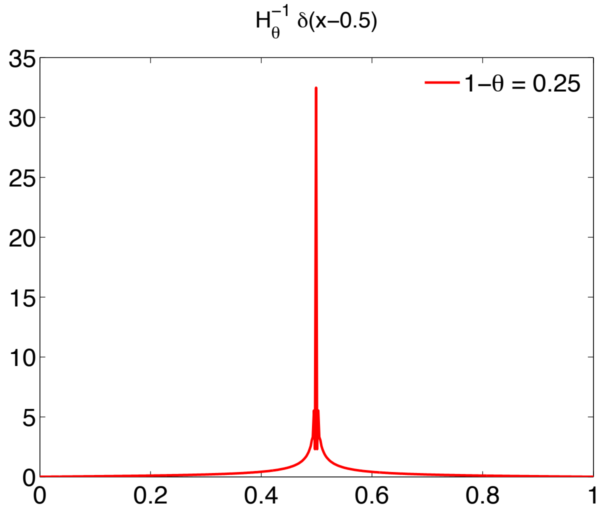
$$\begin{aligned} c\|u_h\|_{[\mathfrak{X},\mathfrak{Y}]_\theta} \leq & \quad \|\mathbf{u}\|_{\mathbf{H}_{\theta,h}} & \leq C\|u_h\|_{[\mathfrak{X},\mathfrak{Y}]_\theta}, \\ c\|u_h\|_{[\mathfrak{X},\mathfrak{Y}]_\theta} \leq & \quad \|\mathbf{u}\|_{\mathbf{H}_\theta} & \leq C\|u_h\|_{[\mathfrak{X},\mathfrak{Y}]_\theta}, \end{aligned}$$

for all $u_h \in [\mathbb{X}_h, \mathbb{Y}_h]_\theta$ and with $\theta \in (0,1)$, $\mathfrak{X} = L^2(\Omega)$, and $\mathfrak{Y} = H_0^1(\Omega)$

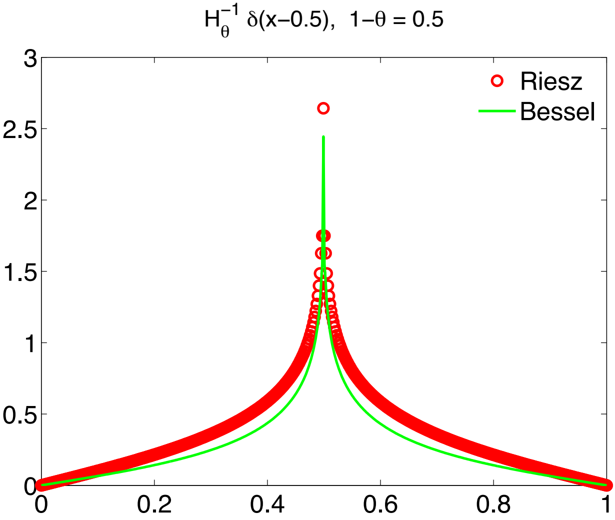
Few examples



Few examples



Few examples



Evaluation of $\mathbf{H}_\theta \mathbf{z}$

- Generalised Lanczos
  $\mathbf{H}\mathbf{V}_k = \mathbf{M}\mathbf{V}_k\mathbf{T}_k + \beta_{k+1}\mathbf{M}\mathbf{v}_{k+1}\mathbf{e}_k^T, \quad \mathbf{V}_k^T\mathbf{M}\mathbf{V}_k = \mathbf{I}_k$
  ($\mathbf{T}_k$ tridiagonal).

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  ($\mathbf{T}_k$ tridiagonal).
- $\mathbf{v}_0 = \mathbf{z}$
  $\mathbf{H}_\theta\mathbf{z} \approx \mathbf{M}\mathbf{V}_k\mathbf{T}_k^{1-\theta}\mathbf{e}_1\|\mathbf{z}\|_\mathbf{M}$ and
  $\mathbf{H}_{\theta,h}\mathbf{z} \approx \mathbf{M}\mathbf{V}_k(\mathbf{I}_k + \mathbf{T}_k^{1-\theta})\mathbf{e}_1\|\mathbf{z}\|_\mathbf{M}$.

