# Solving sequences of linear systems by preconditioned iterative methods

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joint work with

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# Outline

1. Motivation
2. Our goal: Reuse of matrix approximations / preconditioners
3. Reuse of structural information (preconditioner patterns)
4. Reuse of structural $+$ numerical information (preconditioners)
5. Conclusions
6. Advertisement

# 1. Motivation / Newton's method

## 1. Solving systems of nonlinear equations

$$F(x) = 0$$

$$\Downarrow$$

Sequences of linear systems of the form

$$J(x_k)\Delta x = -F(x_k),\ J(x_k) \approx F'(x_k)$$

solved until for some $k, k = 1, 2, \ldots$

$$\|F(x_k)\| < tol$$

$J(x_k)$ may change at points influenced by nonlinearities

## 2. Solving nonlinear convection-diffusion problems

$$-\Delta u + u\nabla u = f$$

$$\Downarrow$$

E.g., from the upwind discretization in 2D, with $u \geq 0$ we get for grid internal nodes $(i, j)$

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij} + hu_{ij}(2u_{ij} - u_{i-1,j} - u_{i,j-1}) = h^2 f_{ij}$$

It is a matrix with five diagonals

Entries in its three diagonals may change in subsequent linear systems

# 1. Motivation / Nonlinear convection-diffusion

## 2. Solving nonlinear convection-diffusion problems (continued)

$$-\Delta u + u \nabla u = f$$

$$\begin{pmatrix}
* & * & \dots & * & & & & \\
* & * & * & \dots & * & & & \\
\vdots & * & * & * & \dots & * & & \\
* & \vdots & * & * & * & \dots & * & \\
 & * & \vdots & * & * & * & \dots & * \\
 & & * & \vdots & * & * & * & \dots \\
 & & & * & \vdots & * & * & * \\
 & & & & * & \vdots & * & *
\end{pmatrix}$$

## 2. Solving nonlinear convection-diffusion problems (continued)

$$-\Delta u + u \nabla u = f$$

$$\begin{pmatrix}
\spadesuit & \spadesuit & \ldots & \spadesuit & & & & \\
* & \spadesuit & \spadesuit & \ldots & \spadesuit & & & \\
\vdots & * & \spadesuit & \spadesuit & \ldots & \spadesuit & & \\
* & \vdots & * & \spadesuit & \spadesuit & \ldots & \spadesuit & \\
 & * & \vdots & * & \spadesuit & \spadesuit & \ldots & \spadesuit \\
 & & * & \vdots & * & \spadesuit & \spadesuit & \ldots \\
 & & & * & \vdots & * & \spadesuit & \spadesuit \\
 & & & & * & \vdots & * & \spadesuit
\end{pmatrix}$$

## 3. Solving equations with a parabolic term

$$\frac{\partial u}{\partial t} - \Delta u = f$$

$$\Downarrow$$

E.g., 2D problem with $2^{nd}$ order centered differences in space and backward Euler time discretization for grid internal nodes $(i,j)$ and time step $t+1$

$$h^2(u_{ij}^{t+1} - u_{ij}^{t}) + \tau(u_{i+1,j}^{t+1} + u_{i-1,j}^{t+1} + u_{i,j+1}^{t+1} + u_{i,j-1}^{t+1} - 4u_{ij}^{t+1}) = h^2 \tau f_{ij}^{t+1}$$

Again, we get a matrix with five diagonals

Diagonal entries may change with time steps

# 2. Our goal / Reuse of approximations: I.

Reuse of approximations of matrices in sequences of linear systems

Solving sequences of systems of linear equations

$$A^0 x = b^0,\ A^1 x = b^1,\ \ldots$$

by preconditioned iterative methods with preconditioners

$$M^0, M^1, \ldots$$

Goal: computing $M^{i+k}$ from $A^{i+k}$ for some $k \geq 1$, and possibly reuse additional information from $M^i$ and $A^i$.

