ALGEBRAIC CONNECTIVITY OF FINITE-ELEMENT HYPERGRAPHS

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1. INTRODUCTION

This paper generalizes results that relate the connectivity of a weighted graph to the smallest nonzero eigenvalue of the graph's Laplacian matrix. We generalize these results to hypergraphs with vector-valued vertices and matrix-valued edges. Our definitions of hypergraphs and their connectivity are designed to model finite-element meshes.

The physical interpretation of our results is as follows. We say that a structure modeled by a finite-element model is *weak* if its vertices can be displaced by a unit perturbation orthogonal to its rigid motions with only a small investment of energy (or work, virtual work, etc.) Given a partition of the vertices of a finite-element model, a displacement is a *cut displacement* if under that displacement, energy is expanded only in elements incident on vertices in both subsets of the partition. Our goal is to show that a structure is weak if and only if it has a weak cut. Our analysis is purely algebraic and can be applied to finite-elements models from several application domains.

Our ultimate objective is to find ways to construct so-called support preconditioners ([2],[3]) for finite-element matrices. The analysis of existing support preconditioners is based on relating generalized eigenvalues of the Laplacians to complex combinatorial properties of pairs of their graphs. We believe that studying the simpler relationship of a simple eigenvalue to the connectivity of a hypergraph will eventually help us develop support preconditioners for hypergraphs.

2. FINITE-ELEMENT HYPERGRAPHS

Hypergraphs generalize graphs by allowing an edge to connect more than two vertices. Let $G = (V, E)$ be a hypergraph, where $V = \{1, 2, \ldots, n\}$ is the set of vertices and E is the set of hyperedges. A hyperedge $e \in E$ is defined by a set of vertices incident to it, $e \subseteq V$. In this paper we do not allow loops (hyperedges that contain a vertex multiple times).

We associate a d-dimensional real vector x_v with each vertex $v \in V$; These vectors induce a nd-dimensional vector which we denote by $x = (x)_{v_\ell}$, where $v_\ell = (v - 1)d + \ell$, $1 \le \ell \le d$, and $1 \le v \le n$. Given a vector v, we define the characteristic projection matrix $P^{(v)} \in \mathbb{R}^{nd \times nd}$ by $(P^{(v)})_{ij} = 1$ if and only if $(v-1)d < i = j \le vd$, and zero otherwise. We also associate local matrices

with hyperedges. Each edge $e \subseteq V$ is associated with a characteristic projection matrix $P^{(e)} = \sum_{v \in e} P^{(v)}$, and with an edge matrix $K^{(e)} \in \mathbb{R}^{nd \times nd}$ of the form $P^{(e)} \tilde{K}^{(e)} P^{(e)}$. The matrix $\tilde{K}^{(e)}$ is of no interest; only $K^{(e)}$ is; we use this notation to ensure that the nonzero structure of $K^{(e)}$ corresponds to the vertices of e. We call the sum K of all the edge matrices, $K = \sum_{e \in E} K^{(e)}$, the *stiffness matrix* of the hypergraph.

Our notation aims to model finite-element discretizations of partial differential equations. The vertices of the hypergraph correspond to nodes in the finite-element mesh. The edges of the hypergraph correspond to elements in the mesh; each element (edge) is incident to a set of nodes (vertices). Each node is associated with a vector of real quantities, which model potential (e.g., electric potential, temperature, infinitesimal spatial displacement in a certain direction). Applying a vector x of potentials to the finite-element model generates forces on the nodes given by Kx. In most applications, the goal is to find the potential x that corresponds to a given vector f of forces; this is achieved by solving $Kx = f$ for x. The element matrices $K^{(e)}$ describe the force $K^{(e)}x$ that a substructure generate when subject to a potential x. (In practice, K and f also represent boundary conditions, not just the forces that the structure generate, but this is irrelevant to our discussion.) In linear elasticity $K^{(e)}$ and K are called element and global stiffness matrices, respectively. We adopt this terminology in this paper.