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  $\mathbf{H}_{\theta,h}\mathbf{z} \approx \mathbf{M}\mathbf{V}_k(\mathbf{I}_k + \mathbf{T}_k^{1-\theta})\mathbf{e}_1\|\mathbf{z}\|_\mathbf{M}$.
- $\mathbf{v}_0 = \mathbf{M}^{-1}\mathbf{z}$
  $\mathbf{H}_\theta^{-1}\mathbf{z} \approx \mathbf{V}_k\mathbf{T}_k^{\theta-1}\mathbf{e}_1\|\mathbf{z}\|_{\mathbf{M}^{-1}}$ and
  $\mathbf{H}_{\theta,h}^{-1}\mathbf{z} \approx \mathbf{V}_k(\mathbf{I}_k + \mathbf{T}_k^{1-\theta})^{-1}\mathbf{e}_1\|\mathbf{z}\|_{\mathbf{M}^{-1}}$.
- Alternative: N. Hale, and N. J. Higham and L. N. Trefethen,

  SIAM J. Numer. Anal.

K-Method (∞ dimensional case)

Let $t > 0$ and $u \in \mathbb{H} + \mathbb{M}$ ($\mathbb{H}$ and $\mathbb{M}$ Hilbert spaces, and $\mathbb{H} \subset \mathbb{M}$ embedding compact)

$$K(t, u, \mathbb{H}, \mathbb{M})^2 = \inf_{v \in \mathbb{H}} \left\{ \|u - v\|_{\mathbb{M}}^2 + t^2 \|v\|_{\mathbb{H}}^2 \right\}.$$

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$$K(t, u, \mathbb{H}, \mathbb{M})^2 = \inf_{v \in \mathbb{H}} \left\{ \|u - v\|_{\mathbb{M}}^2 + t^2 \|v\|_{\mathbb{H}}^2 \right\}.$$

$$\left[\mathbb{M}, \mathbb{H}\right]_s = \left\{ u : u \in \mathbb{M}, t^{-(s+\frac{1}{2})} K(t, u, \mathbb{H}, \mathbb{M}) \in L^2(0, \infty) \right\}$$

Moreover,

$$\begin{aligned} \|u\|_s^2 &= \int_0^\infty t^{-(2s+1)} K(t, u, \mathbb{H}, \mathbb{M})^2 \, \mathbf{d}t \\ &= c \|\mathscr{E}^{1-s} u\|_{\mathbb{M}}^2 \quad \left(c = \int_0^\infty \frac{\mu^{1-2s}}{1 + \mu^2} \mathbf{d}\mu \right). \end{aligned}$$

(Lunardi, 1999)

K-Method

$$K(t, \mathbf{u}, \mathbb{H}, \mathbb{M})^2 = \inf_{\mathbf{v}} \left\{ \|\mathbf{u} - \mathbf{v}\|_{\mathbf{M}}^2 + t^2 \|\mathbf{v}\|_{\mathbf{H}}^2 \right\}$$
$$= \inf_{\mathbf{v}} \left\{ \mathbf{u}^T \mathbf{M} \mathbf{u} - 2\mathbf{v}^T \mathbf{M} \mathbf{u} + \mathbf{v}^T \mathbf{M} \mathbf{v} + t^2 \mathbf{v}^T \mathbf{H} \mathbf{v} \right\}.$$

K-Method

$$K(t, \mathbf{u}, \mathbb{H}, \mathbb{M})^2 = \inf_{\mathbf{v}} \left\{ \|\mathbf{u} - \mathbf{v}\|^2_{\mathbf{M}} + t^2 \|\mathbf{v}\|^2_{\mathbf{H}} \right\}$$
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The Euler equation is

$$-2\mathbf{M}\mathbf{u} + 2\left[\mathbf{M} + t^2\mathbf{H}\right]\mathbf{v} = 0.$$

