Algorithms for Vertex-Weighted Matching

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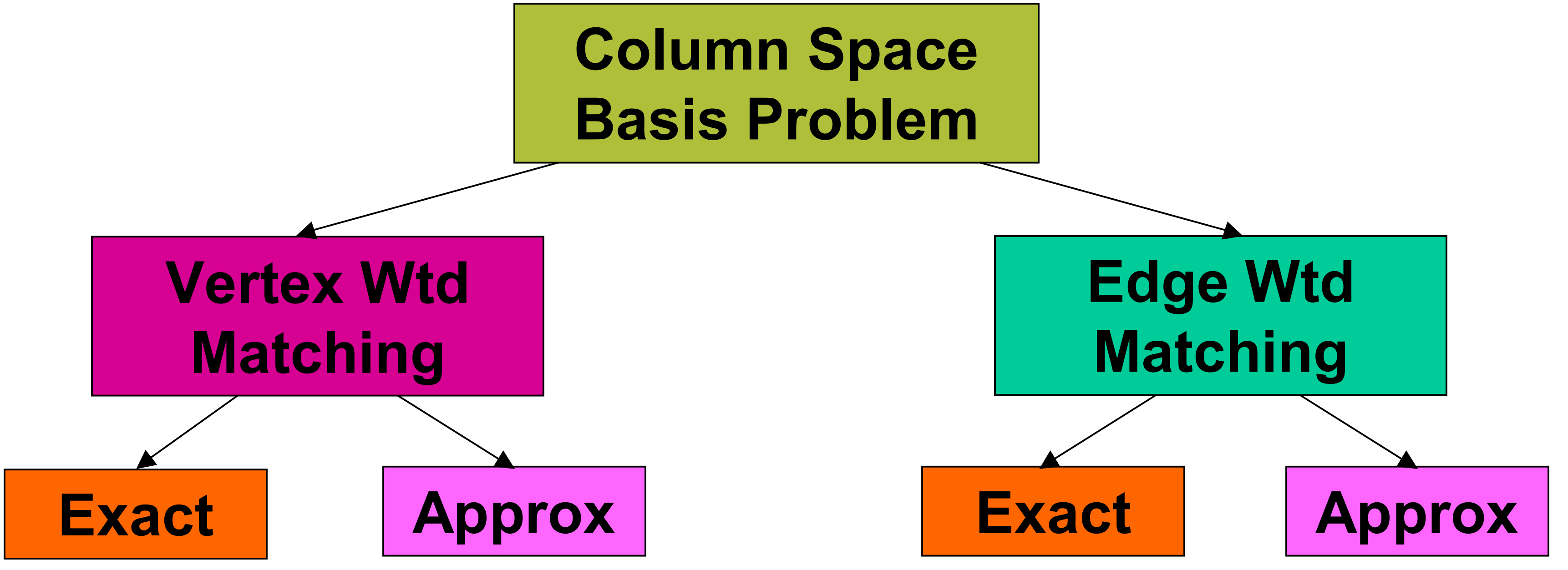
Support: NSF and DOE.

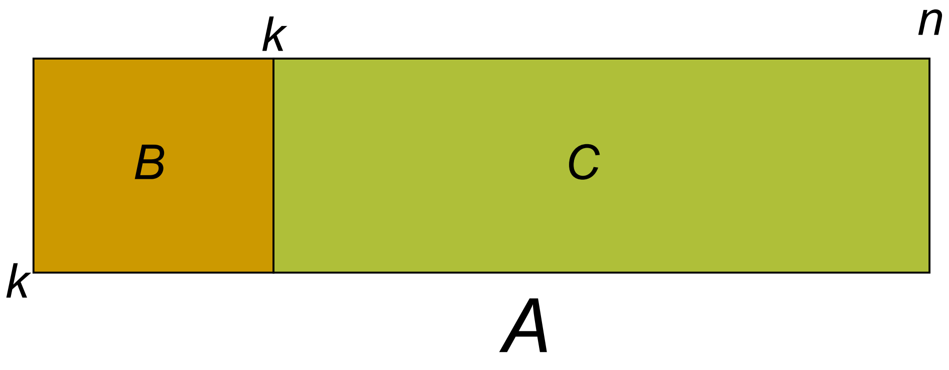
Overview:



Column Space Basis Problem
Vertex Wtd Matching
Edge Wtd Matching
Exact
Approx
Exact
Approx

Motivation: Column-Space Basis Problem

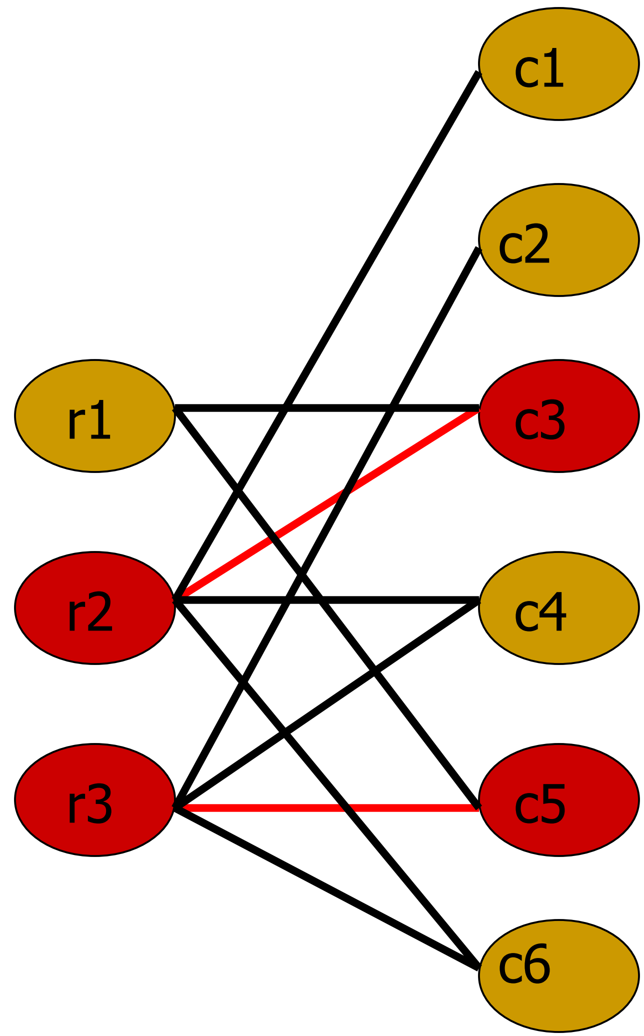
- The Sparsest Basis Problem:
  - Find a sparsest basis *B* for the column space of a sparse matrix A.
  - Solved by a matroid greedy algorithm.
  - Can be computed by a weighted matching.



Matching:

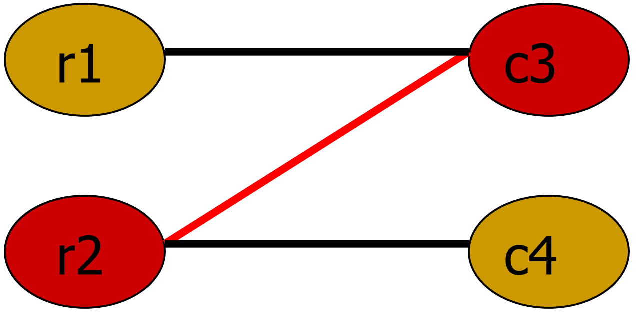
A subset of edges *M* such that no two edges in *M* are incident on the same vertex.

Cardinality/Weighted



(red)	Matched
(black)	Unmatched

Augmenting Paths



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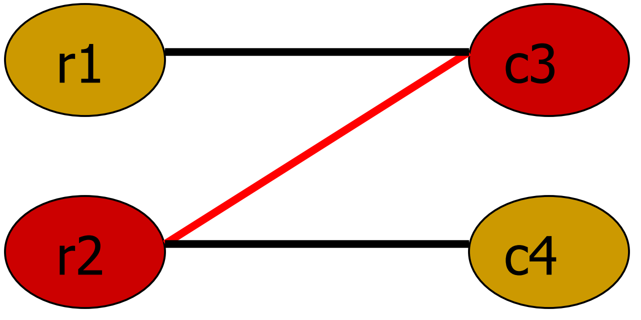
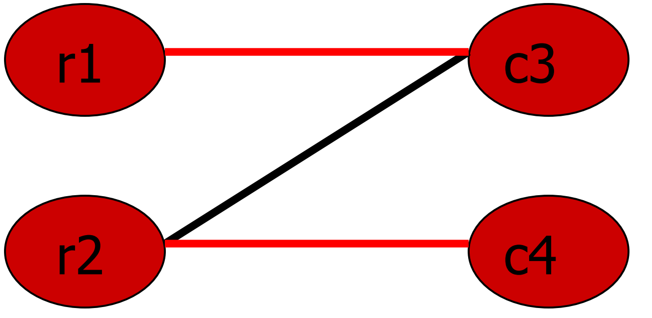
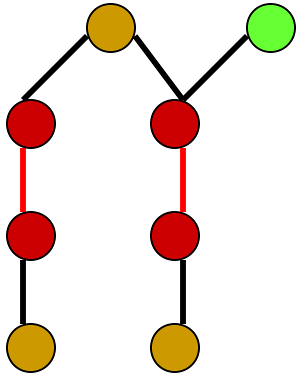


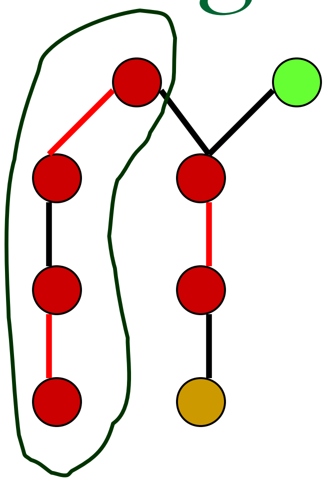


Finding Augmenting Paths:



Single Source/
Single Path

Finding Augmenting Paths:



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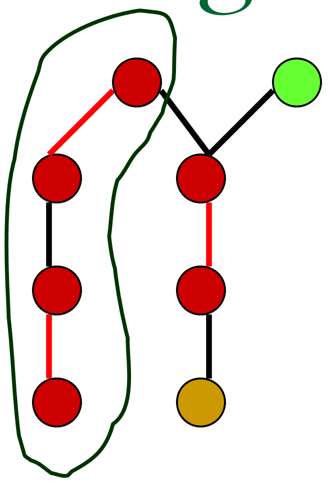

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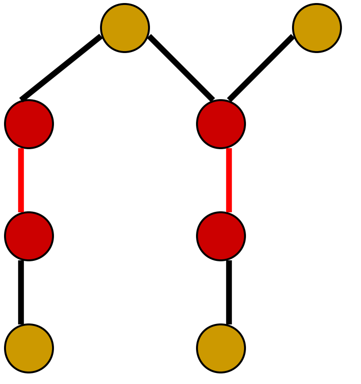

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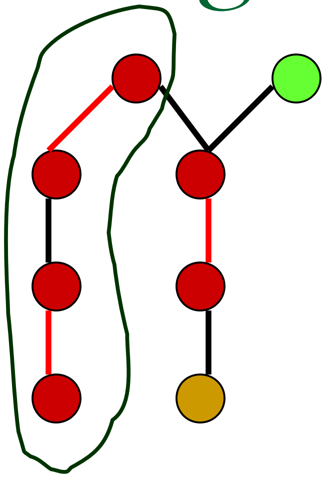

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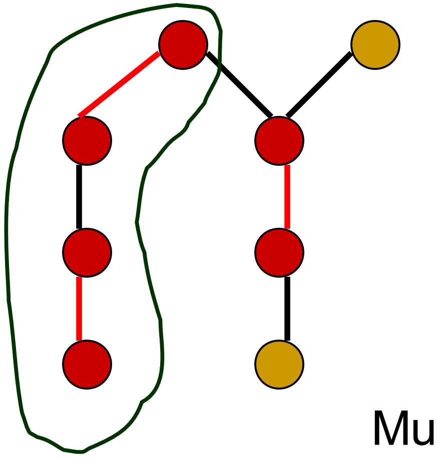

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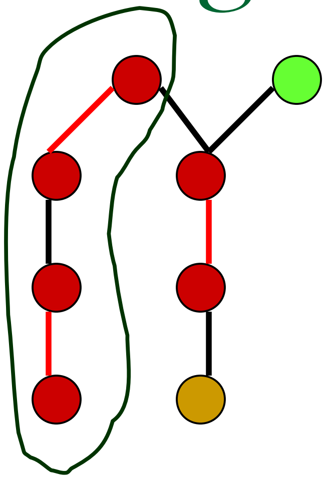

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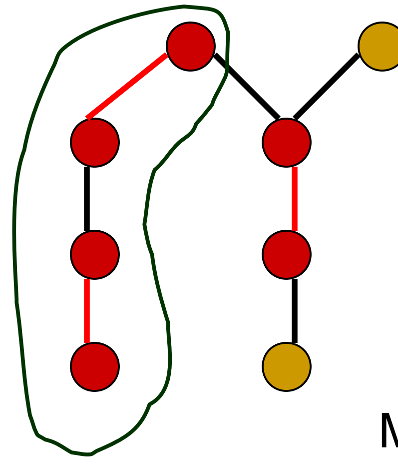

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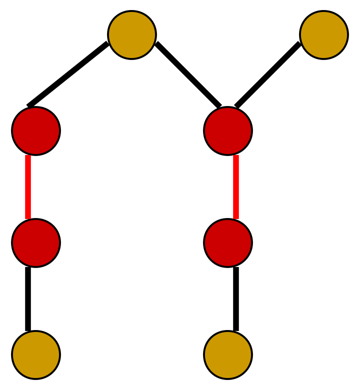

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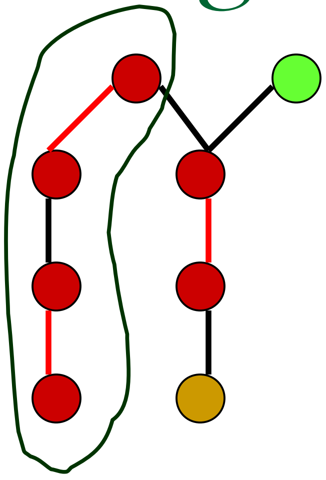

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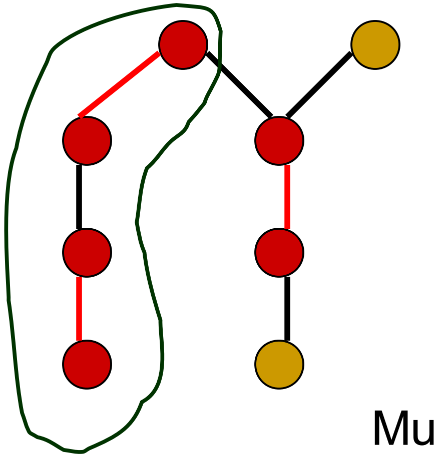

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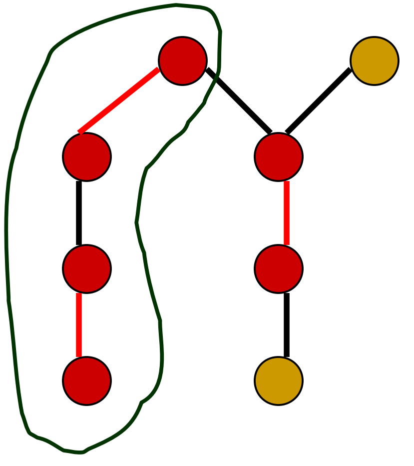

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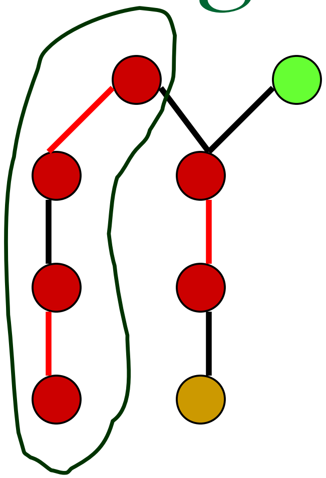

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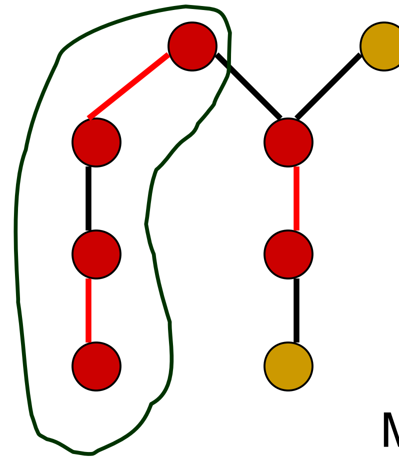

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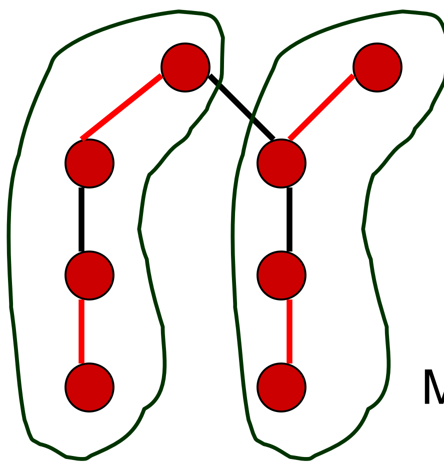
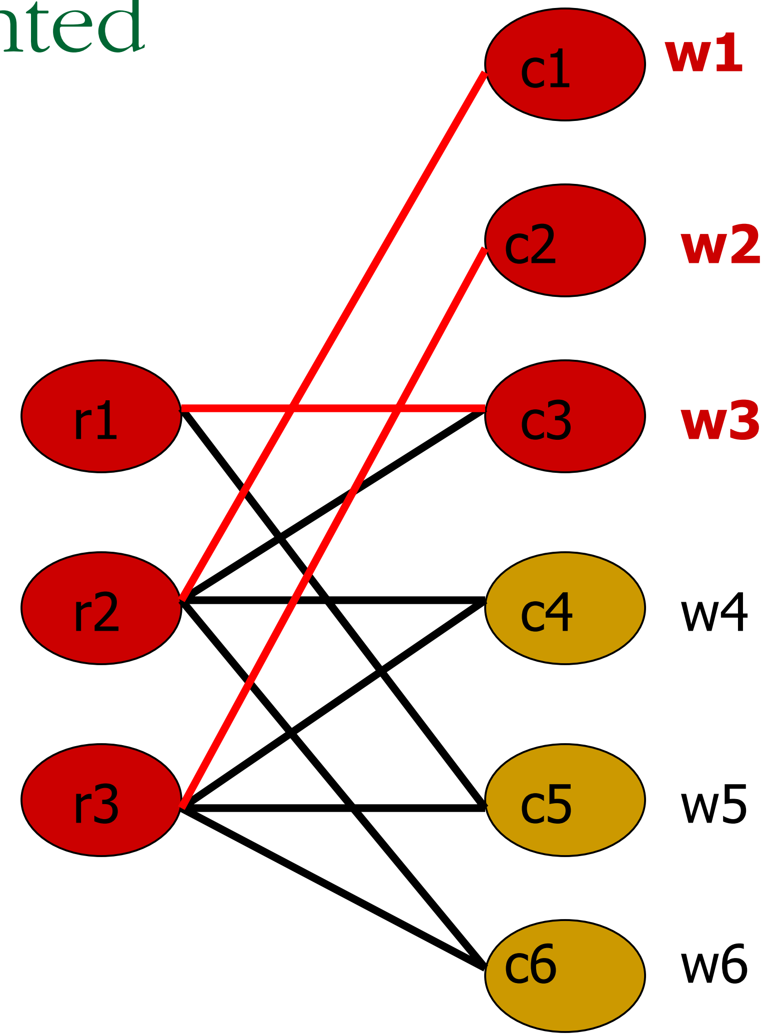