# 2. Our goal / Reuse of approximations: II.

Two basic strategies for the information reuse

# 2. Our goal / Reuse of approximations: II.

Two basic strategies for the information reuse

1. Reuse of matrix patterns
   - Using pattern of $M^i$.
   - Using pattern of $\widehat{A}_i$ ($A_i$, or a part of $A_i$).

# 2. Our goal / Reuse of approximations: II.

Two basic strategies for the information reuse

1. Reuse of matrix patterns
   - Using pattern of $M^i$.
   - Using pattern of $\widehat{A}_i$ ($A_i$, or a part of $A_i$).

2. Reuse of both patterns and values
   - Using entries of $M^i$.
   - Using entries of $\widehat{A}_i$ ($A_i$, or a part of $A_i$).

# 2. Our goal / Related work: I.

## Some related work

- Preconditioners from quasi-Newton updates (Morales, Nocedal, 2000)
- Freezing approximate Jacobians over a couple of subsequent systems (MNK: Shamanskii, 1967; Brent, 1973).
- Freezing the preconditioner over a couple of subsequent systems (MFNK: Knoll, McHugh, 1998).
- Some simple preconditioners (e.g., Jacobi, ILU(0) for PDEs) may be readily available even in parallel and/or matrix-free environment
- Preconditioners from a related matrix, operator (e.g., based on orthogonal grid, Truchas code, LANL, 2003; cf. Knoll, Keyes, 2004) a lot of approaches)
- Solving systems in adaptive filtering by incomplete factorizations + iterative refinement (Comon, Trystram, 1987)
- Approximate diagonal updates (Benzi, Bertaccini, 2003; Bertaccini, 2004)

# 2. Our goal / Related work: II.

## Some related work (continued)

- World of updates of Krylov subspaces (e.g., Morgan 1995-2002); Baglama, Calvetti, Golub, Reichel, 1999; Carpentieri et. al., 2003; de Sturler, 1996; Erhel, Burrage, Pohl, 1996; Duff, Giraud, Langou, Martin, 2005; Giraud et. al. 2004–2005; Parks et al. 2004)
- Dense updates of decompositions (Bartels, Golub, Saunders, 1970; Gill, Golub, Murray, Saunders, 1974)
- Sparse updates of decompositions (Hager, Davis, 1999–2004).

# 2. Our goal / Related work: III.

## Some related work (continued)

- Use of cheap matrix estimations based on graph coloring techniques in matrix free-environment if we know the matrix structure. This is a classical field; a (very restricted) selection of references: Curtis, Powell; Reid,1974; Coleman, Moré, 1983; Coleman, Moré, 1984; Coleman, Verma, 1998;

  The procedure

  - ⁎ Estimate the matrix $A_i$ by a few matvecs
  - ⁎ Get the preconditioner $M_i$ directly from $A_i$

- extensions to SPD (Hessian) approximations; extensions to use both $A$ and $A^T$ in automatic differentiation; more sophisticated estimation of resulting entries (substitution methods)
- to get only a part of the matrix which changes in the outer iterations: partial graph coloring, Gebremedhin, Manne, Pothen, 2004.

# 2. Our goal: An example

The 2D nonlinear convection-diffusion problem (Kelley, 1995); 5-point finite diferences; uniform grid $70 \times 70$; first 8 systems; ILUT(0.1,5)

$$\Delta u - Ru\nabla u = 2000x(1-x)y(1-y),\ R = 500$$

| A-matrix | M-matrix | $CG - its$ |
|---|---|---|
| $A^1$ | $M^1$ | 25 |
| $A^2$ | $M^1$ | 98 |
| $A^3$ | $M^1$ | 90 |
| $A^4$ | $M^1$ | 135 |
| $A^5$ | $M^1$ | 179 |
| $A^6$ | $M^1$ | 229 |
| $A^7$ | $M^1$ | 275 |
| $A^8$ | $M^1$ | 345 |
| $A^{1-8}$ | $M^{1-8}$ | 25 $\pm$ 10 |

Freezing the preconditioner may not be enough

# 3. Reuse of matrix patterns

What do we gain if we use a sparsified pattern for preconditioner computation?