Thus, its solution is

$$\mathbf{v}^* = \left[\mathbf{M} + t^2\mathbf{H}\right]^{-1}\mathbf{M}\mathbf{u} = \mathbf{G}^{-1}\mathbf{M}\mathbf{u}.$$

K-Method

The Euler equation is

$$-2\mathbf{Mu} + 2\left[\mathbf{M} + t^2\mathbf{H}\right]\mathbf{v} = 0.$$

Thus, its solution is

$$\mathbf{v}^* = \left[\mathbf{M} + t^2\mathbf{H}\right]^{-1}\mathbf{Mu} = \mathbf{G}^{-1}\mathbf{Mu}.$$

$$\begin{aligned}
K(t, \mathbf{u}, \mathbb{H}, \mathbb{M})^2 &= \mathbf{u}^T\left[\mathbf{M} - \mathbf{MG}^{-1}\mathbf{M}\right]\mathbf{u} \\
&= \mathbf{u}^T\mathbf{M}\left[\mathbf{I} - \mathbf{G}^{-1}\mathbf{M}\right]\mathbf{u} \\
&= \mathbf{u}^T\mathbf{MG}^{-1}\left[\mathbf{G} - \mathbf{M}\right]\mathbf{u} \\
&= t^2\mathbf{u}^T\mathbf{MG}^{-1}\mathbf{Hu} \\
\Big(&= \mathbf{u}^T\left[\mathbf{I} - \mathbf{G}^{-1}\mathbf{M}\right]\mathbf{Mu} \\
&= \mathbf{u}^T\left[\mathbf{G} - \mathbf{M}\right]\mathbf{G}^{-1}\mathbf{Mu} \\
&= t^2\mathbf{u}^T\mathbf{HG}^{-1}\mathbf{Mu},
\end{aligned}$$

i.e the matrix is SPD$\Big)$.

K-Method

$$\|\mathbf{u}\|_s^2 = \mathbf{u}^T \Big(\int_0^{\infty} t^{1-2s}\ \mathbf{H} \big(\mathbf{M} + t^2 \mathbf{H}\big)^{-1} \mathbf{M}\ \mathbf{d}t \Big) \mathbf{u}$$

K-Method

$$\|\mathbf{u}\|_s^2 = \mathbf{u}^T \left(\int_0^\infty t^{1-2s} \, \mathbf{H} \left(\mathbf{M} + t^2 \mathbf{H} \right)^{-1} \mathbf{M} \, \mathbf{d}t \right) \mathbf{u}$$

or

$$\|\mathbf{u}\|_s^2 = \mathbf{u}^T \left(\int_0^\infty t^{1-2s} \, \mathbf{M} \left(\mathbf{M} + t^2 \mathbf{H} \right)^{-1} \mathbf{H} \, \mathbf{d}t \right) \mathbf{u}$$

K-Method

$$\exists \mathbf{W} \text{ s.t. } \quad \mathbf{M} = \mathbf{W}^T\mathbf{W}, \quad \mathbf{H} = \mathbf{W}^T\boldsymbol{\Delta}^2\mathbf{W}, \quad \boldsymbol{\Delta} \quad \text{diagonal}$$

K-Method

$\exists \mathbf{W}$ s.t. $\mathbf{M} = \mathbf{W}^T\mathbf{W}$, $\mathbf{H} = \mathbf{W}^T\mathbf{\Delta}^2\mathbf{W}$, $\mathbf{\Delta}$ diagonal $\mathbf{\Delta} \geq 0$

