Preconditioning of iterative methods for solving sequences of linear systems

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The talk will be devoted to the problem of constructing preconditioners for solving sequences of systems of linear algebraic equations. Such sequences arise in many applications like computational fluid dynamics and structural mechanics, numerical optimization as well as in solving non-PDE problems. For example, solving discretized nonlinear equations $F(x) = 0$ by the Newton or a Broyden-type method for $F : \mathbb{R}^n \to \mathbb{R}^n$ leads to a sequence of problems

$$J(x_k)(x_{k+1} - x_k) = -F(x_k), \tag{1}$$

for $k = 1, \ldots$ where $J(x_k)$ is the Jacobian evaluated in a current iteration x_k or its approximation [6], [7]. The solution of such systems is the main bottleneck in many applications mentioned above. Apart from improving solvers of individual systems there is a strong need for reduction of costs for solving more subsequent linear systems by inexact solvers and by sharing some of the computational effort. In the following we will consider solving the linear systems by a preconditioned iterative method.

One way to reduce the overall cost for solving all systems is to skip some Jacobian evaluations as in the Shamanskii combination of the Newton's method and the Newton-chord method [3], [12]. In this way we get more systems with identical matrices. If an iterative method is used, decrease of work for Jacobian and function evaluations and possible reuse of preconditioners is typically balanced with increase of work spent in more iterations. Another important special case is solving systems with multiple right-hand sides, see, e.g., [10], [14], [5], [13]. Approaches that attempt to recycle Krylov subspaces which are only approximately equal were studied in [11] and [8]. A further possibility to improve a sequence of runs of a preconditioned iterative method for a sequence of systems is to update previous preconditioners by Broyden updates.

Our approach is based on subsequent updates of matrix approximations in a given sequence of linear systems. The approximations are, for example, used to precondition iterative methods. Basically, there are two main strategies to reuse the matrix approximations in a sequence. Consider two matrices of a sequence denoted as A and B. The first strategy uses only the sparsity pattern of an approximation to A to compute an approximation to B. The second strategy considers the whole approximation to A including its numerical values to construct an approximation to B.

In the first strategy, the sparsity pattern used to compute B can be obtained in different possible ways. It can be chosen as the pattern of a preconditioner for A, or it can be the pattern of the sparsified matrix A. If the system matrices are available only implicitly as in a matrix-free environment, the computation of an approximation of B and matrices of further subsequent linear systems can be determined by a smaller number of matrix-vector products than in their individual evaluation. Positions of nonzero entries in the vectors used for computing the approximations are determined by a graph coloring algorithm [4].

The first strategy does not lead to significant savings in cases of subsequent linearized PDEs with simple stencils using approximations with similar simple patterns. If the system matrices are explicitly available (e.g., from finite differencing), we can use the second strategy and update the approximation to A directly, based on the matrix $B - A$. In our work, we generalized diagonal updates from [1] and [2] (motivated by PDEs with time-dependent evolution), see also [9], to more general updates of a factorized preconditioner (motivated also by nonlinear convection-diffusion problems). Another possibility which we will consider is to construct the preconditioner for B as a product form update of the approximation to A.

For direct methods there are many techniques for updating sparse factors which were mainly developed for the simplex method of linear programming. Our case is different since we are interested only in cheap approximate updates. One of the final goals of this ongoing work is to capture the influence of large entries in the difference between A and B by simple updates of the approximation to A.

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