3. LAPLACIANS AND THEIR ALGEBRAIC CONNECTIVITY

Suppose that G is not a hypergraph but a graph, that $d = 1$, and that we associate with each edge $e = (u, v)$ a scalar weight $c_e > 0$. We define

$$\left(K^{(u,v)}\right)_{ij} = \begin{cases} c_e & \text{if } i = j = u \text{ or } i = j = v \\ -c_e & \text{if } i = u \text{ and } j = v \text{ or vice versa} \\ 0 & \text{otherwise}. \end{cases}$$

Under these conditions, K is the so-called Laplacian of the weighted graph G. The Laplacian is symmetric positive semidefinite, weakly diagonally dominant, and with zero row sums. If G is connected, then K has rank $n - 1$ and its smallest nonzero eigenvalue λ_2 satisfies $\lambda_2 = \min_{x \perp \mathbf{1}} x^T K x / x^T x$, where $\mathbf{1}$ is the vector of all 1's.

What is remarkable about the smallest nonzero eigenvalue of a Laplacian is that it is related to the connectivity of G (see, for example, [4],[5],[1]). There are several ways to relate λ_2 to measures of connectivity of G. The measure that we use in this paper is the so-called isoperimetric number of G. A *edge cut* $C = (S, \overline{S})$ is a subset of E induced by a partition of V, $C = \{e \mid e = (u, v), u \in S, v \notin S\}$. The *cut connectivity* (or *cut rigidity*) is traditionally defined to be the sum of its edge weights, $c(C) = c(S, \overline{S}) = \sum_{e \in C} c_e$. The

isoperimetric number of G is defined as

$$\Phi = \min\left\{ \frac{c(C)}{|S|} \mid S \subset V, C = (S, \overline{S}), 0 < |S| \leq \frac{n}{2} \right\} \ .$$

The isoperimetric number and λ_2 are always fairly close to each other,

$$(3.1) \qquad \frac{\lambda_2}{2} \leq \Phi \leq \sqrt{2\Delta\lambda_2} \ ,$$

where Δ is the maximum sum of edge-weights adjacent to a single vertex in the graph (a generalized-degree bound).

4. Measuring Connectivity in Hypergraphs

How do we measure cuts in hypergraphs? We propose here a definition that coincides with the traditional definition for weighted Laplacians, and which will permit us to generalize the isoperimetric inequalities cited in Equation 3.1.

An edge cut $C = (S, \overline{S})$ in a hypergraph $G = (V, E)$ is a subset of E induced by a partition of V, defined as

$$C = \left\{ e \mid e \cap S \neq \phi, e \cap \overline{S} \neq \phi \right\} \ .$$

Given a cut C we define the matrix $K_{C,\alpha}$,

$$K_{C,\alpha} = \sum_{e \in C} K^{(e)} + \alpha \cdot \sum_{e \notin C} K^{(e)} \ .$$

We define the *rigidity* (connectivity) of a cut C as the following limit,

$$c(C) = \lim_{\alpha \to \infty} \min_{x \perp \mathrm{null} K} \frac{x^T K_{C,\alpha} x}{x^T x} \cdot \frac{|S||\overline{S}|}{|V|}$$

where $\mathrm{null} K$ is the null space of the global stiffness matrix K of G.

Theorem 4.1. *Assuming all element matrices $K^{(e)}$ are positive semidefinite, the rigidity of a cut C exists (in the sense that the limit exists) and is finite if and only if*

$$(4.1) \qquad \mathrm{null} K^{\perp} \cap \bigcap_{e \notin C} \mathrm{null} K^{(e)} \neq \{\mathbf{0}\} \ .$$

Furthermore, if the limit exists, it equals $x^T K x / x^T x$ for some $x \perp \mathrm{null} K$, $x \in \bigcap_{e \notin C} \mathrm{null} K^{(e)}$.

See appendix for the proof.

The last theorem reveals the intuition behind the rigidity limit definition. The rigidity limit measures the minimum energy expanded under a unit displacement, when energy can only be expanded in the elements of the cut itself and the displacement is orthogonal to rigid motions (displacement vectors in $\mathrm{null} K$).

Lemma 4.2. *Assuming all element matrices $K^{(e)}$ are positive semidefinite, the following conditions are equivalent:*

   (a) $\mathrm{null} K^{\perp} \cap \bigcap_{e \notin C} \mathrm{null} K^{(e)} \neq \{\mathbf{0}\}$

   (b) $\mathrm{null} K \subsetneq \bigcap_{e \notin C} \mathrm{null} K^{(e)}$

   (c) $\mathrm{null} K \neq \bigcap_{e \notin C} \mathrm{null} K^{(e)}$

See appendix for the proof.

