Interpolation spaces and their applications from a numerical linear algebra perspective

Lecture II: Applications

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Overview of talk

Lecture II

- An example: Domain Decomposition

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- Other examples

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- Summary and open problems

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- An example: Domain Decomposition
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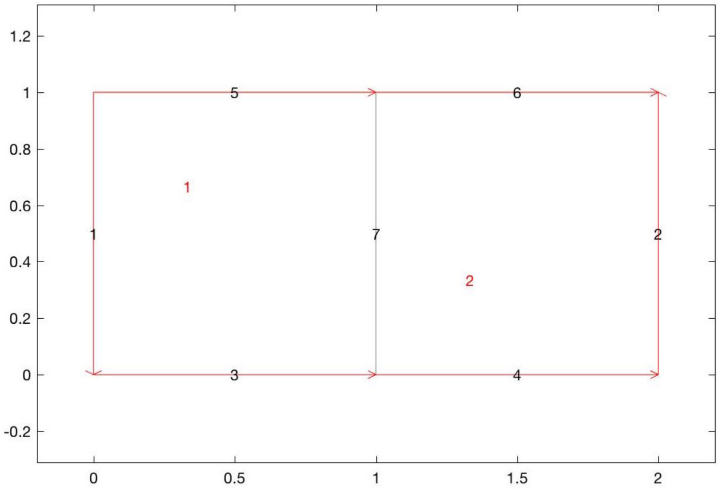
- A. and Loghin SINUM 2009, A., Kourounis, and Loghin IMAJNA 2012.
- Collaborators Drosos Kourounis , Rodrigue Kammogne

Domain Decomposition



A two domains decomposition example

Domain Decomposition

Let Ω be an open subset of $\mathbb{R}^d$ with boundary $\partial\Omega$ and consider the model problem

$$\left\{ \begin{array}{c} \text{Find } u \in H_0^1(\Omega) \text{ such that for all } v \in H_0^1(\Omega), \\ B(u, v) = (f, v). \end{array} \right.$$

$B(\cdot, \cdot)$ continuous and coercive bilinear form on $H_0^1(\Omega)$ and $f \in H^{-1}(\Omega)$.

Domain Decomposition

Problem 1

$$\begin{cases} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla v \mathbf{d}x.$$

Domain Decomposition

Problem 2

$$\begin{cases} -\nu\Delta u + \vec{b}\cdot\nabla u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

$$B(v,w) = (\nu\cdot\nabla v, \nabla w) + \left(\vec{b}\cdot\nabla v, w\right).$$

Steklov-Poincaré operator

Given a partition of Ω into two subdomains $\Omega \equiv \Omega_1 \cup \Omega_2$ with common boundary Γ this problem can be equivalently written as

$$\begin{cases} -\Delta u_1 &= f \qquad \text{in } \Omega_1, \\ u_1 &= 0 \qquad \text{on } \partial\Omega_1 \setminus \Gamma, \end{cases} \qquad \begin{cases} -\Delta u_2 &= f \qquad \text{in } \Omega_2, \\ u_2 &= 0 \qquad \text{on } \partial\Omega_2 \setminus \Gamma, \end{cases}$$

with the 'interface conditions'

$$\begin{cases} u_1 &= u_2 \\ \frac{\partial u_1}{\partial n_1} &= -\frac{\partial u_2}{\partial n_2} \end{cases} \quad \text{on } \Gamma$$

Steklov-Poincaré operator

Given $\lambda_1, \lambda_2 \in H^{1/2}_{00}(\Gamma)$, ψ_1, ψ_2 denote the harmonic extensions of λ_1, λ_2 respectively into Ω_1, Ω_2, i.e., for $i = 1, 2$, ψ_i satisfy

$$\begin{cases} -\Delta\psi_i = 0 & \text{in } \Omega_i, \\ \psi_i = \lambda_i & \text{on } \Gamma, \\ \psi_i = 0 & \text{on } \partial\Omega_i \setminus \Gamma. \end{cases}$$

The Steklov-Poincaré operator $\mathscr{S} : H^{1/2}_{00}(\Gamma) \to H^{-1/2}(\Gamma)$

$$\langle \mathscr{S}\lambda_1, \lambda_2 \rangle_{H^{-1/2}, H^{1/2}_{00}(\Gamma)} = (\nabla\psi_1, \nabla\psi_2)_{L^2(\Omega)} =: s(\lambda_1, \lambda_2).$$

$$c_1 \|\lambda\|^2_{H^{1/2}_{00}(\Gamma)} \leq s(\lambda, \lambda) \leq c_2 \|\lambda\|^2_{H^{1/2}_{00}(\Gamma)}.$$

Steklov-Poincaré operator

$$\text{(i)} \quad \begin{cases} -\Delta u_i^{\{1\}} &= f \ \text{ in } \Omega_i, \\ u_i^{\{1\}} &= 0 \ \text{ on } \partial\Omega_i, \end{cases}$$

$$\text{(ii)} \quad \left\{ \mathscr{S}\lambda = -\frac{\partial u_1^{\{1\}}}{\partial n_1} - \frac{\partial u_2^{\{1\}}}{\partial n_2} \ \text{ on } \Gamma, \right.$$

$$\text{(iii)} \quad \begin{cases} -\Delta u_i^{\{2\}} &= 0 \ \text{ in } \Omega_i, \\ u_i^{\{2\}} &= \lambda \ \text{ on } \partial\Omega_i. \end{cases}$$

The resulting solution is

$$u|_{\Omega_i} = u_i^{\{1\}} + u_i^{\{2\}}.$$

Discrete Formulation

$$\mathbb{V}^h = \mathbb{V}^{h,r} := \left\{ w \in C^0(\Omega) : w|_{\mathfrak{t}} \in P_k \quad \forall \mathfrak{t} \in \mathfrak{T}_h \right\} \subset H^1(\Omega)$$

be a finite-dimensional space of piecewise polynomial functions defined on some subdivision $\mathfrak{T}_h$ of Ω into simplices $\mathfrak{t}$ of maximum diameter h.