$\boxed{0 < s < 1}$

$$\begin{aligned} \|\mathbf{u}\|_s^2 &= \mathbf{u}^T\Big(\int_0^\infty t^{1-2s}\,\mathbf{H}\big(\mathbf{M} + t^2\mathbf{H}\big)^{-1}\mathbf{M}\,\mathbf{d}t\Big)\mathbf{u} \\ &= \mathbf{u}^T\mathbf{W}^T\Big(\int_0^\infty t^{1-2s}\,\mathbf{\Delta}^2\big(\mathbf{I} + t^2\mathbf{\Delta}^2\big)^{-1}\,\mathbf{d}t\Big)\mathbf{W}\mathbf{u} \end{aligned}$$

K-Method

$\exists \mathbf{W}$ s.t. $\quad \mathbf{M} = \mathbf{W}^T\mathbf{W}, \quad \mathbf{H} = \mathbf{W}^T\boldsymbol{\Delta}^2\mathbf{W}, \quad \boldsymbol{\Delta}$ diagonal $\quad \boldsymbol{\Delta} \geq 0$

$0 < s < 1$

$$\|\mathbf{u}\|_s^2 = \mathbf{u}^T\Big(\int_0^\infty t^{1-2s}\, \mathbf{H}\big(\mathbf{M} + t^2\mathbf{H}\big)^{-1}\mathbf{M}\, \mathbf{d}t\Big)\mathbf{u}$$

$$= \mathbf{u}^T\mathbf{W}^T\Big(\int_0^\infty t^{1-2s}\, \boldsymbol{\Delta}^2\big(\mathbf{I} + t^2\boldsymbol{\Delta}^2\big)^{-1}\, \mathbf{d}t\Big)\mathbf{W}\mathbf{u}$$

Change variable $\big(\tau = \boldsymbol{\Delta} t\big)$

$$= \mathbf{u}^T\mathbf{W}^T\Big(\int_0^\infty \tau^{1-2s}\, \boldsymbol{\Delta}^{2s-1}\boldsymbol{\Delta}^2\big(1 + \tau^2\big)^{-1}\boldsymbol{\Delta}^{-1}\, \mathbf{d}\tau\Big)\mathbf{W}\mathbf{u}$$

$$= \mathbf{u}^T\mathbf{W}^T\boldsymbol{\Delta}^{2s}\mathbf{W}\mathbf{u}\int_0^\infty \frac{\tau^{1-2s}}{1+\tau^2}\,\mathbf{d}\tau = \frac{\pi}{2\sin s\pi}\,\mathbf{u}^T\mathbf{H}_{1-\theta}\mathbf{u} \qquad s = 1-\theta$$

Riesz

K-Method:duality (∞ dimensional case)

Let $t > 0$ and $f \in \mathbb{H}^\star + \mathbb{M}^\star$ ($\mathbb{H}$ and $\mathbb{M}$ Hilbert spaces, and $\mathbb{H} \subset \mathbb{M}$ embedding compact) Let $\mathbb{H}^\star$ and $\mathbb{M}^\star$ the corresponding dual spaces.

$$K(t, f, \mathbb{H}^\star, \mathbb{M}^\star)^2 = \inf_{v \in \mathbb{H}^\star} \left\{ \|f - v\|^2_{\mathbb{M}^\star} + t^2 \|v\|^2_{\mathbb{H}^\star} \right\}.$$

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$$\left[\mathbb{M}, \mathbb{H}\right]^\star_s = \left[\mathbb{M}^\star, \mathbb{H}^\star\right]_s$$

Moreover,

K-Method:duality (∞ dimensional case)

Let $t > 0$ and $f \in \mathbb{H}^\star + \mathbb{M}^\star$ ($\mathbb{H}$ and $\mathbb{M}$ Hilbert spaces, and $\mathbb{H} \subset \mathbb{M}$ embedding compact) Let $\mathbb{H}^\star$ and $\mathbb{M}^\star$ the corresponding dual spaces.