Proposition 4.3. *Given a weighted Laplacian, the rigidity limit exists for all of its cuts and equals the respective cut connectivity.*

See appendix for the proof.

5. Isoperimetric Inequalities for Hypergraphs

In this section we generalize the isoperimetric number, Φ, for finite-element hypergraphs. We state and prove some relations between the isoperimetric number and the smallest nonzero eigenvalue of the hypergraph stiffness matrix.

We use our definition of cuts rigidity for hypergraphs to generalize Φ in a natural way. We define Φ in the same way it was defined for graphs in section 3, using the general definition of cut rigidity:

$$\Phi = \min\left\{ \frac{c(C)}{|S|} \mid S \subset V, C = (S, \overline{S}), 0 < |S| \leq \frac{n}{2} \right\}$$

The minimum is taken over all the cuts that have an existing rigidity limit (we call such cuts *admissible cuts*). In the case there are no admissible cuts we define $\Phi = \infty$.

Theorem 5.1. *The smallest nonzero eigenvalue of K is at most 2Φ.*

See appendix for the proof.

Conjecture 5.2. *Assuming all element matrices are symmetric positive semidefinite, the smallest nonzero eigenvalue of K is at least $f(n, \Phi)$, where f is an increasing function of Φ.*

This conjecture is a generalization of Cheeger's inequality from weighted graph Laplacians ($\lambda_2 \geq \Phi^2/2\Delta$) to finite-element hypergraphs.

We hope to prove this conjecture during the following weeks.

References

[1] N. Alon and V. D. Milman, λ_1, *isoperimetric inequalities for graphs, and superconcentrators*, J. Comb. Theory Series B, 38 (1985), pp. 73–88.

[2] M. Bern, J. R. Gilbert, B. Hendrickson, N. Nguyen, and S. Toledo, *Support-graph preconditioners*. Submitted to the *SIAM Journal on Matrix Analysis and Applications*, 29 pages, Jan. 2001.

[3] E. G. Boman and B. Hendrickson, *Support theory for preconditioning*, SIAM Journal on Matrix Analysis and Applications, 25 (2004), pp. 694–717.

[4] J. Cheeger, *A lower bound for the smallest eigenvalue of the laplacian*, in Problems in analysis, R. C. Gunning, ed., Princeton Univ. Press, 1970, pp. 195–199.

[5] M. Fiedler, *Algebraic connectivity of graphs*, Czech. Math. Journal, 23 (1973), pp. 298–305.

APPENDIX: PROOFS OF THE LEMMAS AND THEOREMS

Proof of Theorem 4.1:

Proof. We first show that if condition (4.1) holds, then the limit exists and is finite. We first focus on an easy case, in which $\mathrm{null}K^{\perp} \subseteq \bigcap_{e \notin C} \mathrm{null}K^{(e)}$. In this case, if $\hat{x} \perp \mathrm{null}K$, then the ratio $\hat{x}^T K_{C,\alpha} \hat{x}/\hat{x}^T \hat{x}$ is independent of α, so $\min_{x \perp \mathrm{null}K} x^T K_{C,\alpha} x/x^T x$ is finite and independent of α, so the limit exists. It is easy to see that the ratio is indeed independent of α. If $\hat{x} \perp \mathrm{null}K$, then $\hat{x} \in \mathrm{null}K^{(e)}$ for all $e \notin C$, and therefore

$$\frac{\hat{x}^T K_{C,\alpha} \hat{x}}{\hat{x}^T \hat{x}} = \frac{\sum_{e \in C} \hat{x}^T K^{(e)} \hat{x} + \alpha \cdot \sum_{e \notin C} \hat{x}^T K^{(e)} \hat{x}}{\hat{x}^T \hat{x}} =$$

$$= \frac{\sum_{e \in C} \hat{x}^T K^{(e)} \hat{x} + \sum_{e \notin C} \hat{x}^T K^{(e)} \hat{x}}{\hat{x}^T \hat{x}} = \frac{\hat{x}^T K \hat{x}}{\hat{x}^T \hat{x}} \, .$$

In the more general case, $\mathrm{null}K^{\perp} \nsubseteq \bigcap_{e \notin C} \mathrm{null}K^{(e)}$. Our intermediate claim is that for any large enough α, the minimum of the ratio $x^T K_{C,\alpha} x/x^T x$ is realized by an $\hat{x}$ which is in $\bigcap_{e \notin C} \mathrm{null}K^{(e)}$.