Discrete Formulation

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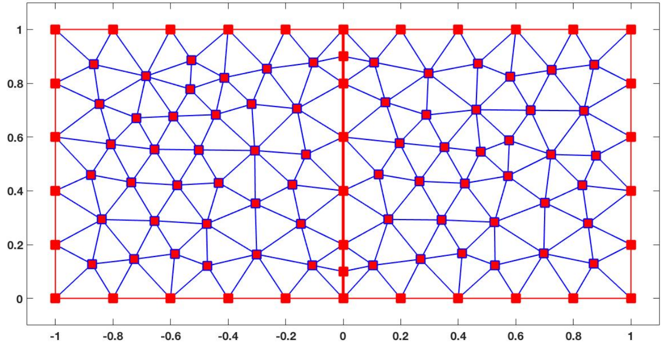


Discrete Formulation

Let further $\mathbb{V}^h_{iI}, \mathbb{V}^h_{iB} \subset \mathbb{V}^h_i$ satisfy $\mathbb{V}^h_{iI} \oplus \mathbb{V}^h_{iB} \equiv \mathbb{V}^h_i$. Let also

$$\mathbb{V}^h_{iI} = \mathrm{span}\left\{\phi^i_k, k = 1 \dots n^I_i\right\}, \quad \mathbb{V}^h_{iB} = \mathrm{span}\left\{\psi^i_k, k = 1 \dots n^B_i\right\}$$

and set $n_I = \sum_i n^I_i$ and $n_B = \sum_i n^B_i$. Let further

$$\mathbb{V}^h_B = \bigcup_{i=1}^{N} \mathbb{V}^h_{iB}$$

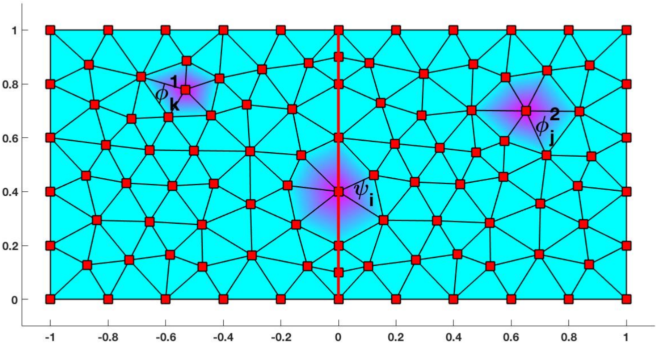
and let $\{\psi_k\}_{1\le k\le n_B}$ denote a basis for $\mathbb{V}^h_B$. Let

$$\mathbb{S}^h = \mathrm{span}\left\{\gamma_0(\Gamma)\psi_k\right\}_{1\le k\le n_B}$$

and set

$$\mathbb{V}^h = \bigcup_{i=1}^{N} \mathbb{V}^h_i \subset H^1_0(\Omega).$$

Discrete Formulation



Matrix formulation: 2 sub-domains

$$\mathbf{A}\mathbf{u} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ \mathbf{A}_{BI} & \mathbf{A}_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_B \end{pmatrix} = \mathbf{f}$$

$\mathbf{A} \in \mathrm{R}^{n \times n}, \mathbf{u} \in \mathrm{R}^n$ where $n = n_I + n_B$ and $\mathbf{A}_{II} \in \mathrm{R}^{n_I \times n_I}, \mathbf{A}_{IB}, \mathbf{A}_{BI}^T \in \mathrm{R}^{n_I \times n_B}, \mathbf{A}_{BB} \in \mathrm{R}^{n_B \times n_B}$ are given by

$$\mathbf{A}_{II} = \begin{pmatrix} \mathbf{A}_{II}^1 & \\ & \mathbf{A}_{II}^2 \end{pmatrix}, \quad \mathbf{A}_{IB} = \begin{pmatrix} \mathbf{A}_{IB}^1 \\ \mathbf{A}_{IB}^2 \end{pmatrix}$$

$$\mathbf{A}_{BI} = \begin{pmatrix} \mathbf{A}_{BI}^1 & \mathbf{A}_{BI}^2 \end{pmatrix}$$

with

$$\begin{aligned} (\mathbf{A}_{II}^i)_{k\ell} &= B_i(\phi_k^i, \phi_\ell^i), & (\mathbf{A}_{IB}^i)_{kj} &= B_i(\phi_k^i, \psi_j), \\ (\mathbf{A}_{BI}^i)_{jk} &= B_i(\psi_j, \phi_k^i), & (\mathbf{A}_{BB})_{pq} &= B(\psi_p, \psi_q), \end{aligned}$$

for all $k, \ell = 1, \ldots, n_i^I, j, p, q = 1, \ldots, n_B, i = 1, 2.$

Matrix formulation: 2 sub-domains

$$\mathbf{Au} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ \mathbf{A}_{BI} & \mathbf{A}_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_B \end{pmatrix}$$

Matrix formulation: 2 sub-domains

$$\mathbf{A}\mathbf{u} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ \mathbf{A}_{BI} & \mathbf{A}_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_B \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ \mathbf{A}_{BI} & \mathbf{A}_{BB} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \\ \mathbf{A}_{BI}\mathbf{A}_{II}^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ & \mathbf{S} \end{pmatrix}$$

$$\boxed{S = A_{BB} - A_{BI}A_{II}^{-1}A_{IB}.}$$

Matrix formulation: 2 sub-domains

$$\mathbf{A}\mathbf{u} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ \mathbf{A}_{BI} & \mathbf{A}_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_B \end{pmatrix}$$

$$\mathbf{u} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ & \mathbf{S} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I} & \\ -\mathbf{A}_{BI}\mathbf{A}_{II}^{-1} & \mathbf{I} \end{pmatrix} \mathbf{f}$$

Matrix formulation: 2 sub-domains

$$\mathbf{A}\mathbf{u} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ \mathbf{A}_{BI} & \mathbf{A}_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_B \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ & \mathbf{S} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_B - \mathbf{A}_{BI}\mathbf{A}_{II}^{-1}\mathbf{f}_I \end{pmatrix}}$$

Matrix formulation: 2 sub-domains

$$\mathbf{A}\mathbf{u} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ \mathbf{A}_{BI} & \mathbf{A}_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_B \end{pmatrix}$$

$$\boxed{\begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ & \mathbf{S} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_B - \mathbf{A}_{BI}\mathbf{A}_{II}^{-1}\mathbf{f}_I \end{pmatrix}}$$

$\mathbf{S}$ large and dense. How to precondition it for an iterative solver?