1. Traditionally: Sparsified pattern $\longrightarrow$ cheaper preconditioner computation
2. Proposal 1: Sparsified pattern $+$ matrix-free environment $\longrightarrow$ cheaper but approximate matrix estimation via matvecs (Cullum, T., 2004)
3. Proposal 2: Freezing the matrix pattern in a matrix-free environment: use of the sparsified pattern of a different matrix from a sequence.
4. Proposal 3: Freezing the preconditioner pattern (from a different preconditioner)

# 3. Reuse of matrix patterns: II.

Traditionally: Sparsified pattern leads to cheaper preconditioner computation.

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Traditionally: Sparsified pattern leads to cheaper preconditioner computation.

original matrix $\longrightarrow$ large ($\spadesuit$) $\&$ small ($\clubsuit$) entries

$$\begin{pmatrix} \spadesuit & \spadesuit & & \spadesuit & & \\ \spadesuit & \spadesuit & & & \spadesuit & \\ & \spadesuit & \spadesuit & & \spadesuit & \\ \spadesuit & & & \spadesuit & \spadesuit & \\ & & & \spadesuit & \spadesuit & \spadesuit \\ & \spadesuit & & & \spadesuit & \spadesuit \end{pmatrix} \quad \begin{pmatrix} \spadesuit & \spadesuit & & \clubsuit & & \\ \spadesuit & \clubsuit & & & \spadesuit & \\ & \clubsuit & \spadesuit & & \spadesuit & \\ \spadesuit & & & \spadesuit & \clubsuit & \\ & & & \clubsuit & \spadesuit & \spadesuit \\ & \spadesuit & & & \clubsuit & \spadesuit \end{pmatrix}$$

# 3. Reuse of matrix patterns: II.

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$$
\begin{pmatrix}
\spadesuit & \spadesuit & & \spadesuit & & \\
\spadesuit & \spadesuit & & & \spadesuit & \\
& \spadesuit & \spadesuit & & \spadesuit & \\
\spadesuit & & & \spadesuit & \spadesuit & \\
& & & \spadesuit & \spadesuit & \spadesuit \\
& \spadesuit & & & \spadesuit & \spadesuit
\end{pmatrix}
\begin{pmatrix}
\spadesuit & \spadesuit & & & & \\
\spadesuit & & & & \spadesuit & \\
& & \spadesuit & & \spadesuit & \\
\spadesuit & & & \spadesuit & & \\
& & & & \spadesuit & \spadesuit \\
& \spadesuit & & & & \spadesuit
\end{pmatrix}
$$

# 3. Reuse of matrix patterns: Proposal 1

Proposal 1: Sparsified pattern $+$ matrix-free environment $\rightarrow$ cheaper but approximate matrix estimation via matvecs (Cullum, T., 2004)

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Proposal 1: Sparsified pattern $+$ matrix-free environment $\rightarrow$ cheaper but approximate matrix estimation via matvecs (Cullum, T., 2004)

First, a gentle introduction into matrix estimation

# 3. Reuse of matrix patterns: Proposal 1

$$\begin{pmatrix}
\color{red}\spadesuit & * & & & & & & \\
\color{red}\spadesuit & * & * & & & & & \\
& * & * & \color{red}\spadesuit & & & & \\
& & * & \color{red}\spadesuit & * & & & \\
& & & \color{red}\spadesuit & * & * & & \\
& & & & * & * & \color{red}\spadesuit & \\
& & & & & * & \color{red}\spadesuit & * \\
& & & & & & \color{red}\spadesuit & *
\end{pmatrix}$$

Columns with “red spades” can be computed at the same time in one matvec since sparsity patterns of their rows do not overlap.

Namely, $A(e_1 + e_4 + e_7)$ computes entries in the columns $1$, $4$ and $7$.