We define $A = \{e \notin C \mid \exists x, x \perp \mathrm{null}K, x \notin \mathrm{null}K^{(e)}\}$. By our assumption, A is not empty (and in most cases A will contain all the edges that are not in C). We define

$$\beta = \min_{e \in A} \min_{x \perp \mathrm{null}K, x \notin \mathrm{null}K^{(e)}} \frac{x^T K^{(e)} x}{x^T x} \, .$$

Let z be a nonzero vector in $\mathrm{null}K^{\perp} \cap \bigcap_{e \notin C} \mathrm{null}K^{(e)}$ and set $\gamma = z^T K z/z^T z = z^T K_{C,\alpha} z/z^T z$. Let $\alpha > \gamma/\beta$ and let $\hat{x} = \mathrm{argmin}_{x \perp \mathrm{null}K} x^t K_{C,\alpha} x/x^t x$. We claim that $\hat{x} \in \bigcap_{e \notin C} \mathrm{null}K^{(e)}$. Assume for contradiction that there exists an $\hat{e} \notin C$ such that $\hat{x} \notin \mathrm{null}K^{(\hat{e})}$. We have

$$\frac{\hat{x}^T K_{C,\alpha} \hat{x}}{\hat{x}^T \hat{x}} = \frac{\sum_{e \in C} \hat{x}^T K^{(e)} \hat{x} + \alpha \cdot \sum_{e \notin C} \hat{x}^T K^{(e)} \hat{x}}{\hat{x}^T \hat{x}} \geq$$

$$\geq \alpha \frac{\hat{x}^T K^{(\hat{e})} \hat{x}}{\hat{x}^T \hat{x}} \geq \alpha \beta > \gamma = \frac{z^T K_{C,\alpha} z}{z^T z} \, .$$

This contradicts the definition of $\hat{x}$ as the minimizer of the ratio. Therefore, the minimizer $\hat{x}$ is in $\mathrm{null}K^{(e)}$ for all $e \notin C$. Hence, for all $\alpha > \gamma/\beta$,

$$\min_{x \perp \mathrm{null}K} \frac{x^T K_{C,\alpha} x}{x^T x} = \frac{\hat{x}^T K_{C,\alpha} \hat{x}}{\hat{x}^T \hat{x}} = \frac{\hat{x}^T K_{C,\alpha=1} \hat{x}}{\hat{x}^T \hat{x}} = \frac{\hat{x}^T K \hat{x}}{\hat{x}^T \hat{x}} \, .$$

This immediately implies convergence to a finite value. This concludes one direction of the proof.

In fact, we proved that if condition (4.1) holds, there exist a vector $\hat{x} \perp \mathrm{null}K$ such that $\hat{x} \in \bigcap_{e \notin C} K^{(e)}$ and the limit equals $\hat{x}^T K \hat{x}/\hat{x}^T \hat{x}$. We showed that both for the easy case and the general case.

We now show that the existence of the limit implies (4.1). Assume that the rigidity limit exists and is finite; we denote the limit by r. The limit expression is only defined when $\mathrm{null}K^{\perp}$ is not empty, so we assume that $\mathrm{null}K^{\perp} \neq \{\mathbf{0}\}$.

If $\mathrm{null}K^{\perp} \subseteq \bigcap_{e \notin C} \mathrm{null}K^{(e)}$ then

$$\mathrm{null}K^{\perp} \cap \bigcap_{e \notin C} \mathrm{null}K^{(e)} \neq \{\mathbf{0}\}$$

Otherwise, we can define the (nonempty) group $A = \{e \notin C \mid \exists x, x \perp \mathrm{null}K, x \notin \mathrm{null}K^{(e)}\}$ and the associated constant β as before.

