Multilevel Algorithms for linear ordering problems

Ilya Safro
Joint work with Dorit Ron and Achi Brandt

Faculty of Mathematics and Computer Science
The Weizmann Institute of Science
POB 26, Rehovot 76100, ISRAEL

Extended abstract

The class of linear ordering problems contains a set of combinatorial optimization problems which deal with minimization of different functionals that are involved by embedding of the set of graph vertices to $[1..n]$. This class contains following graph (or matrix) layout problems : minimum p-sum, workbound, wavefront, envelope, etc. They appear in many applications for solving problems in the large sparse matrix computation, such as finding the minimum linear arrangement [10] or the bandwidth [9]. Some of them are also closely related to the problem of calculating the envelope size of a symmetric matrix or more precisely, to the amount of work needed in the Cholesky factorization of such a matrix [6]. These problems may be motivated as a model used in VLSI design [4] and may be used in several biological applications, graph drawing and other fields (see [5]). Commonly for general graphs (or matrices) these problems are NP-hard and their decision versions are NP-complete.

Since these problems have a practical significance, many heuristic algorithms were developed in order to achieve near optimal solution. Among the most successful are Spectral Sequencing, Optimally Oriented Decomposition Tree, Multilevel based, Simulated Annealing, Genetic Hillclimbing and other. Some of these algorithms have proven themselves superior in solution quality while other in execution time.

One of the most popular strategies for approximating these problems is a spectral approach. According to Hall in [7] the eigenvector v_2 which corresponds to the second smallest eigenvalue of the Laplacian of the graph (provided the graph is connected), is the best nontrivial solution to the unrestricted form of the minimum 2-sum problem (subject to some normalization of the solution). Arrangement of the graph vertices according to v_2 forms the basis step of the spectral approach. The fact of optimality of this solution for non-integer version of the minimum 2-sum problem was used as a basic solution for many ordering problems like the minimum linear arrangement, partitioning, envelope reduction of sparse matrices, etc.

In this talk we present a strategy for development multilevel algorithms especially for linear ordering problems and similarly for some other functionals (like partitioning). In addition, we reinforce our strategy by new concrete algorithms for the minimum p-sum problem, and for the workbound reduction problem. This strategy is based on the Algebraic MultiGrid scheme (AMG) [1, 13]. Recently, using our strategy we have successfully developed and tested the multilevel algorithms for the minimum linear arrangement problem [10] and minimum 2-sum problem [11].

The main objective of a multilevel based algorithm is to create a hierarchy of problems, each representing the original problem, but with fewer degrees of freedom. General multilevel techniques have been successfully applied to various areas of science (e.g. physics, chemistry, engineering, etc.) [2, 3]. AMG methods were originally developed for solving linear systems of equations resulting from the discretization of partial differential equations. Lately they have been applied to various other fields, yielding for example novel methods for image segmentation [12] and for the linear arrangement problem [10]. In the context of graphs it is the Laplacian matrix that represents the related set of equations. The main difference between our approach to other multilevel approaches (related to various graph optimization problems) is the coarsening scheme. While the previous approaches may be viewed as *strict* aggregation process, the AMG coarsening is actually a *weighted* aggregation : each node may be divided into $fractions$, and different fractions belong to different aggregates. This enables more freedom in solving the coarser levels and avoids making hardened local decisions, such as edge contractions, before accumulating the relevant global information.

The disaggregation follows by projecting to a finer level the final arrangement obtained on a coarser level. This initial fine level arrangement is being further improved by applying various local reordering methods. In this talk we would like to introduce an algorithm for the strict minimization, called Windows Minimization, which is based on the *simultaneous* reordering of several vertices. Then our postprocessing is intensified by Simulated Annealing (SA) [8] which is a general method to escape local minima. In the multilevel framework SA is aimed at searching only for *local* changes that guarantees the preservation of large-scale solution features inherited from coarser levels.

We compared the results obtained by our multilevel algorithms with many previously described algorithms. In addition to our previous results for the minimum linear arrangement problem and minimum 2-sum problem, we would like to present the results of bandwidth problem and workbound problem. Our experimental results show that the Algebraic Multilevel framework can be used for the linear ordering problems to obtain high quality results in linear time.

We can refer the reader to articles [10] and [11] to receive the full list of references.

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