The Elimination Tree of a Nonsymmetric Matrix: Theory and Applications

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Schreiber [10] defined the *elimination tree* of a symmetric matrix, a structure that plays an important role in many aspects of sparse elimination, including matrix reordering, symbolic and numerical factorization, and parallelization.

Gilbert and Liu [6] generalized this structure to unsymmetric matrices by defining a pair of *elimination dags* (directed acyclic graphs), which are the transitive reductions of the graphs of the lower and upper triangular factors.

Here we present a new generalization that uses paths instead of edges to define a single tree with many of the same properties as the elimination tree of a symmetric matrix. We also discuss some important applications of this structure:

- Analogously to the symmetric case, the subtree of the elimination tree rooted at node k is a *strongly connected* component of the subgraph of the original *directed* graph induced by the first k vertices. This property suggests a simple algorithm to construct the elimination tree. Quotient graphs, path-preserving edge reduction, and a new, incremental algorithm for finding strongly connected components lead to an efficient implementation. The same approach can be used in diagonal Markowitz ordering [1, 9].

- Existing algorithms for symbolic LU factorization use either symmetric pruning [5] or pruning based on the elimination dags [6] to improve performance. Pruning based on the elimination tree is generally more effective than symmetric pruning, yet the tree can be constructed more easily than the dags.

- The unsymmetric multifrontal method [2, 3, 4, 7] is a powerful approach to solving sparse linear systems. By adding cross edges to the elimination tree we can model how data flows from update to frontal matrix. This model is more refined than the symmetric pattern approach [4], which uses the elimination tree of the symmetrized matrix $A + A^t$; and is somewhat simpler than the task-dag/data-dag approach [7, 8].

- A symmetric reordering of the matrix based on a postorder traversal of the elimination tree gives a recursive, upper *bordered block triangular* (BBT) form. The reordered matrix has further desirable properties. For example, the tree now captures the data dependencies among the rows of the filled matrix and constrains the flow of updates in the unsymmetric multifrontal method, which has implications for parallelization.

References

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