Multiple Source/
Multiple Path

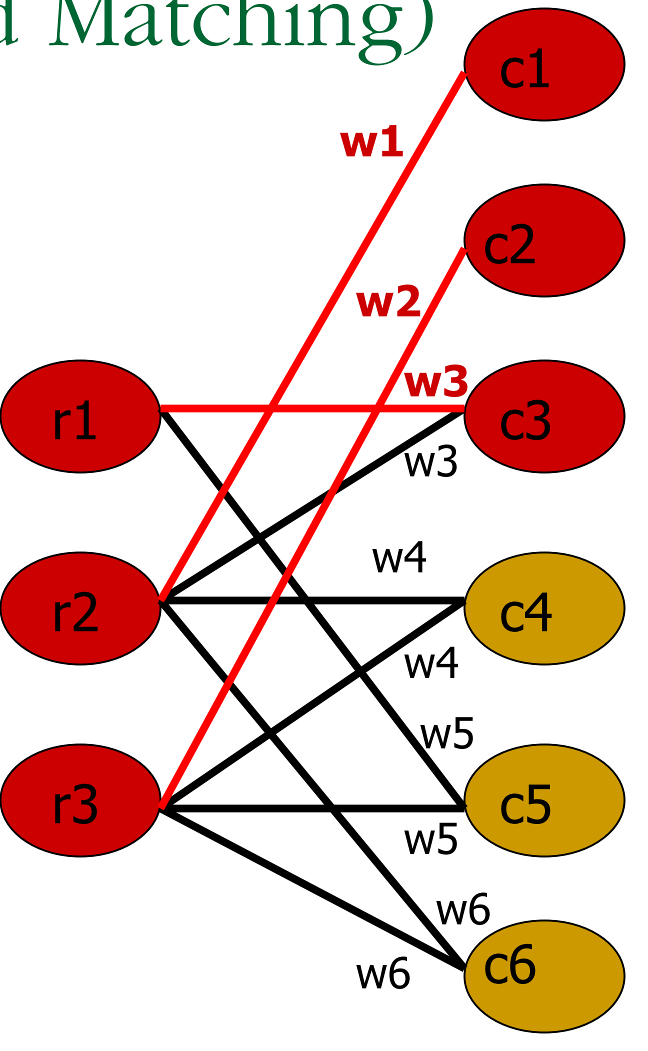
Column-space Basis: (Model as Vertex-Weighted Matching)

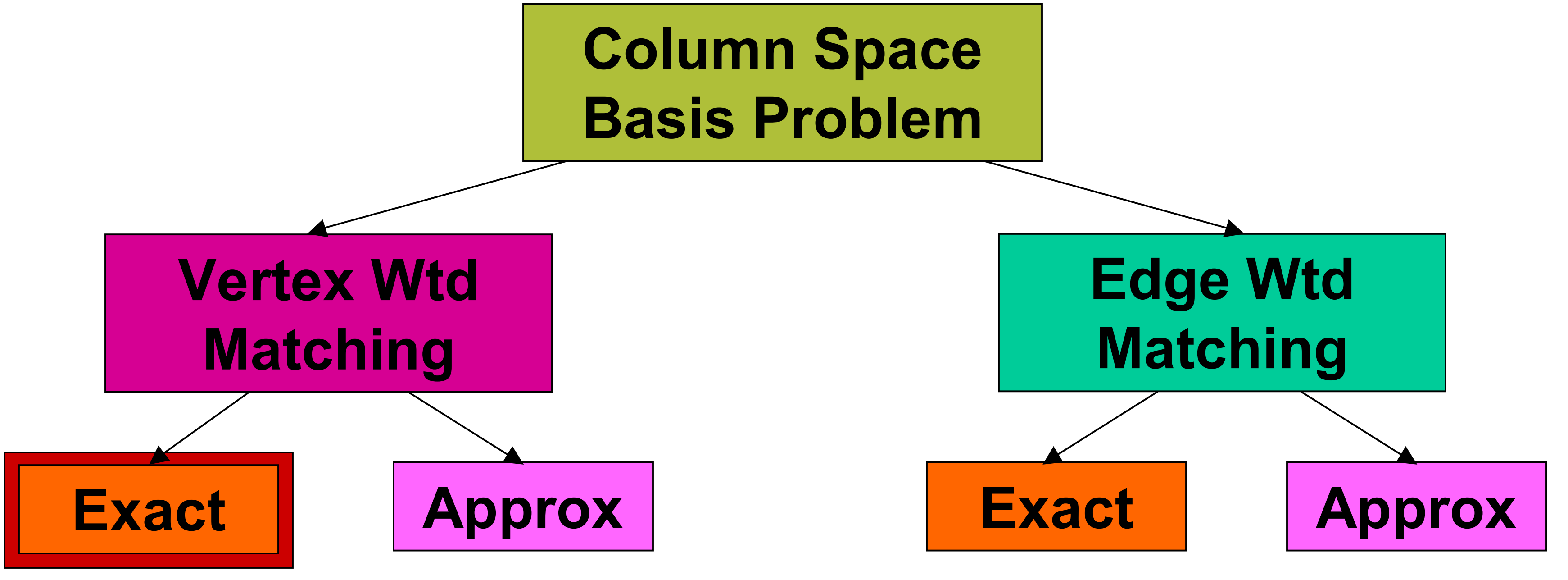
- Find matching in *G* that maximizes the sum of weights on the matched **vertices**.
- Special Case: Bipartite graph with weights on only one set of vertices.



Column-space Basis: (Model as Edge-Weighted Matching)

Find matching in *G* that maximizes the sum of weights on the matched **edges**.




Column Space Basis Problem
Vertex Wtd Matching
Edge Wtd Matching
Exact
Approx
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Max Vertex-Weight Matching:

Algorithm VWM:

- **Find Aug Path from an unmatched vertex of largest weight**

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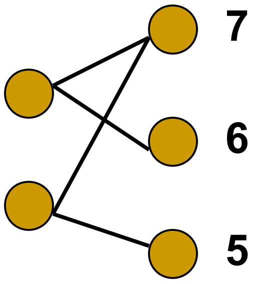
- **Augment And Repeat**

Theoretical Results:

- A max vertex-weight matching has max cardinality.
- A lexicographically optimal matching has max weight. (Mulmuley, Vazirani, and Vazirani. '87)
- Algorithm **VWM** computes a lexicographically optimal matching.
  - And therefore, of max weight and max cardinality.

Example:



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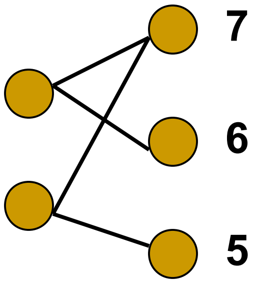

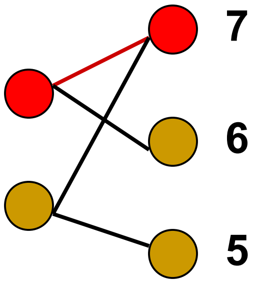

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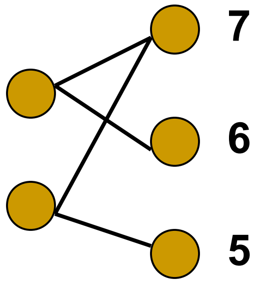

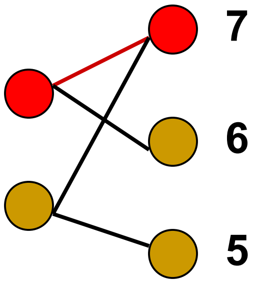

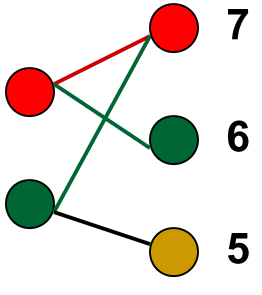

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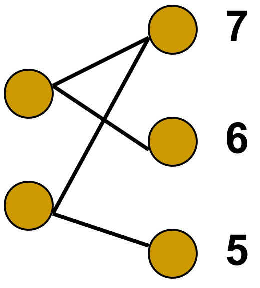
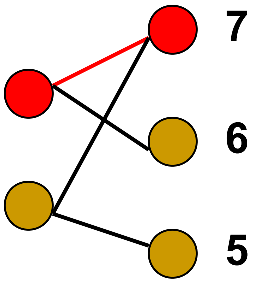
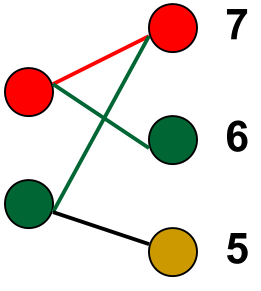
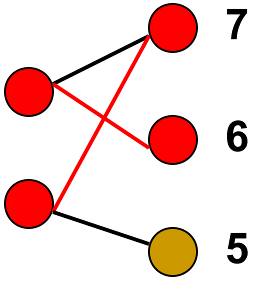