Matrix formulation: N sub-domains

$$\mathbf{A}\mathbf{u} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ \mathbf{A}_{BI} & \mathbf{A}_{BB} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_B \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_B \end{pmatrix} = \mathbf{f}$$

$\mathbf{A} \in \mathrm{R}^{n\times n}, \mathbf{u} \in \mathrm{R}^n$ where $n = n_I + n_B$ and
$\mathbf{A}_{II} \in \mathrm{R}^{n_I \times n_I}, \mathbf{A}_{IB}, \mathbf{A}_{BI}^T \in \mathrm{R}^{n_I \times n_B}, \mathbf{A}_{BB} \in \mathrm{R}^{n_B \times n_B}$ are given by

$$\mathbf{A}_{II} = \begin{pmatrix} \mathbf{A}_{II}^1 & & & & \\ & \ddots & & & \\ & & \mathbf{A}_{II}^i & & \\ & & & \ddots & \\ & & & & \mathbf{A}_{II}^N \end{pmatrix}, \quad \mathbf{A}_{IB} = \begin{pmatrix} \mathbf{A}_{IB}^1 \\ \vdots \\ \mathbf{A}_{IB}^i \\ \ddots \\ \mathbf{A}_{IB}^N \end{pmatrix}$$

$$\mathbf{A}_{BI} = \begin{pmatrix} \mathbf{A}_{BI}^1 & \cdots & \mathbf{A}_{BI}^i & \cdots & \mathbf{A}_{BI}^N \end{pmatrix}$$

with

$$\begin{aligned} (\mathbf{A}_{II}^i)_{k\ell} &= B_i(\phi_k^i, \phi_\ell^i), & (\mathbf{A}_{IB}^i)_{kj} &= B_i(\phi_k^i, \psi_j), \\ (\mathbf{A}_{BI}^i)_{jk} &= B_i(\psi_j, \phi_k^i), & (\mathbf{A}_{BB})_{pq} &= B(\psi_p, \psi_q), \end{aligned}$$

for all $k, \ell = 1, \ldots, n_i^I, j, p, q = 1, \ldots, n_B, i = 1, \ldots, N.$

Discrete Formulation

$$\begin{array}{ll} \text{(i)} & \mathbf{A}_{II,i}\mathbf{u}_i^{\{1\}} = \mathbf{f}_{I,i}, \\ \text{(ii)} & \mathbf{S}\mathbf{u}_B = \mathbf{f}_B - \mathbf{A}_{IB,1}^T\mathbf{u}_1^{\{1\}} - \mathbf{A}_{IB,2}^T\mathbf{u}_2^{\{2\}}, \\ \text{(iii)} & \mathbf{A}_{II,i}\mathbf{u}_i^{\{2\}} = -\mathbf{A}_{IB,1}^T\mathbf{u}_B - \mathbf{A}_{IB,2}^T\mathbf{u}_B, \end{array}$$

where $\mathbf{S}$ is the Schur complement corresponding to the boundary nodes

$$\mathbf{S} = \mathbf{A}_{BB} - \sum_i \mathbf{A}_{IB,i}^T\mathbf{A}_{II,i}^{-1}\mathbf{A}_{IB,i}.$$

The resulting solution is $(\mathbf{u}_{I,1},\ \mathbf{u}_{I,2},\ \mathbf{u}_B)$ where

$$\mathbf{u}_{I,i} = \mathbf{u}_i^{\{1\}} + \mathbf{u}_i^{\{2\}}.$$

$H_{00}^{1/2}$-preconditioners

Let

$$\mathbb{S}^h = \mathrm{span}\left\{\gamma_0(\Gamma)\psi_k\right\}_{1\le k\le n_B}$$

be defined as above and let

$$(\mathbf{L}_k)_{ij} = \left(\gamma_0(\Gamma)\psi_i, \gamma_0(\Gamma)\psi_j\right)_{H_0^k(\Gamma)}$$

for $k = 0, 1$ $(H_0^0(\Gamma) = L^2(\Gamma)$ *). Let*

$$\mathbf{H}_{1/2} := \mathbf{L}_0(\mathbf{L}_0^{-1}\mathbf{L}_1)^{1/2}.$$

$$\mathbf{H}_{1/2,h} := \mathbf{L}_0 + \mathbf{L}_0(\mathbf{L}_0^{-1}\mathbf{L}_1)^{1/2}.$$

Then for all $\boldsymbol{\lambda} \in R^m \setminus \{\mathbf{0}\}$

$$\kappa_1 \le \frac{\boldsymbol{\lambda}^T\mathbf{S}\boldsymbol{\lambda}}{\boldsymbol{\lambda}^T\mathbf{H}_{1/2}\boldsymbol{\lambda}} \le \kappa_2$$

with κ_1, κ_2 *independent of* h*.*

Discrete DD and Preconditioning

$$\mathbf{P} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ 0 & \mathbf{P}_S \end{pmatrix}$$

With P_S we denote the approximation of $\mathbf{H}_{1/2}^{-1}$ (or $\mathbf{H}_{1/2,h}^{-1}$) by a vector.

With this choice, the right-preconditioned system is

$$\mathbf{A}\mathbf{P}^{-1} = \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{A}_{BI}\mathbf{A}_{II}^{-1} & \mathbf{S}\mathbf{H}_{1/2}^{-1} \end{pmatrix}.$$

Then we use FGMRES.

Discrete DD and Preconditioning

$$\mathbf{P} = \begin{pmatrix} \mathbf{A}_{II} & \mathbf{A}_{IB} \\ 0 & \mathbf{P}_S \end{pmatrix}$$

With P_S we denote the approximation of $\mathbf{H}_{1/2}^{-1}$ (or $\mathbf{H}_{1/2,h}^{-1}$) by a vector.

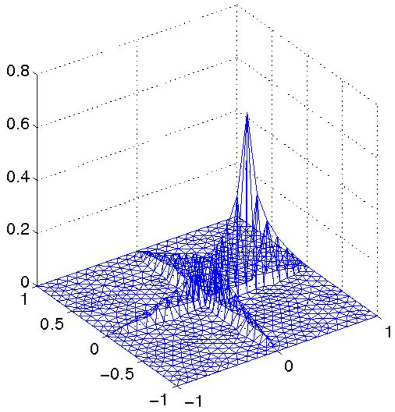
With this choice, the right-preconditioned system is

$$\mathbf{AP}^{-1} = \begin{pmatrix} \mathbf{I} & 0 \\ \mathbf{A}_{BI}\mathbf{A}_{II}^{-1} & \mathbf{SH}_{1/2}^{-1} \end{pmatrix}.$$

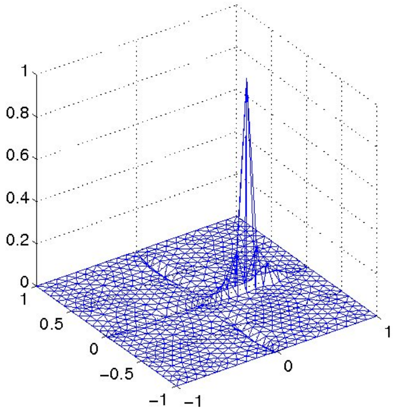
Then we use FGMRES.

