New Evaluation Index of Orderings in Incomplete Factorization Preconditioning

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Abstract— **It is well known that ordering of unknowns greatly affects convergence in Incomplete LU (ILU) factorization preconditioned iterative methods. The authors recently proposed a simple evaluation way for orderings in ILU preconditioning. The evaluation index, which has a simple relationship with a norm of a remainder matrix, is easily computed without additional memory requirement. The computational cost of the index is about the same as that of ILU factorization. The effectiveness of the method is examined by numerical tests using coefficient matrix data obtained from Matrix Market.**

I. Introduction

The ILU factorization preconditioning is one of the most popular preconditioning techniques for Krylov subspace iterative methods [1]. In this preconditioning method, it is well-known that the preconditioning effect is significantly affected by the ordering of the unknowns. Furthermore, since reordering technique has been a well-known parallelization way of ILU preconditioning [2], the relationship between ordering and convergence has been intensively investigated by several researchers [3], [4].

Most of previous investigations on orderings were mainly performed in finite difference analyses. In these studies, the early and important work was done by Duff and Meurant [3]. They indicated the significant effect of orderings on convergence in ILU-preconditioned iterative solvers, and then proposed a use of the norm of a remainder matrix for evaluation of orderings. The remainder matrix $\boldsymbol{R}$ is given by $\boldsymbol{R} = \boldsymbol{M} - \boldsymbol{A}$, where $\boldsymbol{M}$ is the preconditioning matrix and $\boldsymbol{A}$ is the coefficient matrix. While the evaluation method has been confirmed by various numerical tests, the remainder matrix has been commonly used as a tool for examining convergence in subsequent research works [5], [6]. Following Duff and Meurant's work, Doi, Lichnewsky and Washio performed a series of works paying a special attention to "incompatible nodes" [5], [7]. In this research, they proposed a evaluation index for orderings, which is called "incompatibility ratio". The incompatibility ratio, which has a unique value for a fixed ordering, is easily calculated. Next, the authors recently proposed a new evaluation index for orderings, which is called "S.R.I. (Simple Remainder Index)" [8]. This evaluation index estimates the effects of all nodes including non-incompatible nodes, which are not evaluated in incompatibility ratio.

In contrast to studies on ordering of nodes in finite difference analyses, the effect of orderings in unstructured analyses has been rarely discussed. In the unstructured analysis, the effect of individual property of each problem is not trivial. Therefore, it is not easy to evaluate orderings in simple way. We have, however, tried to propose a evaluation method for orderings in unstructured analyses by permitting a small range of errors. We used the result of Duff and Meurant's research for a base of our method, and we finally proposed a new evaluation index "P.R.I. (Precise Remainder Index)" [8]. This index has a simple relationship with the remainder matrix norm under limited conditions. Both of computational cost and memory requirement for computing the index are much smaller than those for calculating the Frobenius norm of the remainder matrix. In this paper, we introduce the two evaluation indices and show the numerical results obtained in unstructured analyses.

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II. New Evaluation Index for Orderings

In order to construct a new evaluation index, we consider the remainder matrix in ILU preconditioning. Fig. 1 shows the algorithm for computing the remainder matrix $\boldsymbol{R}$ associated with ILU factorization. Here, we focus on the update operations of $\boldsymbol{R}$ in the algorithm. The S.R.I. is defined as the total count of the updates operations. In a finite difference analysis based on the 7-point discretization scheme, the S.R.I. value is simply given as follows:

$$I_{rs} = \sum {}_l C_2, \tag{1}$$

where l denotes the number of outgoing directed graphs of each node in ordering graph [7].

Next, we propose a more precise evaluation index (P.R.I.) in which effects of problem parameters are taken into account. In the P.R.I. evaluation, we use a summation of the absolute values of the updates of the remainder matrix. The algorithm of calculating the P.R.I. value I_{rp} is shown in Fig. 1. The additional memory requirement for the calculation is for only one variable. When the coefficient matrix $\boldsymbol{A}$ is symmetric and has the same signs in all diagonal entries, a simple relationship between the remainder matrix norm and the P.R.I. is given as follows:

$$\|\boldsymbol{R}\|_A = I_{rp}, \tag{2}$$

where $\|\boldsymbol{R}\|_A$ is defined as a sum of the absolute values of all entries of $\boldsymbol{R}$, and is given by

$$\|\boldsymbol{R}\|_A = \sum_{IJ} |r_{IJ}|. \tag{3}$$

The operator $\|\cdot\|_A$ satisfies the definition of the matrix norm shown in [8].