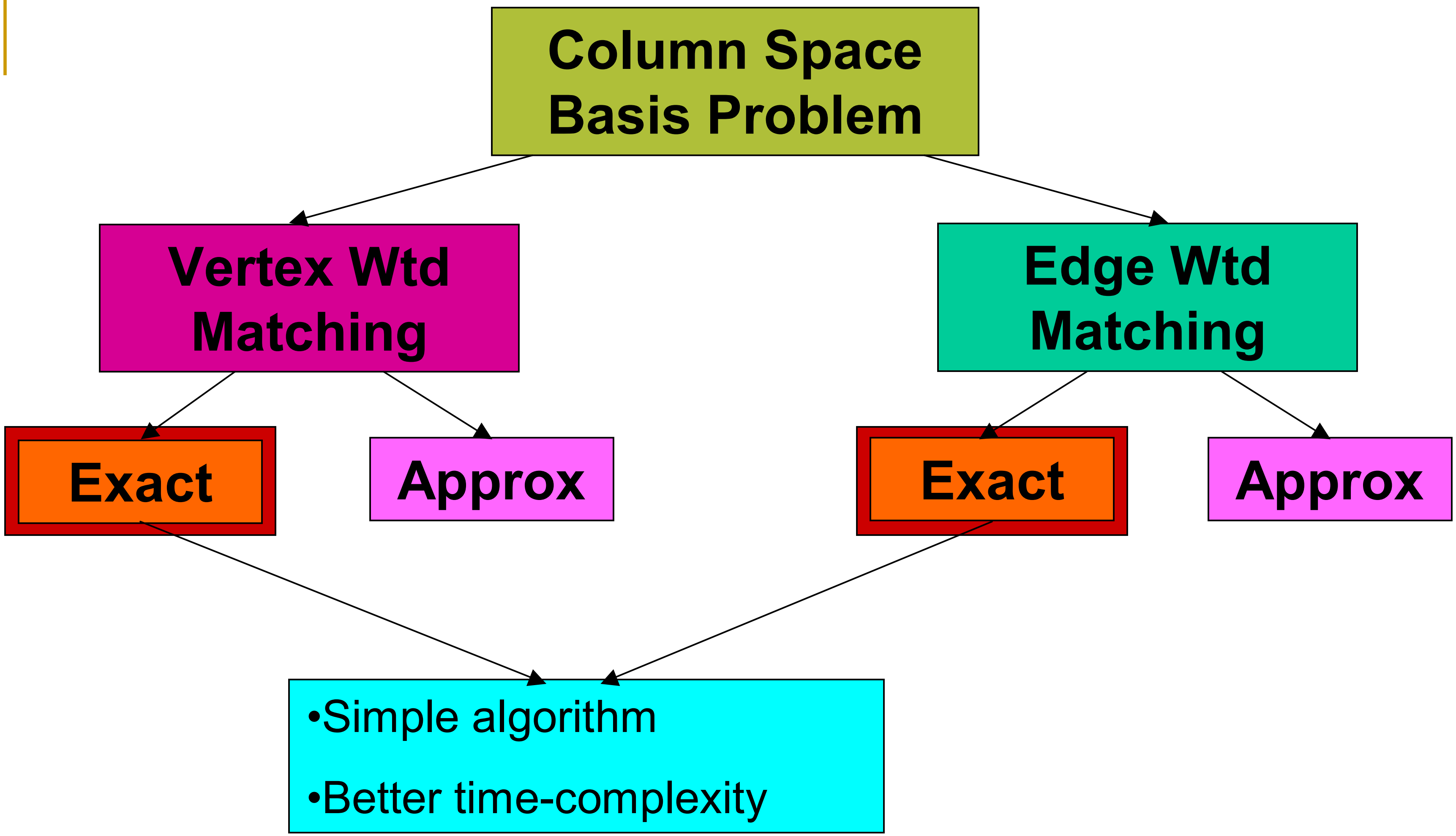




Complexity Analysis:

(Graph: G(U,V, E), |V| < |U|)

Type	Complexity	Remark
Vertex Wtd.	$O(\|U\|\|E\|)$	Single Path(SP)
Edge Wtd.	$O(\|V\|^3)$	SP(Array)
	$O(\|V\|\|E\|\log\|V\|)$	SP(Binary Heap)
Cardinality	$O(\|V\|\|E\|)$	Single Path(SP)
	$O(\sqrt{\|V\|}\|E\|)$	Multiple Path(MP)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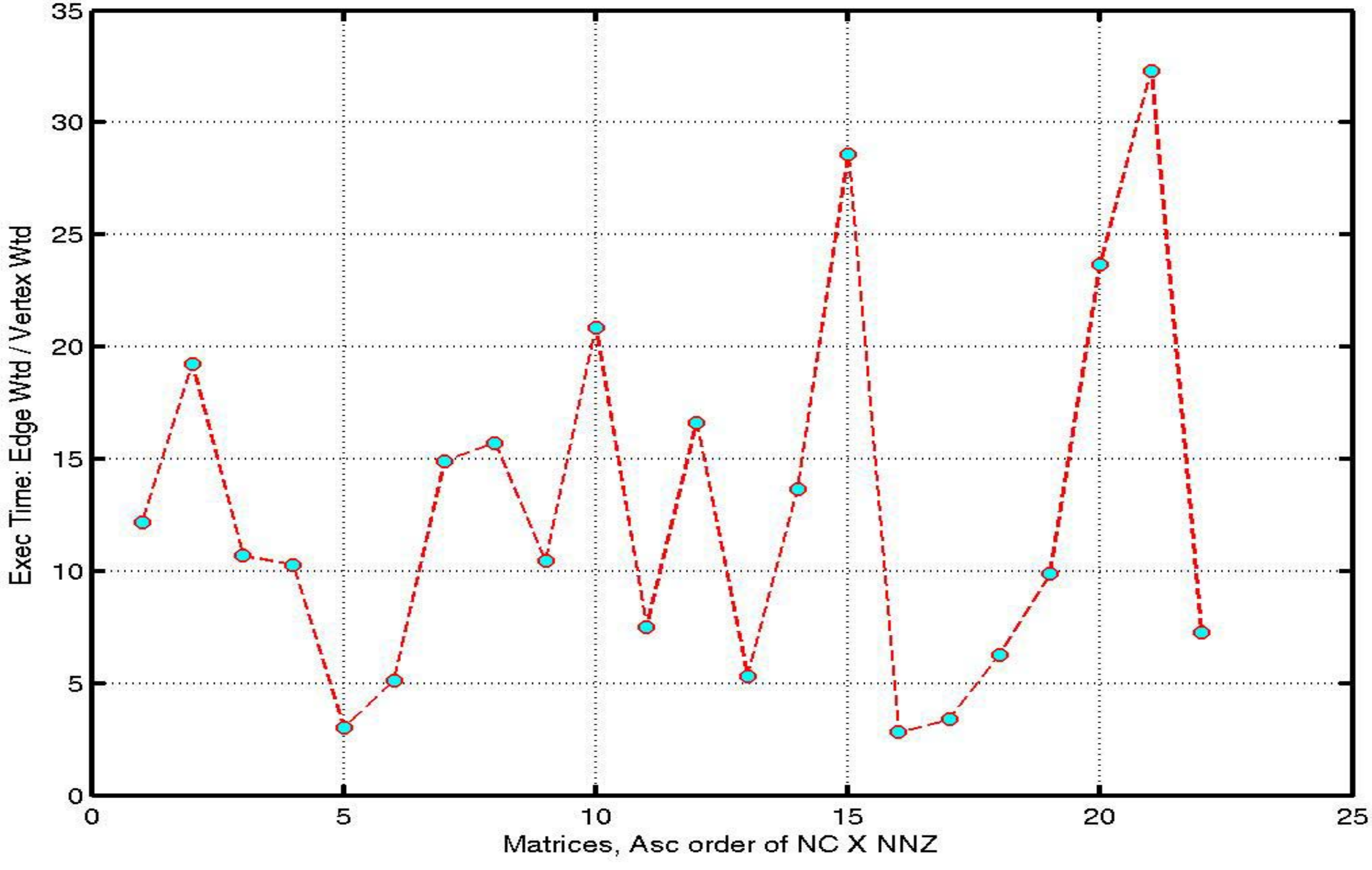
Approx
•Simple algorithm
•Better time-complexity

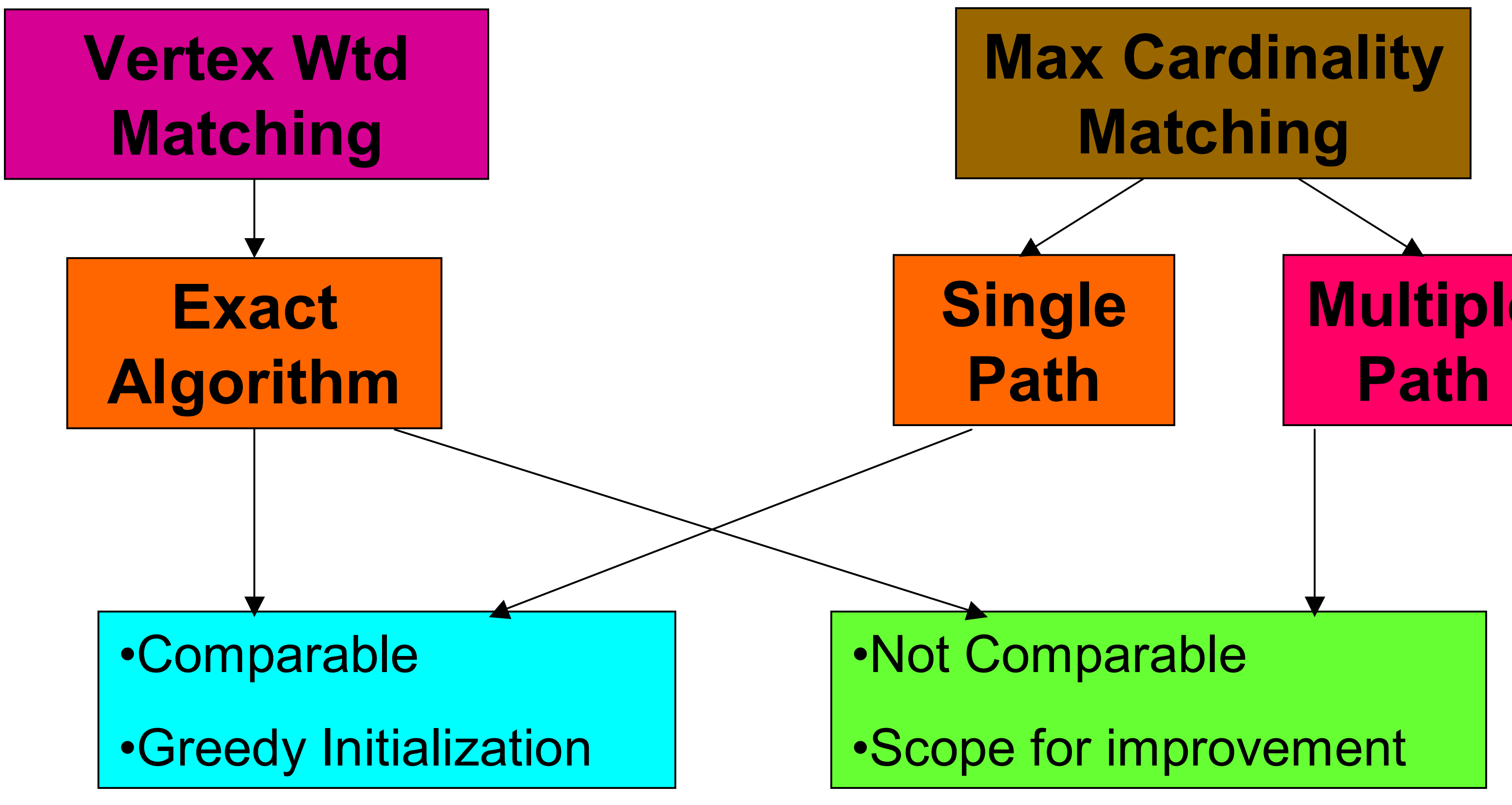
Subset of Input Matrices:
(sorted by NC.NNZ)