For Problem 2 we denote with $\mathbf{A}_{II} = \nu \mathbf{L}_{II} + \mathbf{N}_{II}$ where $\mathbf{L}_{II}$ is the direct sum of Laplacians assembled on each subdomain and $\mathbf{N}_{II}$ is the direct sum of the convection operator $\vec{b} \cdot \nabla$ assembled also on each subdomain.

Green functions on wirebasket

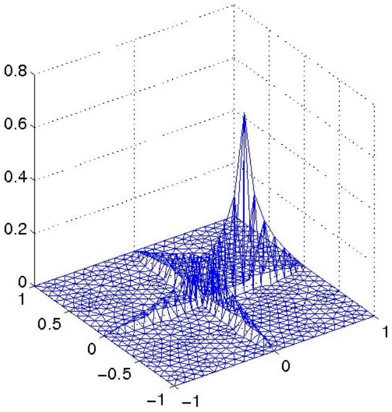


Steklov-Poincaré

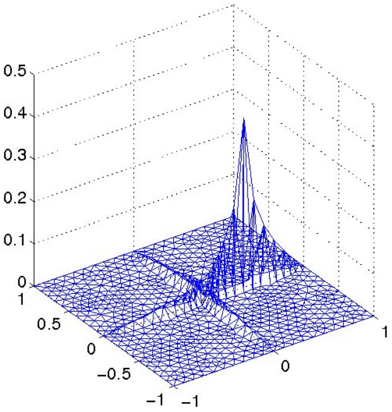


Neumann-Neumann

Green functions on wirebasket



Steklov-Poincaré



$H^{1/2}$

Numerical results: Poisson equation

			Linear			Quadratic		
#*dom*	*n*	*m*	$\mathbf{H}_{1/2,h}$	$\mathbf{H}_{1/2}$	$\widehat{\mathbf{H}}_{1/2}$	$\mathbf{H}_{1/2,h}$	$\mathbf{H}_{1/2}$	$\widehat{\mathbf{H}}_{1/2}$
4	45,377	449	10	9	9	11	11	11
	180,865	897	10	10	10	11	11	11
	722,177	1793	11	11	11	11	11	11
16	45,953	1149	13	12	12	13	13	13
	183,041	2301	13	13	13	13	13	13
	730,625	4605	13	13	13	13	13	13
64	66,049	3549	16	14	14	16	15	15
	263,169	7133	16	15	15	16	15	15
	1,050,625	14,301	17	16	15	17	15	15

FGMRES iterations for model problem (in $\widehat{\mathbf{H}}_{1/2}$ *we dumped* $\mathbf{L}_0$*)* .

Numerical results: an other problem

			Linear			Quadratic		
#dom	n	m	$\nu = 1$	$\nu = 0.1$	$\nu = 0.01$	$\nu = 1$	$\nu = 0.1$	$\nu = 0.01$
4	45,377	449	10	12	21	12	13	20
	180,865	897	11	11	20	12	13	19
	722,177	1793	11	11	19	12	12	18
16	45,953	1149	12	17	37	13	17	35
	183,041	2301	13	17	35	13	16	32
	730,625	4605	12	15	32	12	15	30
64	66,049	3549	16	22	55	17	21	51
	263,169	7133	17	22	52	16	20	46
	1,050,625	14,301	15	19	47	16	19	43

FGMRES iterations for 2nd model problem

Laplace-Beltrami

We can extend everything to an interface that is the union of manifolds $\mathfrak{m}_k$ by using the Lapace-Beltrami operator and interpolating between $L^2(\Gamma)$ and $H^1_{\partial\Omega}(\Gamma)$ with the norm

$$\|u\|_{H^1_{\partial\Omega}(\Gamma)} = \left(\sum_{k=1}^{K} \|u_k\|^2_{H^1_{\partial\Omega}(\mathfrak{m}_k)} \right)^{1/2}.$$

using $H^1_0(\mathfrak{m}_k)$ with

$$|v|^2_{H^1_0(\mathfrak{m}_k)} = \int_{\mathfrak{m}_k} \left| \nabla^k_\Gamma v \right|^2 \mathrm{d}s(\mathfrak{m}_k)$$

where ∇^k_Γ denote the tangential gradient of v with respect to $\mathfrak{m}_k$

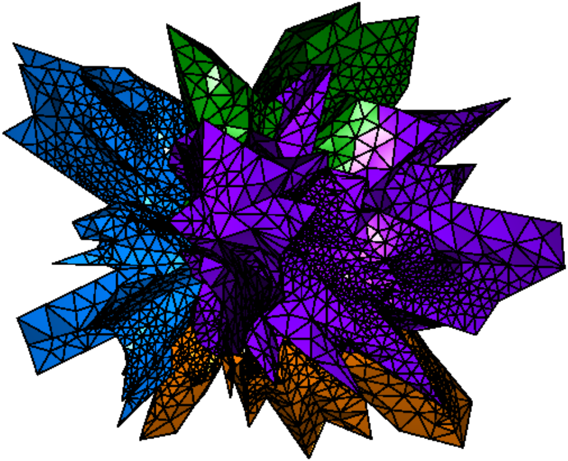
$$\nabla^k_\Gamma v(\mathbf{x}) := \nabla v(\mathbf{x}) - \mathbf{n}_k(\mathbf{x})(\mathbf{n}_k(\mathbf{x}) \cdot \nabla v(\mathbf{x})),$$

where $\mathbf{n}_k(\mathbf{x})$ is the normal to $\mathfrak{m}_k$ at $\mathbf{x}$.

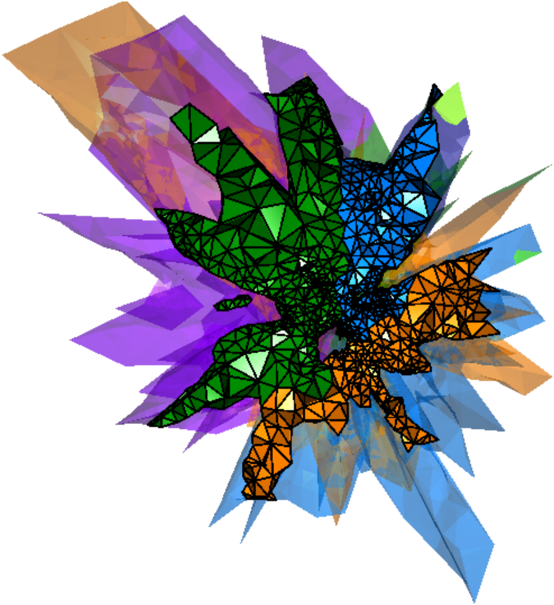
A. , Kourounis, Loghin IMA J. Num. Anal. 2012