$\boldsymbol{R} = \boldsymbol{O}$
$I_{rp} = 0$
$for\ I = 1\ to\ n-1$
 $for\ J = I+1\ to\ n$
  $for\ K = I+1\ to\ n$
   $if\ \tilde{a}_{J,I} \neq 0\ \&\ \tilde{a}_{I,K} \neq 0\ \&\ \tilde{a}_{J,K} \neq 0\ then$
    $\tilde{a}_{J,K} = \tilde{a}_{J,K} - \tilde{a}_{J,I} * \tilde{a}_{I,K} / \tilde{a}_{I,I}$
     // (ILU factorization)
   endif
   $if\ \tilde{a}_{J,I} \neq 0\ \&\ \tilde{a}_{I,K} \neq 0\ \&\ \tilde{a}_{J,K} = 0\ then$
    $r_{J,K} = r_{J,K} + \tilde{a}_{J,I} * \tilde{a}_{I,K} / \tilde{a}_{I,I}$
     // (Computation of $\boldsymbol{R}$)
    $I_{rp} = I_{rp} + |\tilde{a}_{J,I} * \tilde{a}_{I,K} / \tilde{a}_{I,I}|$
     // (Computation of P.R.I.)
   endif
  end for
 end for
end for

Fig.1 Algorithm of ILU factorization with computing remainder matrix and P.R.I.

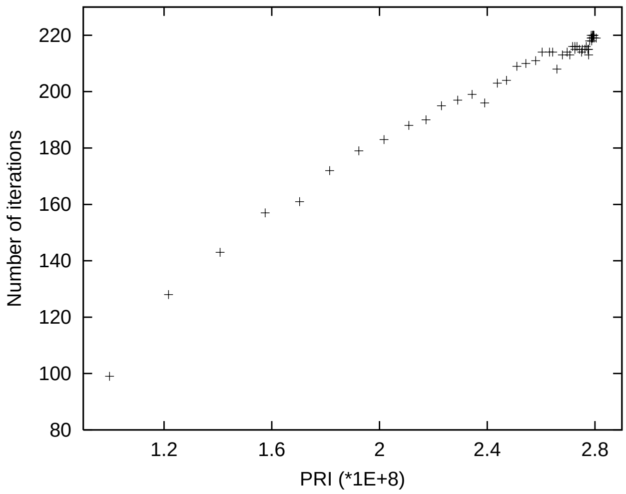

Fig. 2. Relationship between number of iterations and P.R.I. (S1RMQ4M1 from Matrix Market)

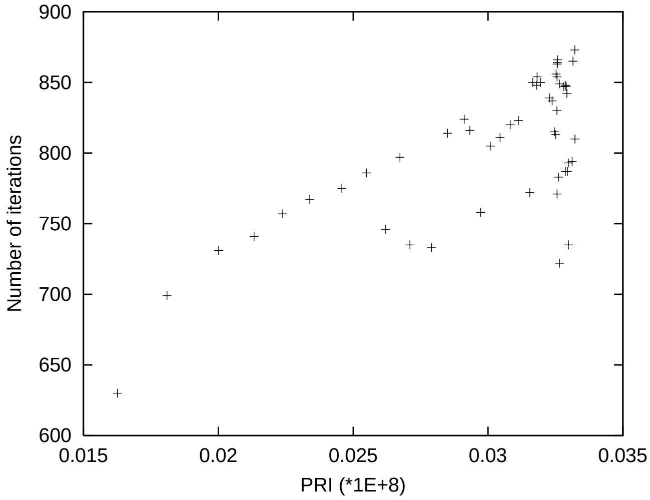

Fig. 3. Relationship between number of iterations and P.R.I. (S3RMT3M1 from Matrix Market)

III. Numerical Results

In this extended abstract, we show two numerical results using coefficient matrix data downloaded from Matrix Market [10]. Since the coefficient matrices are symmetric, the ICCG (Incomplete Cholesky Conjugate Gradient) method is used. The convergence criterion of the iterative method is given by $||\boldsymbol{r}||_2/||\boldsymbol{b}||_2 < 10^{-7}$ where $\boldsymbol{r}$ and $\boldsymbol{b}$ are the residual vector and the right-hand side vector, respectively. The correlation between the P.R.I. and convergence is examined by using 50 random orderings.

Fig. 2 shows the relationship between the P.R.I. and the number of iterations in the numerical test for CYLSHELL S1RMQ4M1 data from Matrix Market. The coefficient matrix arises from a finite element structure analysis with shell type elements. In this case, our evaluation index gives good estimates of convergence. The correlation coefficient reaches 0.86.

Fig. 3 depicts the numerical results using CYLSHELL S3RMT3M1 data from Matrix Market. Since the matrix is derived from a finite element analysis with an unstructured triangular mesh, many spikes are observed in the convergence behavior of the residual. Therefore, the convergence estimate by the P.R.I. includes some errors especially when orderings have high P.R.I. values. However, a high correlation coefficient value, which is over 0.8, is obtained in the whole random ordering test.

IV. Conclusion

The present paper introduces the convergence evaluation indices in ILU preconditioned iterative solvers. The evaluation indices are easily computed without additional memory requirement. The effectiveness of the method is examined by numerical tests using coefficient matrix data from Matrix Market. Other numerical results will be presented in the conference.

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