Matrix	NR	NC	NNZ
lp_truss (5)	1.00E+03	8.81E+03	2.78E+04
lp_df1001 (7)	6.07E+03	1.22E+04	3.56E+04
lp_ken_11 (8)	1.47E+04	2.13E+04	4.91E+04
lp_maros_r7 (10)	3.14E+03	9.41E+03	1.45E+05
lp_ken_13 (14)	2.86E+04	4.27E+04	9.72E+04
tbdmatlab (15)	5.98E+03	1.99E+04	4.30E+05
lp_nug20 (18)	1.52E+04	7.26E+04	3.05E+05
lp_ken_18 (20)	1.05E+05	1.55E+05	3.58E+05
tbdlinux (21)	2.02E+04	1.13E+05	2.16E+06
lp_nug30 (22)	5.23E+04	3.79E+05	1.57E+06

Time: Edge Wt. vs. Vertex Wt.

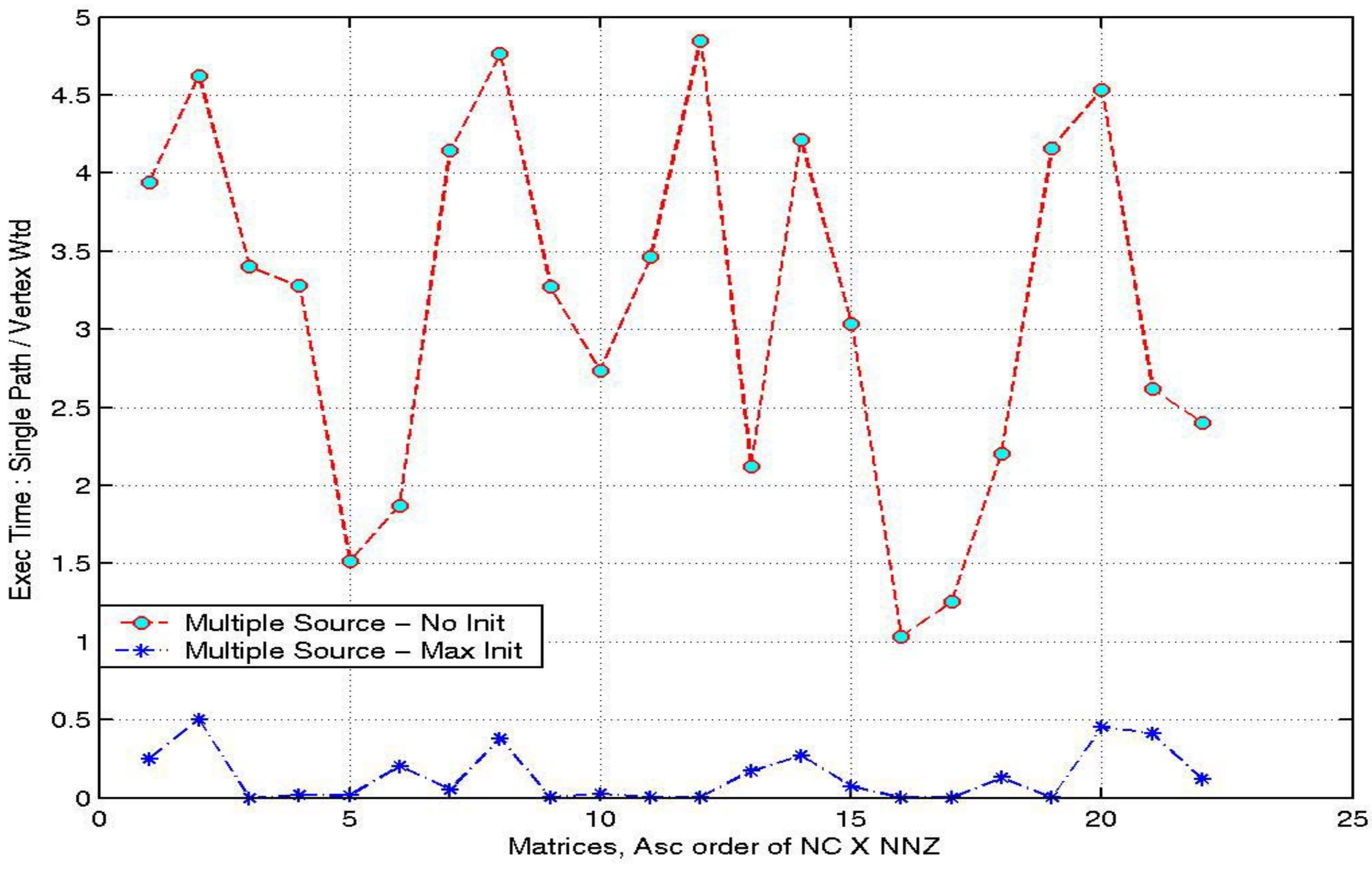
(Multiple Source-No Initialization $O(n^3)$)



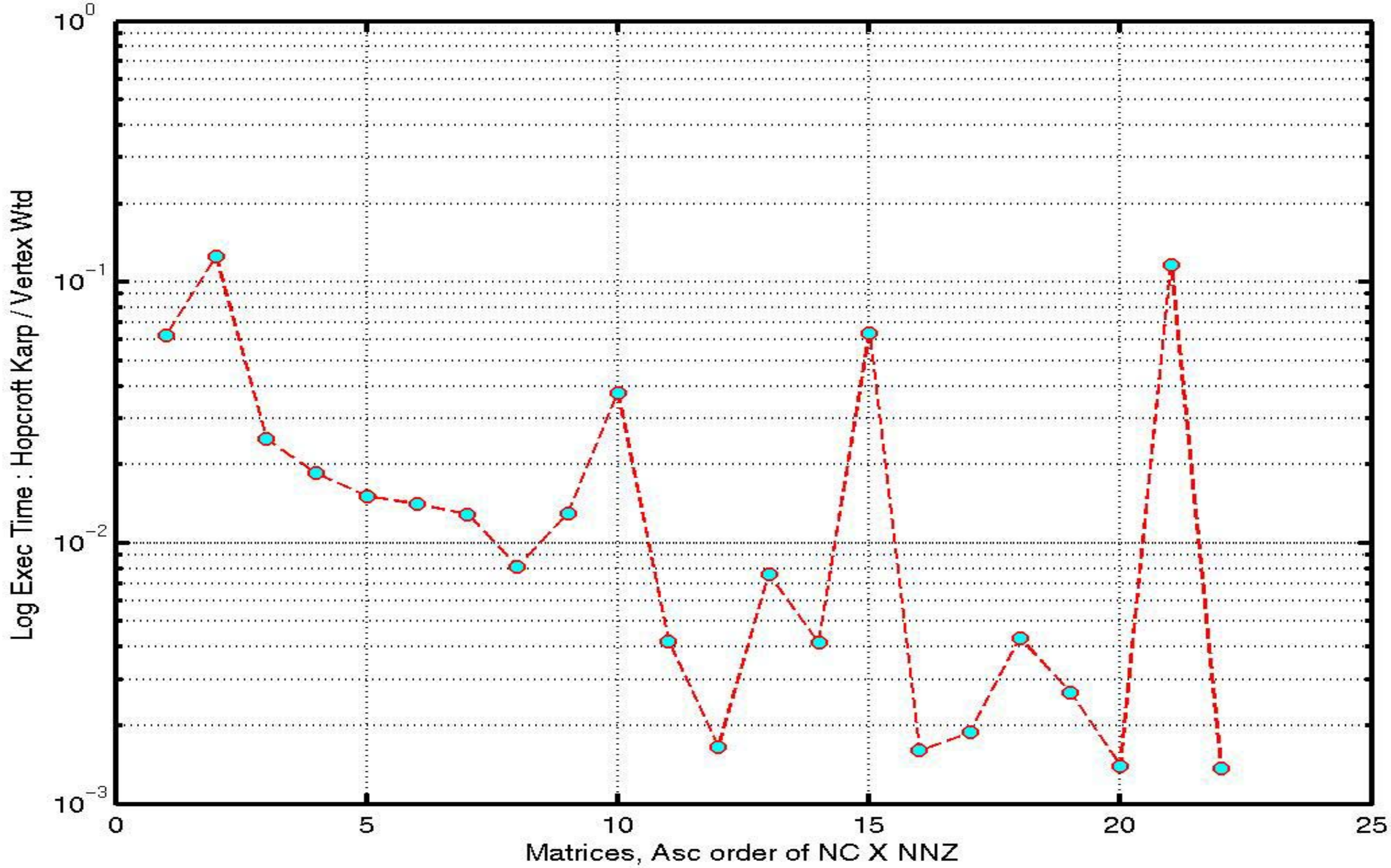

Vertex Wtd Matching
Max Cardinality Matching
Exact Algorithm
Single Path
Multiple Path
•Comparable
•Greedy Initialization
•Not Comparable
•Scope for improvement