Other Domains: CRYSTAL

A. , Kourounis, Loghin IMA J. Num. Anal. 2012



Other Domains: CRYSTAL

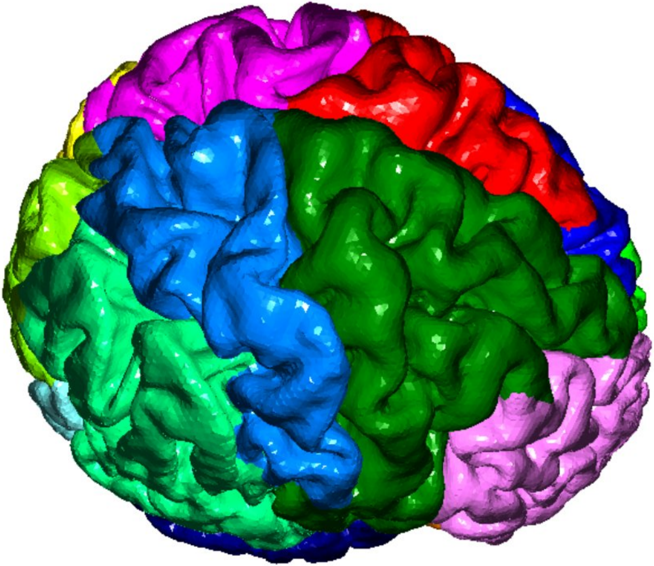


Other Domains: CRYSTAL

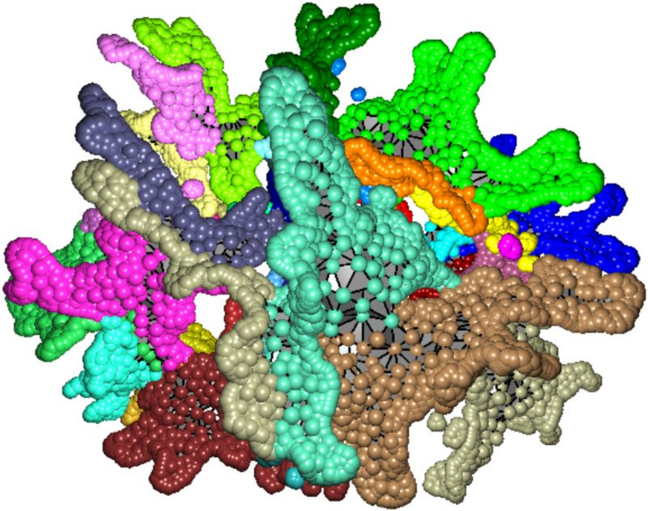
	$(L, M)_{1/2}$			$(L, I)_{1/2}$		
n	$N = 16$	$N = 64$	$N = 256$	$N = 16$	$N = 64$	$N = 256$
240,832	17	24	31	110	104	141
2,521,753	21	25	29	28	71	65

Iterations for the crystal problem with and without the mass matrix.

Other Domains: “BRAIN”



Other Domains: “BRAIN”



Other Domains: "BRAIN"

	$n = 5120357$	$n = 25973106$
$N = 1024$	$n_b = 679160$ it $= 22$	$n_b = 2067967$ it $= 22$
$N = 2048$	$n_b = 895170$ it $= 22$	$n_b = 2737064$ it $= 23$
$N = 4096$	$n_b = 1172815$ it $= 24$	$n_b = 3602083$ it $= 23$

Results for reaction-diffusion PDE on Brain (N number of subdomains, n_b number of nodes in interface, it FGMRES iteration number, and $\theta = 0.7$).

Two Domains

Various analyses going back to the 1980s considered the discrete Steklov-Poincaré operator arising from two-domain decomposition methods for linear systems resulting from finite element or finite difference discretizations of model diffusion or, more generally, scalar elliptic problems Dryja 82 and 84, Golub 84, Bjorstad 86, Chan 87.

The discrete anisotropic Steklov-Poincaré operator

Let $a \in \mathbb{R}_+$ and let $\Omega = (-1,1) \times (-a,a)$. Consider the finite element solution of

$$\begin{cases} -\Delta u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

using a partition of Ω into two equal domains separated by a horizontal boundary $\Gamma = \{(x,0) : -1 \le x \le 1\}$ and subdivided uniformly into equal triangles with sides $h_x = 2/(n+2), h_y = 2a/(n+2)$ with $n = 2m+1, m \in N$. Let

$$T_k := \text{tridiag}[-1, 2, -1] \in \mathbb{R}^{k \times k}$$

denote a scaled FEM discretisation of $-\mathrm{d}^2/\mathrm{d}x^2$ on a mesh with k interior points and let $I_k \in \mathbb{R}^{k \times k}$ denote the identity matrix. With this notation, the 2D discrete Laplacian matrix $L \in \mathbb{R}^{n^2 \times n^2}$ is given by

$$L = \frac{h_x}{h_y} T_n \otimes I_n + \frac{h_y}{h_x} I_n \otimes T_n = \frac{1}{a} T_n \otimes I_n + a I_n \otimes T_n := L_a.$$

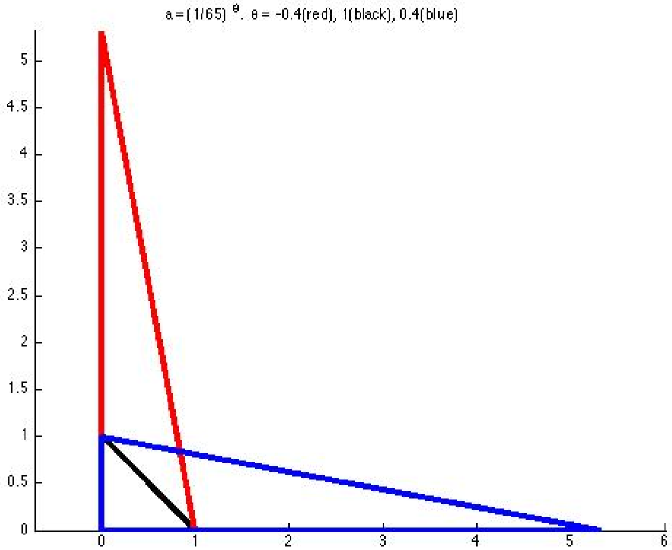
remark

The above expression corresponds also to a uniform discretisation of the anisotropic diffusion problem

$$\begin{cases} -\left(a\partial_{xx} + \frac{1}{a}\partial_{yy}\right) u &= g \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is any square.