# 3. Reuse of matrix patterns: Proposal 1

four matvecs needed $\rightarrow$ large (coloured $\spadesuit$) $\&$ small ($\color{magenta}\clubsuit$) entries

$$
\begin{pmatrix}
\color{red}\spadesuit & \color{blue}\spadesuit & & \color{orange}\spadesuit & & \\
\color{red}\spadesuit & \color{blue}\spadesuit & & & \color{green}\spadesuit & \\
& \color{blue}\spadesuit & \color{red}\spadesuit & & \color{green}\spadesuit & \\
\color{red}\spadesuit & & & \color{orange}\spadesuit & \color{green}\spadesuit & \\
& & & \color{orange}\spadesuit & \color{green}\spadesuit & \color{red}\spadesuit \\
& \color{blue}\spadesuit & & & \color{green}\spadesuit & \color{red}\spadesuit
\end{pmatrix}
\qquad
\begin{pmatrix}
\color{red}\spadesuit & \color{blue}\spadesuit & & \color{magenta}\clubsuit & & \\
\color{red}\spadesuit & \color{magenta}\clubsuit & & & \color{green}\spadesuit & \\
& \color{magenta}\clubsuit & \color{red}\spadesuit & & \color{green}\spadesuit & \\
\color{red}\spadesuit & & & \color{orange}\spadesuit & \color{magenta}\clubsuit & \\
& & & \color{magenta}\clubsuit & \color{green}\spadesuit & \color{red}\spadesuit \\
& \color{blue}\spadesuit & & & \color{magenta}\clubsuit & \color{red}\spadesuit
\end{pmatrix}
$$

# 3. Reuse of matrix patterns: Proposal 1

large (coloured $\spadesuit$) $\&$ small ($\textcolor{magenta}{\clubsuit}$) entries $\quad \longrightarrow \quad$ only two matvecs needed

$$\begin{pmatrix} \textcolor{red}{\spadesuit} & \textcolor{blue}{\spadesuit} & & \textcolor{orange}{\spadesuit} & & \\ \textcolor{red}{\spadesuit} & \textcolor{blue}{\spadesuit} & & & \textcolor{green}{\spadesuit} & \\ & \textcolor{blue}{\spadesuit} & \textcolor{red}{\spadesuit} & & \textcolor{green}{\spadesuit} & \\ \textcolor{red}{\spadesuit} & & & \textcolor{orange}{\spadesuit} & \textcolor{green}{\spadesuit} & \\ & & & \textcolor{orange}{\spadesuit} & \textcolor{green}{\spadesuit} & \textcolor{red}{\spadesuit} \\ & \textcolor{blue}{\spadesuit} & & & \textcolor{green}{\spadesuit} & \textcolor{red}{\spadesuit} \end{pmatrix} \quad \begin{pmatrix} \textcolor{red}{\spadesuit} & \textcolor{blue}{\spadesuit} & & & & \\ \textcolor{red}{\spadesuit} & & & & \textcolor{green}{\spadesuit} & \\ & & \textcolor{red}{\spadesuit} & & \textcolor{green}{\spadesuit} & \\ \textcolor{red}{\spadesuit} & & & \textcolor{orange}{\spadesuit} & & \\ & & & & \textcolor{green}{\spadesuit} & \textcolor{red}{\spadesuit} \\ & \textcolor{blue}{\spadesuit} & & & & \textcolor{red}{\spadesuit} \end{pmatrix}$$

# 3. Reuse of matrix patterns: Proposal 1

large (coloured $\spadesuit$) $\&$ small ($\color{magenta}\clubsuit$) entries $\quad \longrightarrow \quad$ only two matvecs needed