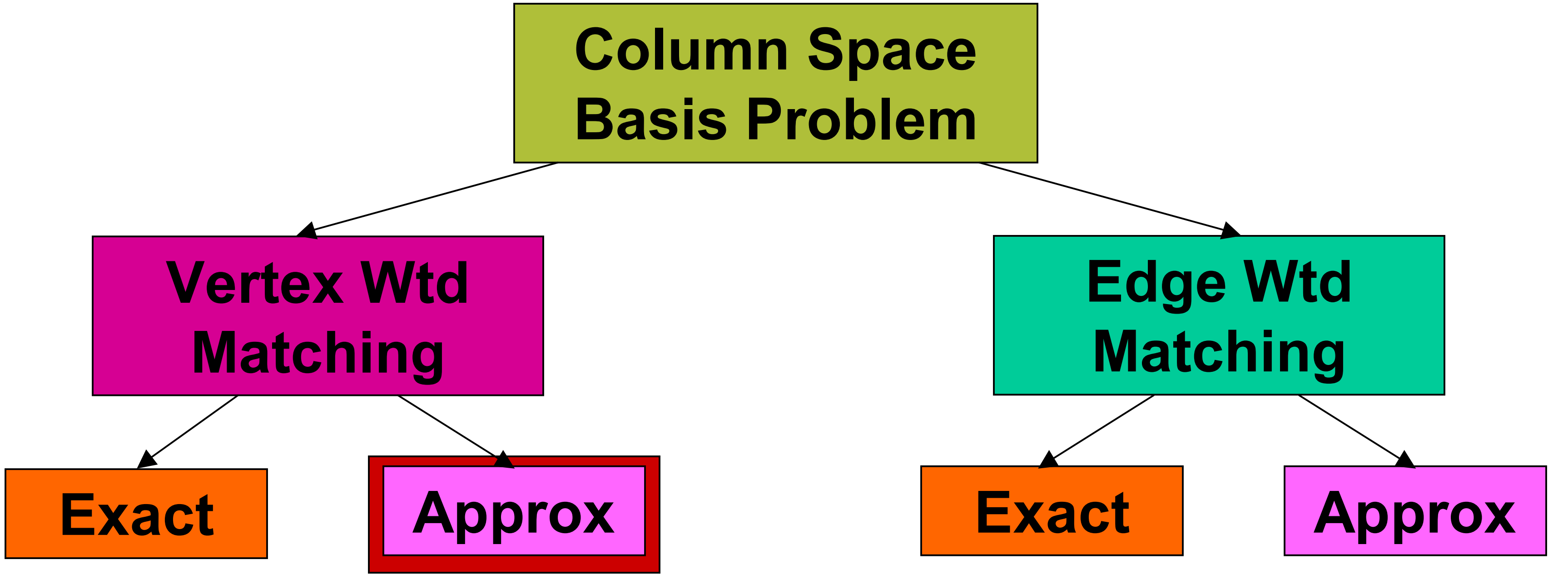
Time: Cardinality vs. Vertex Wt.

(Multiple Source-Single Path)



Time: Cardinality vs. Vertex Wt.
(Multiple Source-Multiple Path (Hopcroft-Karp))

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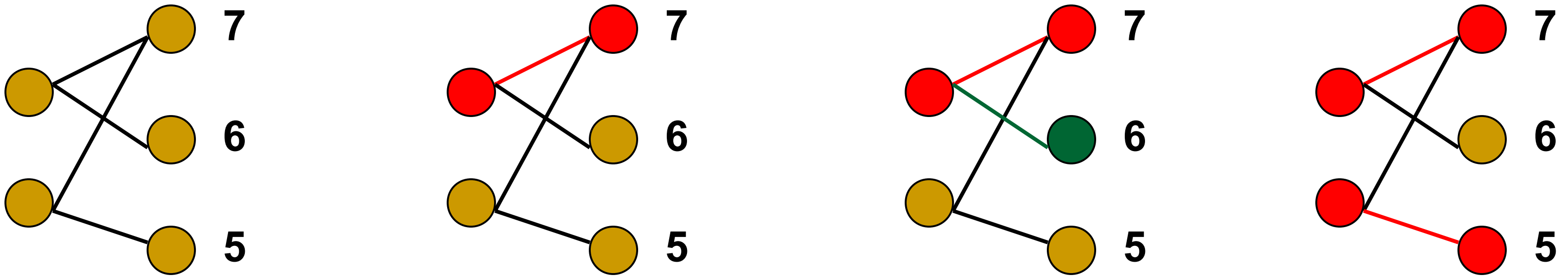
Example:

½ Approx, Max Path Length 1



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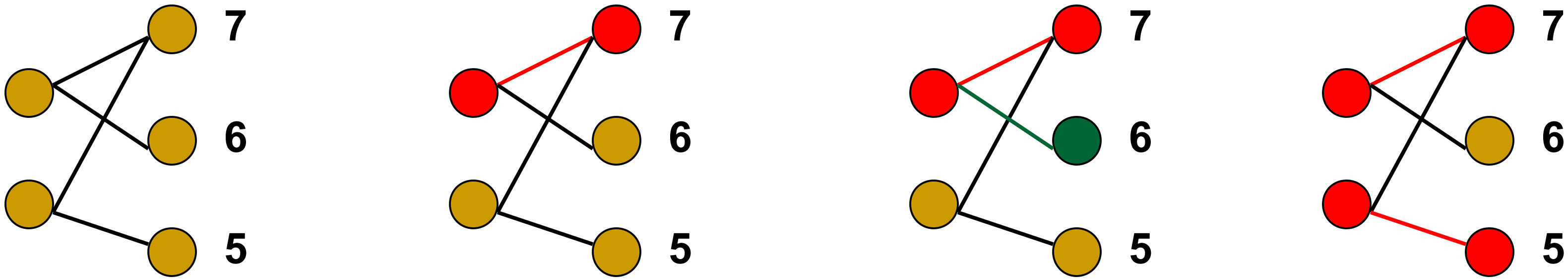
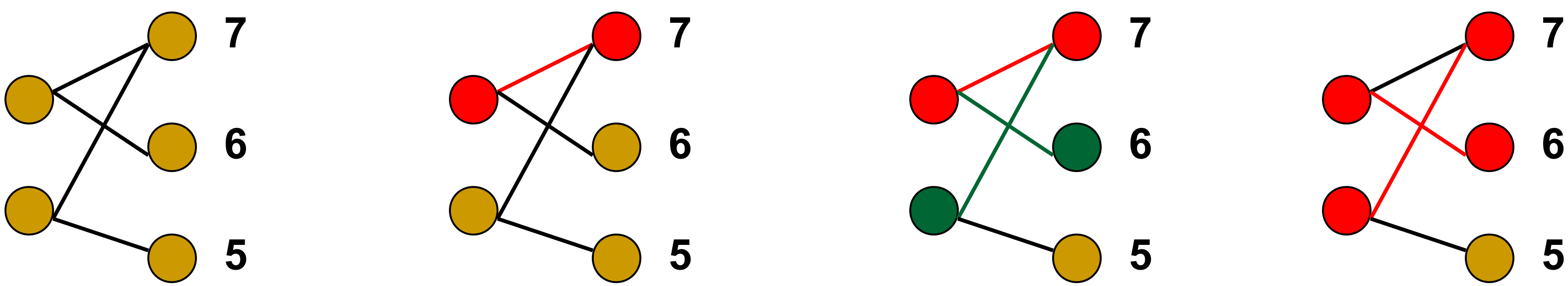


⅔ Approx, Max Path Length 3



Approximation Ratio:

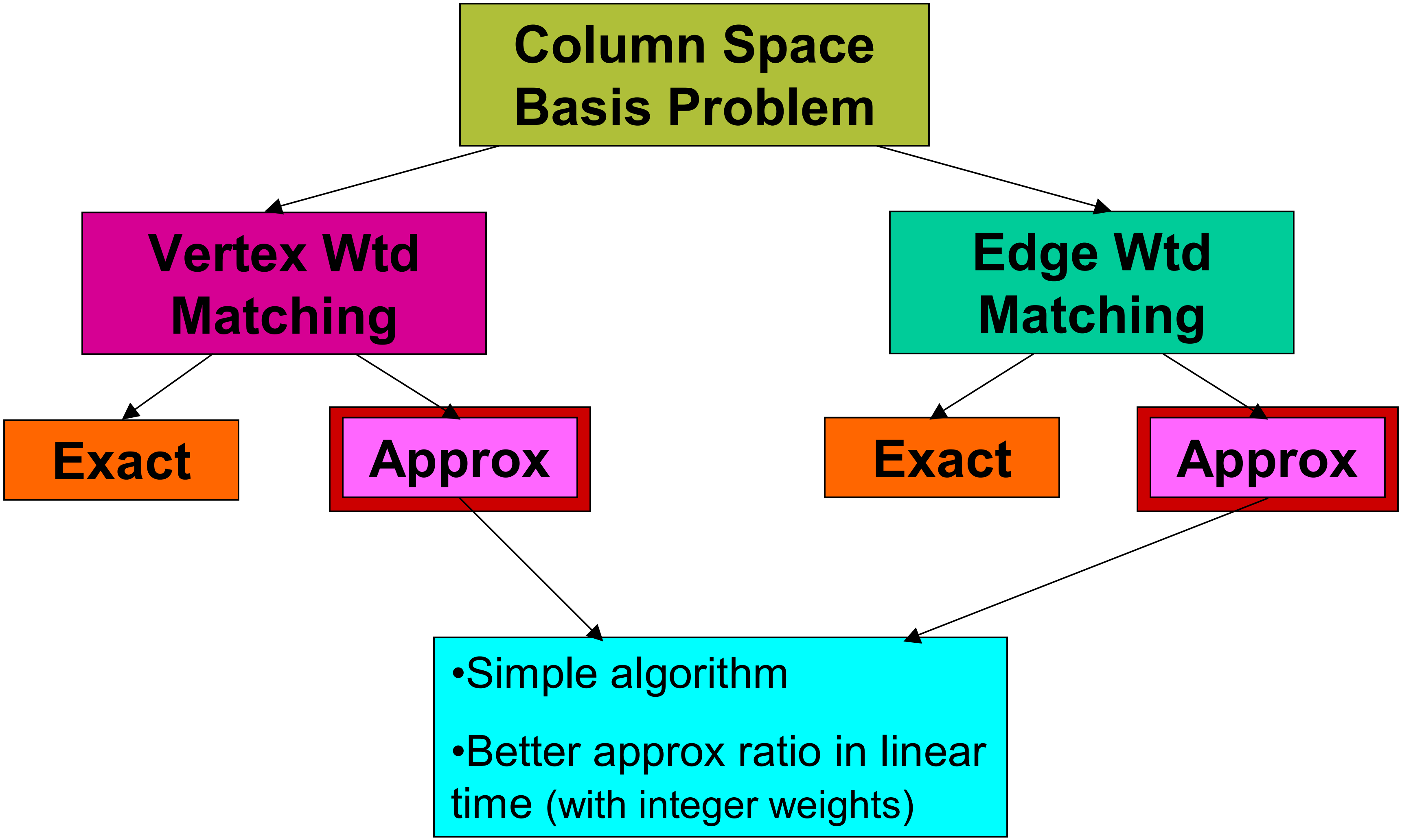
If *no* M-aug. path of length *(2k+1)* edges exists, then approx ratio of M to M_{opt} is:

$$\left(\frac{k+1}{k+2}\right)$$

k	0	1	2	…
A.R.	½	⅔	¾	…

Approx Algorithms for Edge-Wtd Matching:

Algorithms/People	App Ratio	Complexity
Global Max (Avis '83)	$1/2$	$O(\|E\|\log\|V\|)$
Local Max (Preis '99)	$1/2$	$O(\|E\|)$
Path Growing (Drake/Hougardy '03)	$1/2$	$O(\|E\|)$
Short Aug. (D/H '03)	$2/3-\varepsilon$	$O(\|E\|\frac{1}{\varepsilon})$
Rand & Det/ (Pettie/Sanders '04)	$2/3-\varepsilon$	$O(\|E\|\log\frac{1}{\varepsilon})$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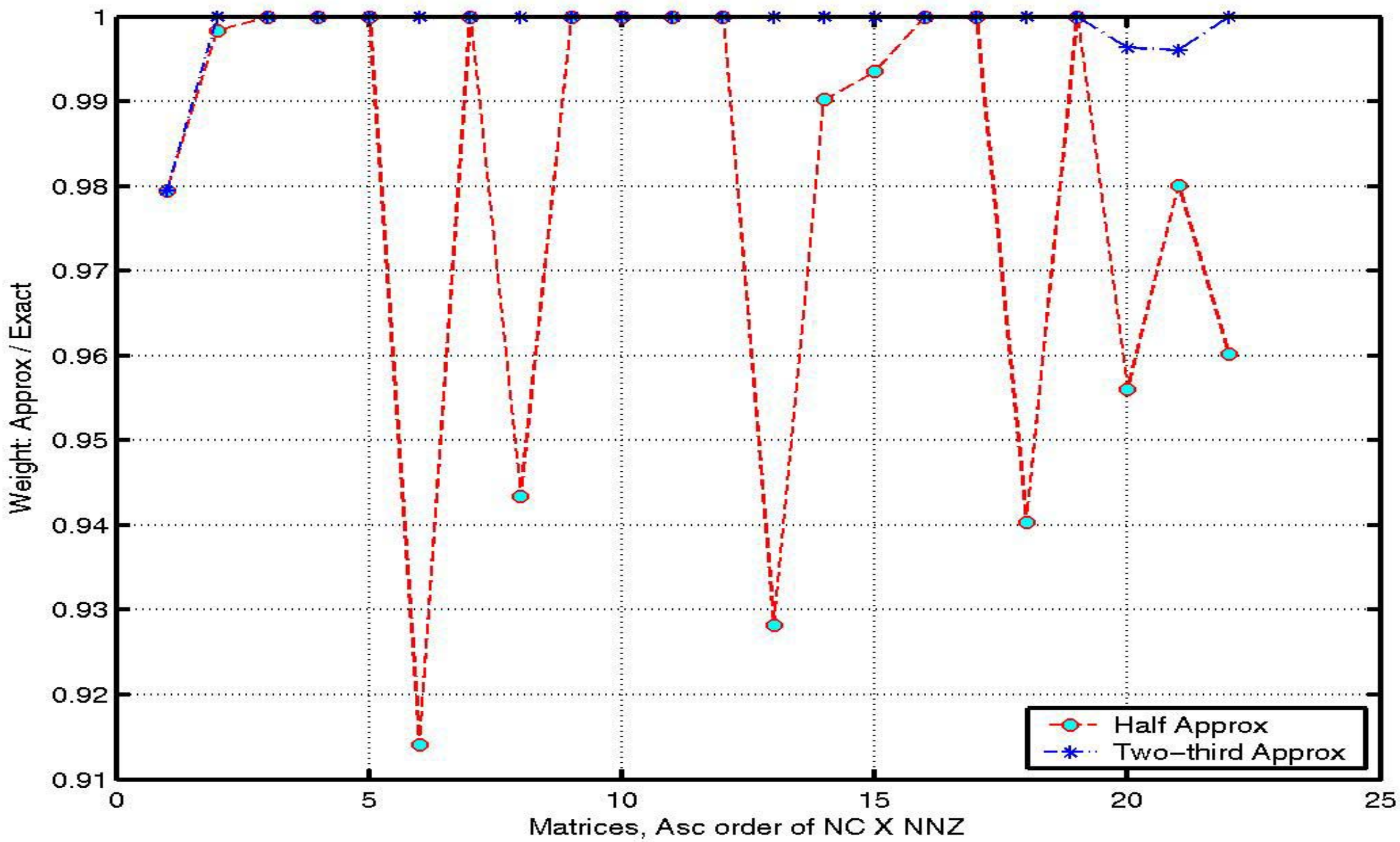
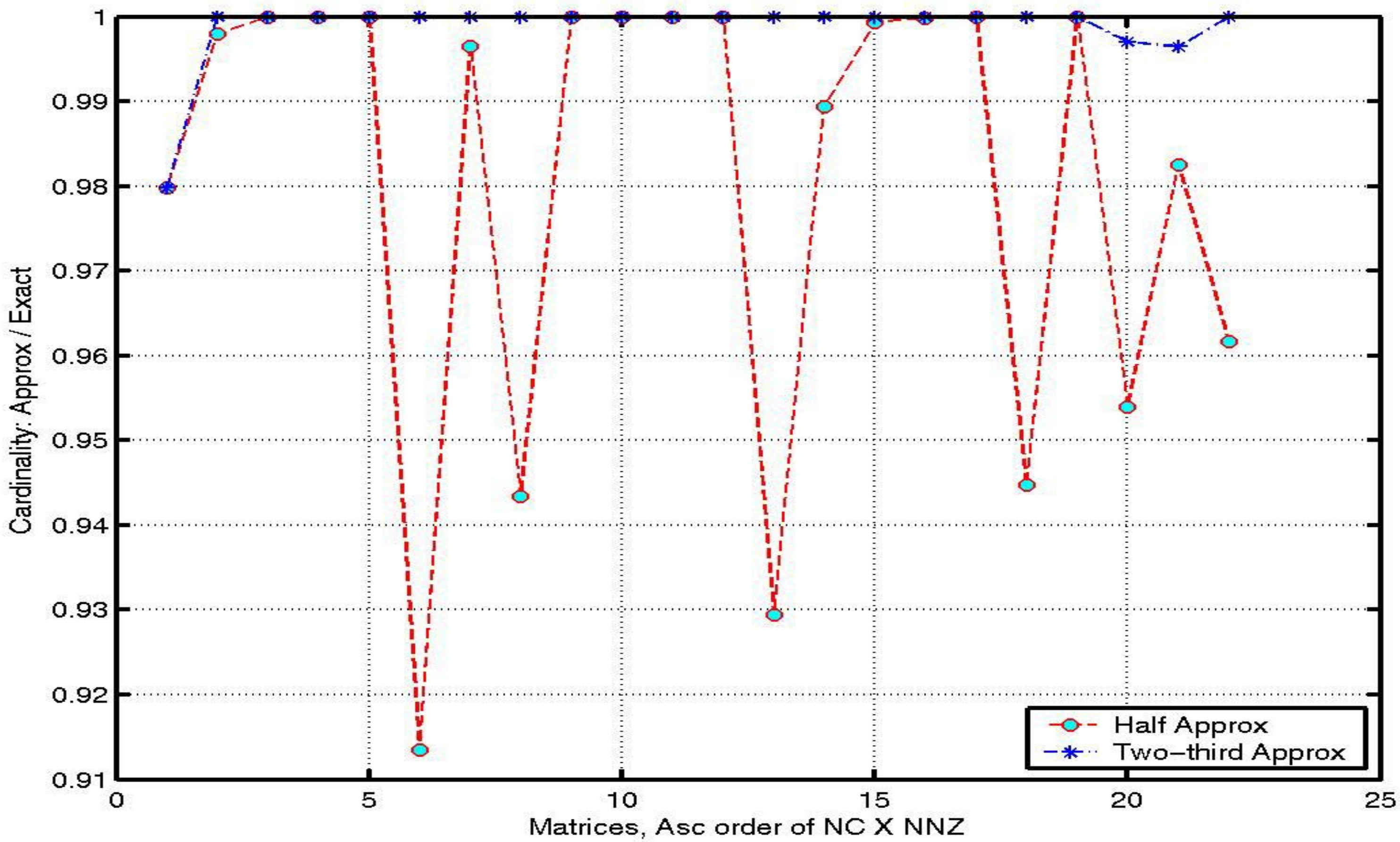
•Better approx ratio in linear time (with integer weights)