$$K(t, f, \mathbb{H}^\star, \mathbb{M}^\star)^2 = \inf_{v \in \mathbb{H}^\star} \left\{ \|f - v\|^2_{\mathbb{M}^\star} + t^2 \|v\|^2_{\mathbb{H}^\star} \right\}.$$

$$\left[\mathbb{M}, \mathbb{H}\right]^\star_s = \left[\mathbb{M}^\star, \mathbb{H}^\star\right]_s \left(= H^{-s}\right)$$

Moreover,

$$\|f\|^2_{-s} = \int_0^\infty t^{-(2s+1)} K(t, u, \mathbb{H}^\star, \mathbb{M}^\star)^2 \, \mathbf{d}t$$
$$= c \|\mathscr{E}^{s-1} f\|^2_{\mathbb{M}} \quad \left(c = \int_0^\infty \frac{\mu^{1-2s}}{1+\mu^2} \mathbf{d}\mu\right).$$

(Lions-Magenes, 1968, Bergh-Löfström Thorem 3.7.1)

K-Method: Duality

$$\exists \mathbf{W} \text{ s.t. } \quad \mathbf{M}^{-1} = \mathbf{W}^{-1}\mathbf{W}^{-T}, \quad \mathbf{H}^{-1} = \mathbf{W}^{-1}\boldsymbol{\Delta}^{-2}\mathbf{W}^{-T}, \ \boldsymbol{\Delta} \geq 0 \text{ diagonal}$$

K-Method: Duality

$\exists \mathbf{W}$ s.t. $\quad \mathbf{M}^{-1} = \mathbf{W}^{-1}\mathbf{W}^{-T}, \quad \mathbf{H}^{-1} = \mathbf{W}^{-1}\mathbf{\Delta}^{-2}\mathbf{W}^{-T}, \ \mathbf{\Delta} \geq 0$ diagonal

$\boxed{0 < s < 1}$

$$\|\mathbf{f}\|_{-s}^2 = \mathbf{f}^T \left(\int_0^\infty t^{1-2s} \, \mathbf{H}^{-1} \left(\mathbf{M}^{-1} + t^2 \mathbf{H}^{-1} \right)^{-1} \mathbf{M}^{-1} \, \mathbf{d}t \right) \mathbf{f}$$

K-Method: Duality

$\exists \mathbf{W}$ s.t. $\mathbf{M}^{-1} = \mathbf{W}^{-1}\mathbf{W}^{-T}$, $\mathbf{H}^{-1} = \mathbf{W}^{-1}\boldsymbol{\Delta}^{-2}\mathbf{W}^{-T}$, $\boldsymbol{\Delta} \geq 0$ diagonal

$\boxed{0 < s < 1}$

$$\|\mathbf{f}\|_{-s}^2 = \mathbf{f}^T \left(\int_0^\infty t^{1-2s}\, \mathbf{H}^{-1} \left(\mathbf{M}^{-1} + t^2 \mathbf{H}^{-1}\right)^{-1} \mathbf{M}^{-1}\, \mathbf{d}t \right) \mathbf{f}$$

$$= \mathbf{f}^T \mathbf{W}^{-1} \boldsymbol{\Delta}^{-2s} \mathbf{W}^{-T} \mathbf{f} \int_0^\infty \frac{\tau^{1-2s}}{1+\tau^2}\, \mathbf{d}\tau$$

$$= \frac{\pi}{2 \sin s\pi}\, \mathbf{H}_{1-\theta}^{-1} \quad \boxed{s = 1 - \theta} \qquad \text{Riesz}$$

Fractional Laplacian operator:$(-\Delta)^s$

Let $\boxed{s \in (0,1)}$ and f a smooth enough function:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

Fractional Laplacian operator:$(-\Delta)^s$

Let $\boxed{s \in (0,1)}$ and f a smooth enough function:

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The operator is non-local and non-linear

Fractional Laplacian operator:$(-\Delta)^s$

Let $\boxed{s \in (0,1)}$ and f a smooth enough function:

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases}$$

We will use a spectral approach

$$-\Delta : L^2(\Omega) \to L^2(\Omega)$$

$$Dom(-\Delta) = \left\{ v \in H_0^1(\Omega) : \Delta v \in L^2(\Omega) \right\}$$

The operator is positive, unbounded, and closed. Its inverse is compact.