Let $\epsilon > 0$, there exists α_0 that for all $\alpha \geq \alpha_0$,

$$(5.1) \qquad \min_{x \perp \mathrm{null}K} \frac{x^T K_{C,\alpha} x}{x^T x} < r + \epsilon \, .$$

Let $\alpha_1 = \max(\alpha_0, (r + \epsilon)/\beta)$ and then define $\hat{x}$

$$\hat{x} = \mathrm{argmin}_{x \perp \mathrm{null}K} \frac{x^T K_{C,\alpha_1} x}{x^T x} \, .$$

Assume for contradiction that $\mathrm{null}K^{\perp} \cap \bigcap_{e \notin C} \mathrm{null}K^{(e)} = \{\mathbf{0}\}$. Therefore, there exists an $\hat{e} \notin C$ for which $\hat{x} \notin \mathrm{null}K^{(\hat{e})}$. We have

$$\frac{\hat{x}^T K_{C,\alpha_1} \hat{x}}{\hat{x}^T \hat{x}} = \frac{\sum_{e \in C} \hat{x}^T K^{(e)} \hat{x} + \alpha_1 \cdot \sum_{e \notin C} \hat{x}^T K^{(e)} \hat{x}}{\hat{x}^T \hat{x}} \geq$$

$$\geq \alpha_1 \frac{\hat{x}^T K^{(\hat{e})} \hat{x}}{\hat{x}^T \hat{x}} \geq \frac{r + \epsilon}{\beta} \beta = r + \epsilon \, .$$

This contradicts equation 5.1. Therefore,

$$\mathrm{null}K^{\perp} \cap \bigcap_{e \notin C} \mathrm{null}K^{(e)} \neq \{\mathbf{0}\} \, .$$

This concludes the proof of the theorem. $\square$

Proof of Lemma 4.2:

Proof. First, we observe that since all $K^{(e)}$ are positive semidefinite, $\mathrm{null}K = \bigcap_{e \in E} \mathrm{null}K^{(e)}$.

We show that $\mathrm{null}K \subseteq \bigcap_{e \in E} \mathrm{null}K^{(e)}$. Let $x \in \mathrm{null}K$, we have $0 = x^T K x = \sum_{e \in E} x^T K^{(e)} x$. Since $x^T K^{(e)} x \geq 0$ for all $e \in E$, we have $x^T K^{(e)} x = 0$ for all $e \in E$. Therefore, $x \in \bigcap_{e \in E} \mathrm{null}K^{(e)}$. This proves the containment in one direction.

The containment in the other direction is even simpler. Let $x \in \bigcap_{e \in E} \mathrm{null}K^{(e)}$, we have

$$x^T K x = \sum_{e \in E} x^T K^{(e)} x = 0 \, .$$

Therefore, $x \in \mathrm{null}K$.

It follows that $\mathrm{null}K = \bigcap_{e \in E} \mathrm{null}K^{(e)} \subseteq \bigcap_{e \notin C} \mathrm{null}K^{(e)}$. Therefore, conditions (b) and (c) are equivalent.

Assuming condition (a), there exist a vector x such that

$$\mathbf{0} \neq x \in \mathrm{null}K^{\perp} \cap \bigcap_{e \notin C} \mathrm{null}K^{(e)} \, .$$

It is clear that $x \notin \mathrm{null}K$ and $x \in \bigcap_{e \notin C} \mathrm{null}K^{(e)}$. We showed that $\mathrm{null}K \subseteq \bigcap_{e \notin C} \mathrm{null}K^{(e)}$. Therefore, $\mathrm{null}K \subsetneq \bigcap_{e \notin C} \mathrm{null}K^{(e)}$, that is, condition (b) holds.

Assuming condition (b), there exist a vector x such that $\mathbf{0} \neq x \in \bigcap_{e \notin C} \mathrm{null}K^{(e)}$ and $x \notin \mathrm{null}K$. We can write x as $x = x_1 + x_2$, where $x_1 \in \mathrm{null}K$ and $\mathbf{0} \neq x_2 \in \mathrm{null}K^{\perp}$. Notice that $x_1 \in \bigcap_{e \notin C} \mathrm{null}K^{(e)}$ since $\mathrm{null}K \subseteq \bigcap_{e \notin C} \mathrm{null}K^{(e)}$. Therefore, $x_2 = x - x_1 \in \bigcap_{e \notin C} \mathrm{null}K^{(e)}$. We have $\mathbf{0} \neq x_2 \in \mathrm{null}K^{\perp} \cap \bigcap_{e \notin C} \mathrm{null}K^{(e)}$, that is, condition (a) holds.

This concludes the proof of the lemma. $\square$

Proof of Proposition 4.3:

Proof. We show now that the cut connectivity of the Laplacian (that is, the sum of the edge weights in the cut) equals the rigidity limit of the respective cut.