$$
\begin{pmatrix}
\color{red}\spadesuit & \color{blue}\spadesuit & & \color{orange}\spadesuit & & \\
\color{red}\spadesuit & \color{blue}\spadesuit & & & \color{green}\spadesuit & \\
& \color{blue}\spadesuit & \color{red}\spadesuit & & \color{green}\spadesuit & \\
\color{red}\spadesuit & & & \color{orange}\spadesuit & \color{green}\spadesuit & \\
& & & \color{orange}\spadesuit & \color{green}\spadesuit & \color{red}\spadesuit \\
& \color{blue}\spadesuit & & & \color{green}\spadesuit & \color{red}\spadesuit
\end{pmatrix}
\qquad
\begin{pmatrix}
\color{red}\spadesuit & \color{blue}\spadesuit & & & & \\
\color{red}\spadesuit & & & & \color{blue}\spadesuit & \\
& & \color{red}\spadesuit & & \color{blue}\spadesuit & \\
\color{red}\spadesuit & & & \color{blue}\spadesuit & & \\
& & & & \color{blue}\spadesuit & \color{red}\spadesuit \\
& \color{blue}\spadesuit & & & & \color{red}\spadesuit
\end{pmatrix}
$$

# 3. Reuse of matrix patterns: Proposal 2: Results

Proposal 2: Freezing the matrix pattern in a matrix-free environment: use of the sparsified pattern of a different matrix from a sequence.

Driven cavity flow, $n = 8603$, $R = 500$, ILUT($1.0 * 10^{-6}, 25$); Newton

| No. A-matrix | No. P-matrix | $CG - its$ |
|---|---|---|
| $A^1$ | $M^1$ via $A^1$ and pattern of $\widehat{A}^1$ | 44 |
| $A^2$ | $M^2$ via $A^2$ and pattern of $\widehat{A}^1$ | 38 |
| $A^3$ | $M^3$ via $A^3$ and pattern of $\widehat{A}^1$ | 41 |
| $A^4$ | $M^4$ via $A^4$ and pattern of $\widehat{A}^1$ | 42 |
| $A^5$ | $M^5$ via $A^5$ and pattern of $\widehat{A}^1$ | 43 |
| $A^2$ | $M^2$ via $A^2$ and pattern of $\widehat{A}^2$ | 42 |
| $A^3$ | $M^3$ via $A^3$ and pattern of $\widehat{A}^3$ | 38 |
| $A^4$ | $M^4$ via $A^4$ and pattern of $\widehat{A}^4$ | 43 |
| $A^5$ | $M^5$ via $A^5$ and pattern of $\widehat{A}^5$ | 42 |

Number of matvecs due to the use of sparsified patterns: 51 $\longrightarrow$ 26

# 3. Reuse of matrix patterns: Proposal 3

Proposal 3: Freezing the preconditioner pattern (for preconditioners with pattern input)

- Determination of a suitable pattern of preconditioner may be sometimes rather difficult

Our approach:

- Find a pattern for $A_0$ by a sophisticated / time-consuming method using both symbolic/numeric information
- Use the pattern to get preconditioners $M_0, M_1, \ldots$
- Restart if necessary

# 3. Reuse of matrix patterns: Proposal 3

old SPAI versus old LS based on old SPAI pattern

The 2D nonlinear convection-diffusion problem (Kelley, 1995); 5-point finite diferences; uniform grid $70 \times 70$; first 8 systems; SPAI(0.0,5,5); preconditioner size = 87300

$$\Delta u - Ru\nabla u = 2000x(1-x)y(1-y),\ R = 500$$

| NO UPDATES | | |
|---|---|---|
| $Matrix$ | $old\_SPAI\_its$ | $old\_LS/SPAI\_its$ |
| $A^1$ / $M^1$ | 50 | 49 |
| $A^2$ / $M^1$ | 76 | 71 |
| $A^3$ / $M^1$ | 89 | 79 |
| $A^4$ / $M^1$ | 113 | 134 |
| $A^5$ / $M^1$ | 138 | 147 |
| $A^6$ / $M^1$ | 157 | 161 |
| $A^7$ / $M^1$ | 201 | 173 |
| $A^8$ / $M^1$ | 238 | 179 |

# 3. Reuse of matrix patterns: Proposal 3

SPAI versus LS with old SPAI pattern

The same 2D nonlinear convection-diffusion problem (Kelley, 1995); SPAI(0.0,5,5)

new SPAI versus new LS with old SPAI pattern

| | |
|---|---|
| Number of iterations in SPAI: | $50 \pm 8$ |
| Number of iterations in LS with old SPAI pattern: | $50 \pm 8$ |
| Time for computing SPAI preconditioner | =0.62 – 0.69 s |
| Time for computing LS preconditioner with old SPAI pattern | =0.22 s |
| Sequential time for 50 iterations | $\approx$ 0.1 s |