remark



Eigenvalue analysis

Let us consider the Schur complement of L_a for a general value of $a > 0$. Let $L_{m,n}$ denote the Laplacian corresponding to a discretisation with m interior nodes in the x-direction, respectively, n in the y-direction. Dropping the subscript a for now, the Laplacian for the original problem is $L = L_{n,n}$ where

$$L_{n,n} = \frac{1}{a}\begin{pmatrix} aL_{m,n} & & -e_m \otimes I_n \\ & aL_{m,n} & -e_1 \otimes I_n \\ -e_m^T \otimes I_n & -e_1^T \otimes I_n & aT \end{pmatrix}$$

where

$$L_{m,n} = \frac{1}{a}T_m \otimes I_n + aI_m \otimes T_n$$

and

$$T = \frac{2}{a}I_n + aT_n.$$

Eigenvalue analysis

Then the Schur complement is (using the block Toeplitz character of $L_{m,n}$)

$$S = T - \frac{1}{a^2} \sum_{i \in \{1,m\}} (e_i^T \otimes I_n) L_{m,n}^{-1} (e_i \otimes I_n) = T - \frac{2}{a^2} (e_m^T \otimes I_n) L_{m,n}^{-1} (e_m \otimes I_n).$$

Eigenvalue analysis

Let now $T_k \in \mathbb{R}^{k\times k}$ have the eigenvalue decomposition

$$T_k = V_k D_k V_k^T$$

with

$$(D_k)_{ii} = 2\left(1 - \cos\frac{i\pi}{k+1}\right) =: \mu_i^{(k)},$$

and

$$V_k = [v_1, \ldots, v_k], \quad (v_j)_i = \sqrt{\frac{2}{k+1}}\sin\frac{ij\pi}{k+1}, \quad i,j = 1,\ldots,k.$$

Let $T_m = V_m D_m V_m^T$, $T_n = V_n D_n V_n^T$; with this notation, the inverse of $L_{m,n}$ is given by

$$\begin{aligned} L_{m,n}^{-1} &= (V_m \otimes V_n)\left[aI_m \otimes D_n + \frac{1}{a}D_m \otimes I_n\right]^{-1}(V_m \otimes V_n)^T \\ &:= (V_m \otimes V_n)\, D_{m,n}^{-1}\,(V_m \otimes V_n)^T \end{aligned}$$

Eigenvalue analysis

Now,

$$(V_m \otimes V_n)(e_m \otimes I_n) = (V_m e_m) \otimes V_n = v_m^T \otimes V_n$$

and similarly

$$(e_m^T \otimes I_n)(V_m \otimes V_n)^T = v_m \otimes V_n^T.$$

Let $(D_{m,n})_i \in \mathbb{R}^{n \times n}$ denote the ith diagonal block of $D_{m,n} \in \mathbb{R}^{mn \times mn}$ with $i = 1, \ldots, m$. It follows that

$$(D_{m,n})_i = aD_n + \frac{1}{a}(D_m)_{ii} I_n.$$

Eigenvalue analysis

Hence,

$$
\begin{aligned}
S &= T - \frac{2}{a^2}(v_m^T \otimes V_n) D_{m,n}^{-1} \left(v_m \otimes V_n^T \right) \\
&= V_n \left[\frac{2}{a} I_n + a D_n \right] V_n^T - \frac{2}{a^2} \sum_{i=1}^{m} (v_m)_i^2 V_n (D_{m,n}^{-1})_i V_n^T \\
&= V_n \left[\frac{2}{a} I_n + a D_n - \frac{2}{a^2} \sum_{i=1}^{m} (v_m)_i^2 (D_{m,n}^{-1})_i \right] V_n^T \\
&= V_n \left[\frac{2}{a} I_n + a D_n - \frac{2}{a^2} \sum_{i=1}^{m} (v_m)_i^2 \left(a D_n + \frac{1}{a} (D_m)_{ii} I_n \right)^{-1} \right] V_n^T .
\end{aligned}
$$

Eigenvalue analysis

Thus, the eigenvalues of S are

$$\lambda_j(S) = a\mu_j^{(n)} + \frac{2}{a}\left[1 - \sum_{i=1}^{m} \frac{(v_m)_i^2}{a^2\mu_j^{(n)} + \mu_i^{(m)}}\right], \quad (j = 1, \dots, n).$$

Note that labeling the eigenvalues of T_n in increasing order, i.e., $\mu_1^{(n)} \leq \cdots \leq \mu_n^{(n)}$, yields a labeling with a similar ordering for the eigenvalues of S: $\lambda_1(S) \leq \cdots \leq \lambda_n(S)$.

Eigenvalue analysis

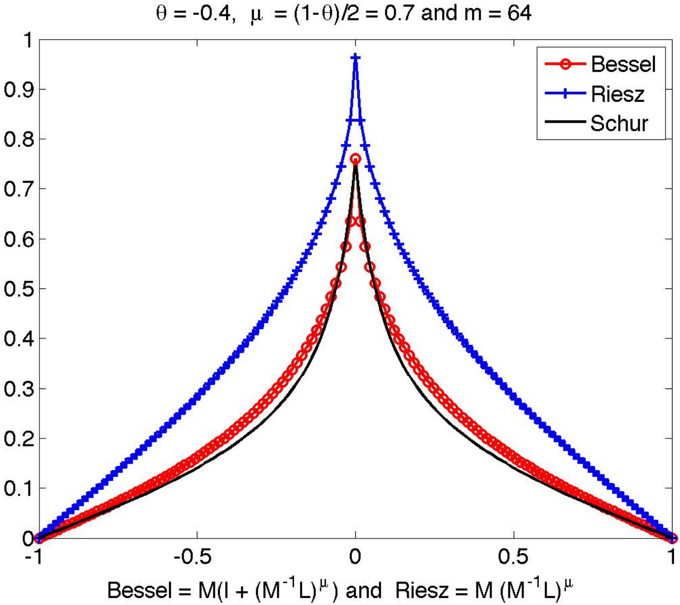
Let $h = \dfrac{1}{m+1}$. Thus, if $a = \mathcal{O}(h^\theta)$ the resulting asymptotics for the extreme eigenvalues of S are included below:

$$\lambda_1(S) = \begin{cases} O(h^{2+\theta}), & \theta \leq -1 \\ O(h), & \theta \in (-1, 0] \\ O(h^{1-\theta}), & \theta > 0. \end{cases} , \quad \lambda_n(S) = \begin{cases} O(h^{\theta}), & \theta \leq 0 \\ O(1), & \theta \in (0, 1] \\ O(h^{1-\theta}), & \theta > 1 \end{cases} .$$

Hence,

$$\kappa_2(S) \sim \begin{cases} O(h^{-2}), & \theta < -1, \\ O(h^{\theta-1}), & \theta \in [-1, 1], \\ O(1), & \theta > 1. \end{cases}$$