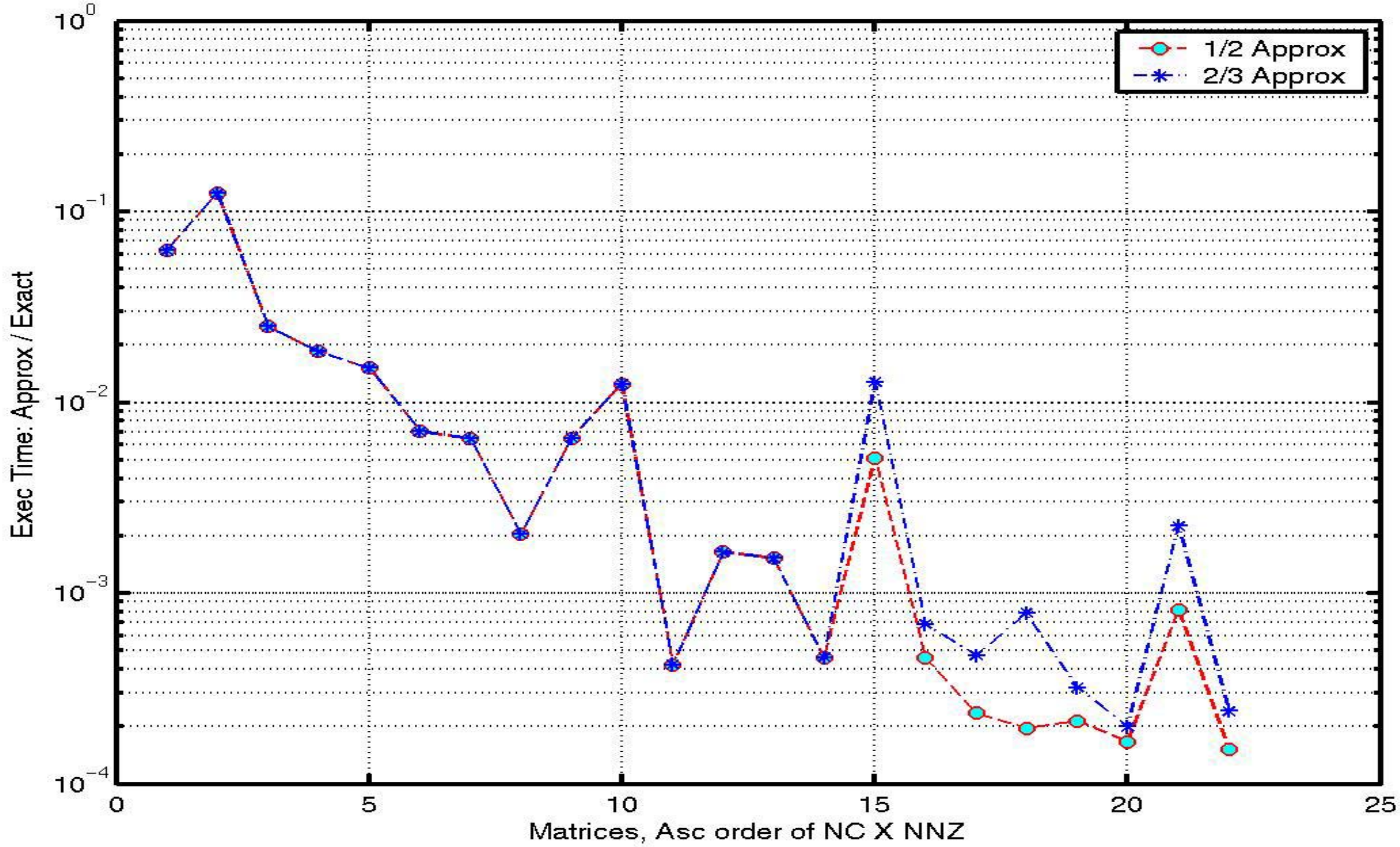
Weight: Approx vs. Exact



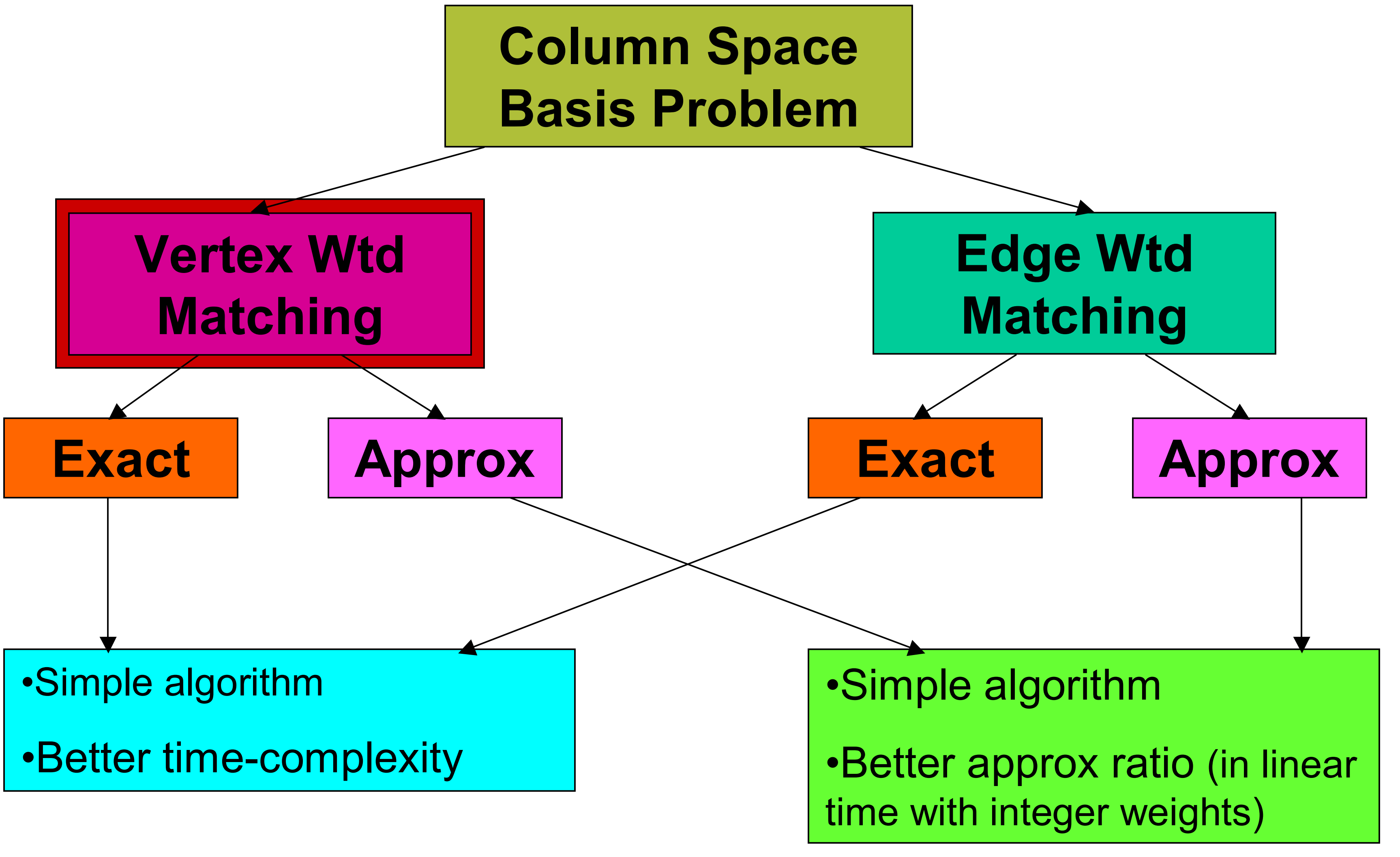
Cardinality: Approx vs. Exact



Time: Approx vs. Exact



Recap


Column Space Basis Problem
Vertex Wtd Matching
Edge Wtd Matching
Exact
Approx
Exact
Approx
•Simple algorithm
•Better time-complexity
•Simple algorithm
•Better approx ratio (in linear time with integer weights)

Future Work:

- We have extended the vertex weighted algorithms to bipartite graphs with weights on both the sets of vertices and nonbipartite graphs. Experimentation is required.
- Use vertex weighted matching algorithms to solve the column-space basis problem.
- Compute 1x1 and 2x2 pivots for symmetric indefinite matrices using max weight matchings in nonbipartite graphs.

References:

1. *Exact and Approx Algorithms for Vertex weighted matching*, in prep.
2. *Combinatorial Algorithms for Computing Column Space Bases that have Sparse Inverses.* Ali Pinar, Edmond Chow, and Alex Pothen, Preprint, 2005.
3. *Matching is as Easy as Matrix Inversion*. K. Mulmuley, U. Vazirani, and V. Vazirani, Proceedings of Symposium on the Theory of Computing, 1987. Combinatorica, Vol. 7, No. 1, 1987.
4. *Linear time ½-approximation algorithm for maximum weighted matching in general graphs*. R. Preis, In Proc. 16th Ann. Symp. On Theoretical Aspects of Computer Science (STACS), LNCS 1563, pages 259-269, 1999.
5. *Improved Linear Time Approximation Algorithms for Weighted Matchings.* D. Drake and S. Hougardy, 7th Workshop on Randomization, and Approximation Techniques in Computer Science, LNCS 2764, pages 14-23, 2003.