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$\exists \{\lambda_k, \phi_k\}_{k\in\mathbb{N}}$ s.t. $\{\phi_k\}_{k\in\mathbb{N}}$ orthonormal basis for $L^2(\Omega)$ and $H_0^1(\Omega)$

$$\begin{cases} -\Delta\phi_k = \lambda_k \phi_k & \text{in } \Omega \\ \phi_k = 0 & \text{on } \Gamma \end{cases}$$

$$(-\Delta)^s u = \sum_{k=1}^{\infty} u_k \lambda_k^s \phi_k, \quad u_k = \int_\Omega u\phi_k.$$

If $f = \sum_{k=1}^{\infty} f_k\phi_k$ then $u_k = \lambda_k^{-s}\phi_k, \ \forall k \geq 1.$

$$Dom(-\Delta)^{\frac{s}{2}} = \left[H_0^1(\Omega), L^2(\Omega)\right]_{1-s}$$

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$$w \in \left[H_0^1(\Omega), L^2(\Omega)\right]_{1-s}, \quad \|w\|_s^2 = \sum_{k=1}^{\infty} \lambda_k^s |w_k|^2$$

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$$\mathbf{H}_{1-\theta}\mathbf{u} = \mathbf{f}, \qquad s = 1-\theta$$

$$\mathbf{u} = \frac{1}{c}\mathbf{H}_{1-s}^{-1}\mathbf{f}, \quad c = \frac{\pi}{2\sin \pi s}$$

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$$c\mathbf{u} = \left(\int_0^\infty t^{1-2s}\, \mathbf{H}^{-1}\left(\mathbf{M}^{-1} + t^2\mathbf{H}^{-1}\right)^{-1}\mathbf{M}^{-1}\, \mathbf{d}t\right)\mathbf{f}.$$

Numerical integration

$$
\begin{aligned}
c\mathbf{u} &= \left(\int_0^\infty t^{1-2s}\ \mathbf{H}^{-1}\left(\mathbf{M}^{-1}+t^2\mathbf{H}^{-1}\right)^{-1}\mathbf{M}^{-1}\ \mathbf{d}t\right)\mathbf{f} \\
&= \left(\int_0^\infty t^{1-2s}\ \left(\mathbf{H}+t^2\mathbf{M}\right)^{-1}\ \mathbf{d}t\right)\mathbf{f} \\
&\approx \sum_{k=1}^{N}\omega_k t_k^{1-2s}\ \left(\mathbf{H}+t_k^2\mathbf{M}\right)^{-1}\mathbf{f}
\end{aligned}
$$

Numerical integration

$$\Big(\int_0^\infty t^{1-2s}\, \mathbf{H}^{-1}\big(\mathbf{M}^{-1} + t^2\mathbf{H}^{-1}\big)^{-1}\mathbf{M}^{-1}\, \mathbf{d}t\Big)\mathbf{f}$$
$$= \Big(\int_0^\infty Q(t)\, \mathbf{d}t\Big)\mathbf{f}$$

Numerical integration

$$t = L\cot^2(y/2) \quad \text{(L is a user-chosen constant)}$$

$$\int_0^\infty Q(t)\,\mathbf{d}t = \int_0^\pi Q(L\cot^2(y/2))\frac{2L\sin(y)}{[1-\cos(y)]^2}\,\mathbf{d}y$$