Let the cut C be defined by the partition $(S, \overline{S})$. If the partition is trivial, i.e. $S = V$ or $S = \phi$, then the rigidity limit and the cut connectivity both equal 0.

Otherwise S ans $\overline{S}$ are both nonempty. For simplicity, we assume that the graph is connected (so null$K = \{\mathbf{1}\}$). Let $\hat{x}$ be a vector of potentials which equals β for all vertices of S and γ for all vertices of $\overline{S}$. We will show that this vector brings the ratio $\hat{x}^T K_{C,\alpha} \hat{x} \backslash \hat{x}^T \hat{x}$ to a minimum for a large enough α.

Let $k = |S|$. We choose β and γ such that $\hat{x} \perp \mathbf{1}$, this means $k\beta + (n-k)\gamma = 0$. We also constrain β and γ to get $\|\hat{x}\| = 1$, this means $k\beta^2 + (n-k)\gamma^2 = 1$. We have

$$\hat{x}^T K \hat{x} = \sum_{e \in E} \hat{x}^T K^{(e)} \hat{x} = \sum_{e \in C} \hat{x}^T K^{(e)} \hat{x} = \sum_{e \in C} c_e (\beta - \gamma)^2 .$$

Combining the last equality with the two former equations we have

$$\hat{x}^T K \hat{x} = (\beta - \gamma)^2 \sum_{e \in C} c_e = \frac{n}{k(n-k)} \sum_{e \in C} c_e = \frac{|V|}{|S||\overline{S}|} \sum_{e \in C} c_e .$$

For all α we have

$$\frac{\hat{x}^T K_{C,\alpha} \hat{x}}{\hat{x}^T \hat{x}} = \hat{x}^T K_{C,\alpha} \hat{x} = \hat{x}^T K \hat{x} = \frac{|V|}{|S||\overline{S}|} \sum_{e \in C} c_e .$$

Let y be a unit vector of potentials that is not in the form of $\hat{x}$ i.e. there are two adjacent vertices in S (or $\overline{S}$) with potentials $\mu \neq \eta$. Observe that for a large enough α we have

$$\frac{y^T K_{C,\alpha} y}{y^T y} \geq \alpha(\mu - \eta)^2 > \frac{|V|}{|S||\overline{S}|} \sum_{e \in C} c_e = \frac{\hat{x}^T K \hat{x}}{\hat{x}^T \hat{x}} .$$

Therefore, the minimum of the ratio is realized by $\hat{x}$ and is independent of α. We have

$$c(C) = \lim_{\alpha \to \infty} \min_{x \perp \text{null} K} \frac{x^T K_{C,\alpha} x}{x^T x} \frac{|S||\overline{S}|}{|V|} =$$

$$= \frac{\hat{x}^T K \hat{x}}{\hat{x}^T \hat{x}} \frac{|S||\overline{S}|}{|V|} = \frac{|V|}{|S||\overline{S}|} \sum_{e \in C} c_e \frac{|S||\overline{S}|}{|V|} = \sum_{e \in C} c_e .$$

The rigidity limit exists and equals the sum of the cut's edge weights. This concludes the proof of the proposition. □

Proof of Theorem 5.1:

Proof. Let k be the dimension of the null-space of the stiffness matrix K. We assume that $k < n$. We denote the smallest nonzero eigenvalue of K by λ_{k+1}.

We assume that there is at least one admissible cut C, otherwise the bound is trivial. Let $C = (S, \overline{S})$ be the cut that minimizes the isoperimetric ratio, that is $\Phi = c(C)/|S|$ and $|S| \leq |V|/2$. The rigidity limit exists for that cut, therefore there exist a vector $\hat{x} \perp$ null$K$ such that $c(C) = \hat{x}^T K \hat{x} / \hat{x}^T \hat{x} \cdot |S||\overline{S}|/|V|$.

We have

$$\lambda_{k+1} = \min_{x \perp \text{null} K} \frac{x^T K x}{x^T x} \leq \frac{\hat{x}^T K \hat{x}}{\hat{x}^T \hat{x}} =$$

$$= c(C) \frac{|V|}{|S||\overline{S}|} = \Phi \frac{|V|}{|\overline{S}|} \leq 2\Phi .$$

In the last inequality we used the fact that $|\overline{S}| \geq |V|/2$. □

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