# 4. Reuse of matrix: Entrywise updates

Exploiting both pattern and numerical values

$$A_0 \to A_1 = A_0 + B$$

# 4. Reuse of matrix: Entrywise updates

Exploiting both pattern and numerical values

$$A_0 \longrightarrow A_1 = A_0 + B$$

Case I: Simple one-entry off-diagonal updates

$$B = uv^T = \alpha e_i e_j^T$$

# 4. Reuse of matrix: Entrywise updates

Exploiting both pattern and numerical values

$$A_0 \longrightarrow A_1 = A_0 + B$$

Case I: Simple one-entry off-diagonal updates

$$B = uv^T = \alpha e_i e_j^T$$

$$A_1^{-1} = (A_0 + B)^{-1} = (A_0 + xy^T)^{-1} = (LDU + xy^T)^{-1} =$$
$$(L(D + L^{-1}xy^T U^{-1})U)^{-1} \approx U^{-1}(I + \alpha D^{-1} e_i e_j^T)^{-1} D^{-1} L^{-1} =$$

$$\frac{U^{-1}(I - \alpha D^{-1} e_i e_j^T) D^{-1} L^{-1}}{1 + \alpha e_j^T D^{-1} e_i}$$

# 4. Reuse of matrix: Entrywise updates

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$$\frac{U^{-1}(I - \alpha D^{-1} e_i e_j^T) D^{-1} L^{-1}}{1 + \alpha e_j^T D^{-1} e_i}$$

- Approximation good if there is a strong matrix diagonal – it can be forced by a weighted matching applied to the matrix graph
- Inversion of a Gauss-Jordan transform
- More Gauss-Jordan transforms can be accumulated in one sweep (using pattern-based conditions) . . . next slide

# 4. Reuse of matrix: Entrywise updates II.

More Gauss-Jordan transforms

- Entries for a sweep with Gauss-Jordan transforms can be (e.g.) found from a weighted spanning forest $T$ of the graph $G_B$ of a (sparsified) difference matrix $B = A_1 - A_0$.

# 4. Reuse of matrix: Entrywise updates II.

More Gauss-Jordan transforms

- Entries for a sweep with Gauss-Jordan transforms can be (e.g.) found from a weighted spanning forest $T$ of the graph $G_B$ of a (sparsified) difference matrix $B = A_1 - A_0$.

The procedure

1. Find the weighted forest $T$ of $G_B$
2. Find the order of Gauss-Jordan transforms corresponding to the edges of $T$ (It can be proved that it is feasible)
3. Add to the product of the Gauss-Jordan transforms other nonzeros entries of $B$ (It can be shown which entries, based on simple structural conditions)

# 4. Reuse of matrix: Entrywise updates III.

Updates of diagonal $+$ subdiagonal representing convection changes in a 2D convection-diffusion problem, $n = 10000$

| 2D CD problems with a 1D - convection shift | | |
|---|---|---|
| $Shift$ | $NO\_updates\_its$ | $CONV\_updates\_its$ |
| 0.1 | 52 | 54 |
| 0.2 | 58 | 60 |
| 0.3 | 67 | 60 |
| 0.4 | 68 | 61 |
| 0.5 | 75 | 65 |
| 0.6 | 81 | 73 |
| 0.7 | 100 | 76 |
| 0.8 | 121 | 78 |
| 0.9 | 146 | 81 |
| 1.0 | 186 | 82 |