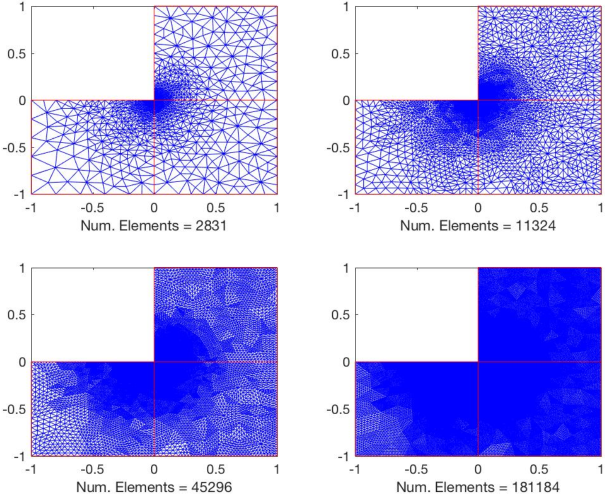
$$\approx \sum_{k=1}^{N} \omega_k Q(L\cot^2(y_k/2))$$

$$y_k \equiv \frac{k\pi}{N+1} \quad k = 1, \ldots, N$$

$$\omega_k \equiv \frac{2L\sin(y_k)}{[1-\cos(y_k)]^2}\left[\frac{2}{N+1}\right]\sum_{j=1}^{N}\frac{\sin(jy_k)[1-\cos(j\pi)]}{j}$$

Boyd 1986

Fractional Laplacian operator: examples



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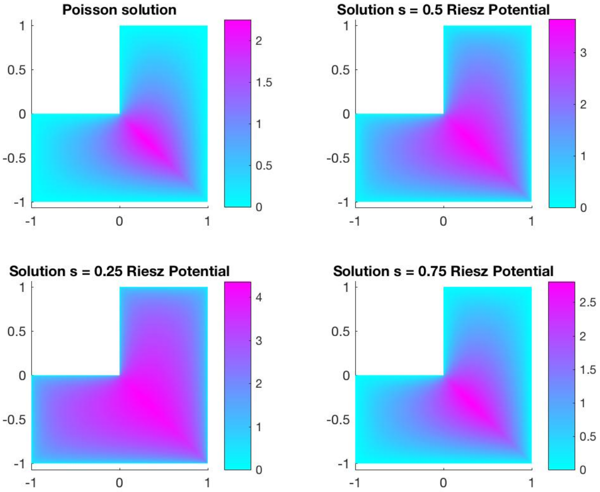

$$-\mathrm{div}(a(x)\nabla u) = 5,\ a(x) = 10^{-6} + |x_1 + x_2|,\ N = 40, \text{ and } L = 0.1.$$

Matrix order $n = 90125$.

Alternative solver for $(-\Delta)^s$

Caffarelli-Silvestre (2007) and Nochetto-Otarola-Salgado (2014) proposed the following alternative formulation

$$\mathcal{C} = \Omega \times (0, \infty) \text{ and } \partial_L \mathcal{C} = \Gamma \times [0, \infty)$$

$$\begin{cases} \text{div}(y^\alpha \nabla \mathfrak{u}) = 0 & \text{in } \mathcal{C} \\ \mathfrak{u} = 0 & \text{on } \partial_L \mathcal{C} \\ \dfrac{\partial \mathfrak{u}}{\partial \nu^\alpha} = d_s f & \text{on } \Omega \times \{0\} \end{cases}$$

where $\mathfrak{u} : (x, y) \in \mathcal{C} \to \mathfrak{u}(x, y) \in \mathrm{R}$ and $\dfrac{\partial \mathfrak{u}}{\partial \nu^\alpha} = -\lim_{y \to 0^+} y^\alpha \mathfrak{u}_y$ is the conormal exterior derivative of $\mathfrak{u}$, and ν unit outer normal to $\mathcal{C}$ at $\Omega \times \{0\}$.

$$\boxed{\alpha = 1 - 2s \in (-1, 1)}$$

$$d_s(-\Delta)^s u = \frac{\partial \mathfrak{u}}{\partial \nu^\alpha} \quad \text{in } \Omega$$

$$u(x) = \mathfrak{u}(x, 0).$$