Numerical examples: $\mathbf{S}^{-1}\delta(0)$, $\mathbf{H}_{00}^{-1/2}\delta(0)$ (Bessel, Riesz)



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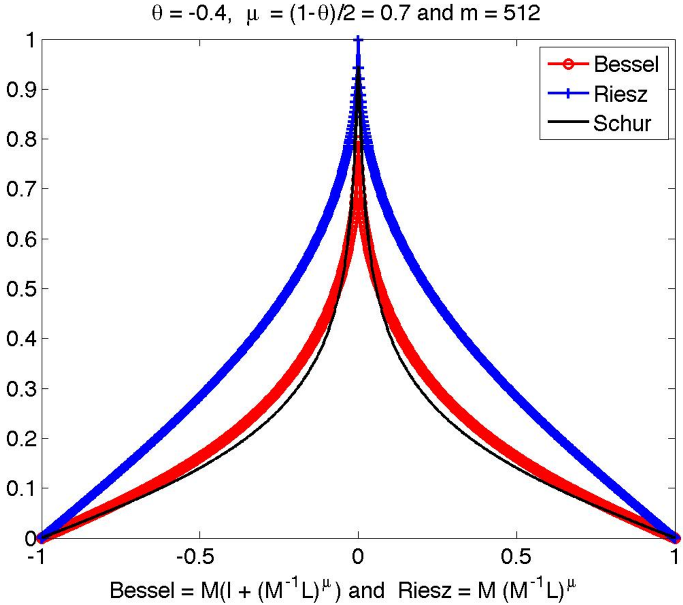
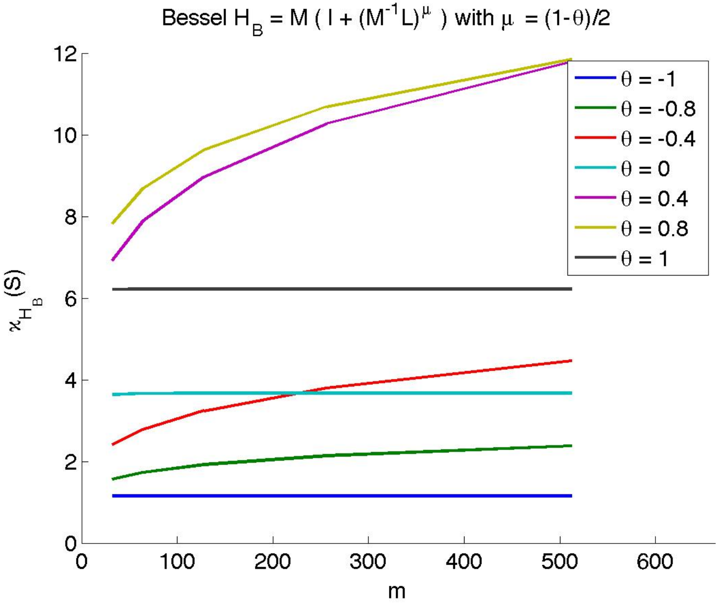
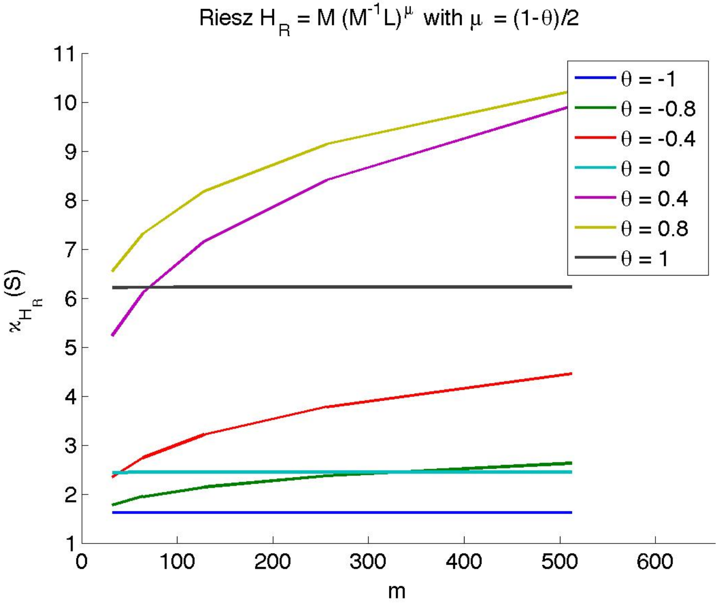


$\kappa_{H_B} = \kappa_2(SH_B^{-1})$



$\kappa_{H_R} = \kappa_2(SH_R^{-1})$



Eigenvalue analysis: proof sketch

Lemma

The following holds:

$$\int \frac{\sin^2(x)}{\varrho^2 + \sin^2(\frac{x}{2})} \mathrm{d}x =$$
$$2\Big[2\varrho^2 x + x + \sin(x) - 4\varrho\sqrt{\varrho^2+1}\tan^{-1}\Big(\frac{\sqrt{\varrho^2+1}\tan(\frac{x}{2})}{\varrho}\Big)\Big].$$

$$(\pi h)\sum_{i=1}^{m} \frac{\sin^2(\pi i h)}{\varrho_j^2 + \sin^2(\frac{\pi}{2} i h)} = \int_0^{\pi} \frac{\sin^2(x)}{\varrho_j^2 + \sin^2(\frac{x}{2})} \mathrm{d}x + \frac{\mathcal{E}}{\pi}$$

where $|\mathcal{E}| \approx 2h$*.*

Eigenvalue analysis: proof sketch

Let now $\varrho_j^2 := \frac{1}{4}a^2\mu_j^{(n)}$. We have

Eigenvalue analysis: proof sketch

$$\begin{aligned}
\lambda_j(S) &= \frac{1}{a}\left[4\varrho_j^2 + 2 - 2\sum_{i=1}^{m} \frac{2h\sin^2(i\pi h)}{a^2\mu_j^{(n)} + 4\sin^2(\frac{\pi}{2}ih)}\right] \\
&= \frac{1}{a}\left[4\varrho_j^2 + 2 - \frac{\pi h}{\pi}\sum_{i=1}^{m} \frac{\sin^2(\pi ih)}{\varrho_j^2 + \sin^2(\frac{\pi}{2}ih)}\right] \\
&= \frac{1}{a}\left[4\varrho_j^2 + 2 - \frac{1}{\pi}\int_0^{\pi} \frac{\sin^2(x)}{\varrho_j^2 + \sin^2(\frac{x}{2})}\mathrm{d}x - \frac{\mathcal{E}}{\pi}\right] \\
&= \frac{1}{a}\left[4\varrho_j^2 + 2 - \frac{2}{\pi}\left(2\pi\varrho_j^2 + \pi - 2\pi\varrho_j\sqrt{\varrho_j^2+1}\right) - \frac{\mathcal{E}}{\pi}\right] \\
&= \frac{1}{a}4\varrho_j\sqrt{\varrho_j^2+1} - \frac{\mathcal{E}}{\pi a}.
\end{aligned}$$

where $|\mathcal{E}| \approx 2h$ and, for $a = \mathcal{O}(h^{\theta})$, $\dfrac{\mathcal{E}}{\pi a} = \mathcal{O}(h^{1-\theta})$:

Reaction-Diffusion Systems

Rodrigue Kammogne, D. Loghin Proceed. DD 2012

Rodrigue Kammogne, D. Loghin Tech. Rep. Univ. Birmingham $\Omega \subset \mathbf{R}^2$

$$\begin{cases} -\mathbf{D}\Delta\mathbf{u} + \mathbf{M}\mathbf{u} = \mathbf{f} & \text{in } \Omega \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega \end{cases}$$

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$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \ \mathbf{M} = \begin{bmatrix} \alpha_(x,y) & \beta_1(x,y) \\ \beta_2(x,y) & \alpha_2(x,y) \end{bmatrix}, \ \mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \ \mathbf{D} = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \text{ (SPD).}$$

$\mathbf{f} \in L^2(\Omega)$ and $\mathbf{M}$ satisfies

$$0 < \gamma_{min} < \frac{\boldsymbol{\xi}^T \mathbf{M} \boldsymbol{\xi}}{\boldsymbol{\xi}^T \boldsymbol{\xi}} \ \forall \boldsymbol{\xi} \in \mathrm{R}^2 \setminus \{\} \ ; \text{ and} ||\mathbf{M}|| < \gamma_{max}.$$

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$$\alpha_1 = \begin{cases} 1 & \text{if } x^2 + y^2 < 1/4 \\ 100 & \text{otherwise} \end{cases} ; \alpha_2 = \begin{cases} 100 & \text{if } x^2 + y^2 < 1/4 \\ 1 & \text{otherwise} \end{cases}$$

$$\beta_1 = \begin{cases} 0.1 & \text{if } x^2 + y^2 < 1/4 \\ 1 & \text{otherwise} \end{cases} \quad \beta_2 = \begin{cases} 1 & \text{if } x^2 + y^2 < 1/4 \\ 0.1 & \text{otherwise} \end{cases}$$

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$d_1 = 1$, $d_2 = 0.1$	
domains=	4 16 64
size=8450	18 24 28
33282	19 25 28
132098	20 26 28

Reaction-Diffusion Systems

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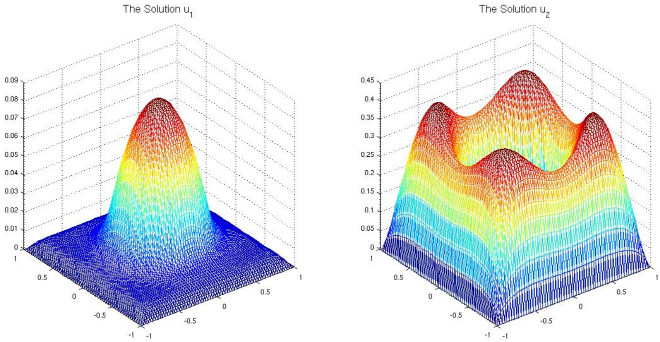

Other applications: Mathematical Finance

L.Silvestre Communications on Pure and Applied Mathematics 2007
Luis A. Caffarelli, Sandro Salsa, Luis Silvestre Invent. math. 2008

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$$\begin{cases} u(x) \geq \phi(x) \text{ in } \mathbf{R}^n, \\ \lambda u + (-\Delta)^s u \geq 0 \text{ in } \mathbf{R}^n, \\ \lambda u + (-\Delta)^s u(x) = 0 \text{ for those } \ x \text{ s.t. } \ u(x) > \phi(x) \\ \lim_{|x| \to +\infty} u(x) = 0. \end{cases}$$

Other applications: Quasi-Geostrophic Equation

L.Silvestre Ann. I. H. Poincare (2010)
L. Caffarelli, L. Silvestre, Comm. Partial Differential Equations (2007)
L. Caffarelli, A. Vasseur, Ann. of Math., (2012)
P. Constantin, J. Wu, Ann. I. H. Poincare Anal. Non Lin. (2008), (2009)
P. Constantin, J. Wu, SIAM J. Math. Anal. (1999)

Other applications: Quasi-Geostrophic Equation

$\theta : \mathbf{R}^2 \times [0, +\infty) \to \mathbf{R}$

$$\partial_t \theta(x,t) + w \cdot \nabla\theta(x,t) + (-\Delta)^{\alpha/2}\theta(x,t) = 0, \qquad \theta(x,0) = \theta_0$$

and

$$w = (R_2\theta, R_1\theta)$$

where R_i are the Riesz transforms

$$R_i\theta(x) = cPV \int_{\mathbf{R}^2} \frac{(y_i - x_i)\theta(y)}{|y-x|^3} dy.$$

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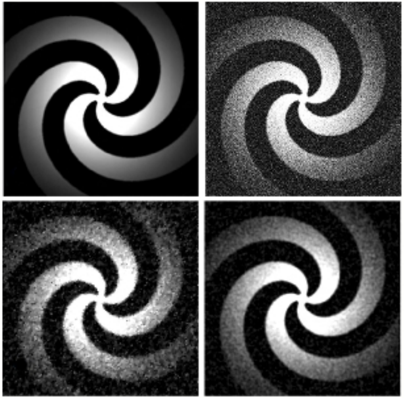
Summary

- Interpolation spaces produce dense matrices i.e. non-local operators BUT we can compute everything using SPARSE LINEAR ALGEBRA
- Interpolation spaces are not only useful in DD
- Link with integro-differential operator such as Riemann-Liouville fractional derivative (M.Riesz 1938,1949).
- In modelling complex phenomena the use of non-local operators is a new promising subject attracting increasing attention.
- Other areas of application that are worth to mention include: BEM and image processing (filtering):

top-left: original top-right:noised



bottom-left: $\min\left\{\int_\Omega |\nabla u(x)|\mathrm{d}x + 1/50\int_\Omega (u_0(x) - u(x))^2 \mathrm{d}x\right\}$

Pascal Getreuer (2007)

bottom-right: $\min\left\{||u||^2_{1/2} + 1/50\int_\Omega (u_0(x) - u(x))^2 \mathrm{d}x\right\}$