New preconditioner: 34 iterations

# 4. Reuse of matrix: Entrywise updates IV.

Updates of diagonal $+$ subdiagonal representing convection changes in a 2D convection-diffusion problem (different from the previous one), $n = 10000$

| 2D CD problems with a 1D - convection shift | | |
|---|---|---|
| $Shift$ | $NO\_updates\_its$ | $CONV\_updates\_its$ |
| 0.1 | 34 | 33 |
| 0.2 | 36 | 35 |
| 0.3 | 39 | 34 |
| 0.4 | 38 | 33 |
| 0.5 | 44 | 34 |
| 0.6 | 56 | 38 |
| 0.7 | 53 | 34 |
| 0.8 | 63 | 34 |
| 0.9 | 69 | 30 |

New preconditioner: 35 iterations

# 4. Reuse of matrix: Triangular updates

Exploiting both pattern and numerical values

$$A_0 \rightarrow A_1 = A_0 + B$$

# 4. Reuse of matrix: Triangular updates

Exploiting both pattern and numerical values

$$A_0 \rightarrow A_1 = A_0 + B$$

Case II: Triangular updates

$$B = L_B + D_B + U_B$$

# 4. Reuse of matrix: Triangular updates

Exploiting both pattern and numerical values

$$A_0 \rightarrow A_1 = A_0 + B$$

Case II: Triangular updates

$$B = L_B + D_B + U_B$$

$$A_1^{-1} = (A_0 + B)^{-1} = (A_0 + L_B + D_B + U_B)^{-1} =$$
$$(LDU + L_B + D_B + U_B)^{-1} = (L(D + L^{-1}(L_B + D_B + U_B)U^{-1})U)^{-1} \approx$$
$$U^{-1}(I + D(D_B + U_B)U^{-1})^{-1}D^{-1}L^{-1} = (DU + D_B + U_B)^{-1}L^{-1}$$

# 4. Reuse of matrix: Triangular updates

Exploiting both pattern and numerical values

$$A_0 \longrightarrow A_1 = A_0 + B$$

Case II: Triangular updates

$$B = L_B + D_B + U_B$$

$$A_1^{-1} = (A_0 + B)^{-1} = (A_0 + L_B + D_B + U_B)^{-1} =$$
$$(LDU + L_B + D_B + U_B)^{-1} = (L(D + L^{-1}(L_B + D_B + U_B)U^{-1})U)^{-1} \approx$$
$$U^{-1}(I + D(D_B + U_B)U^{-1})^{-1}D^{-1}L^{-1} = (DU + D_B + U_B)^{-1}L^{-1}$$

- Approximation good in many practical situations
- It can be shown theoretically that it behaves well if $||L - I||$ is small
- . . . but not only in this case, as the subsequent experiments show
- $L$ or $U$ based

# 4. Reuse of matrix: Triangular updates II.

Triangular update versus no update

The 2D nonlinear convection-diffusion problem (Modified from Kelley, 1995); upwind FD; uniform grid $50 \times 50$; BiCgstab ($10^{-7}$), ILU(0.001)

$$\Delta u - Ru\nabla u = x(1-x)y(1-y), \; R = 50$$

| matrix / precond | CG its / no update | CG its / L-based update | CG its / U-based update |
|---|---|---|---|
| $A^1$ / $M^1$ | 4 | 4 | 4 |
| $A^2$ / $M^1$ | 8 | 6 | 6 |
| $A^3$ / $M^1$ | 11 | 8 | 10 |
| $A^4$ / $M^1$ | 16 | 11 | 16 |
| $A^5$ / $M^1$ | 23 | 17 | 24 |
| $A^6$ / $M^1$ | 32 | 22 | 24 |
| $A^7$ / $M^1$ | 70 | 37 | 32 |
| $A^8$ / $M^1$ | 68 | 40 | 32 |
| $A^9$ / $M^1$ | 67 | 41 | 32 |
| $A^{10}$ / $M^1$ | 74 | 42 | 32 |
| $A^{1-10}$ / $M^{1-10}$ | $4 \pm 2$ | $4 \pm 2$ | $4 \pm 2$ |

# 4. Reuse of matrix: Triangular updates III.

Triangular update versus no update

The 2D nonlinear convection-diffusion problem (Modified from Kelley, 1995); upwind FD; uniform grid $50 \times 50$; BiCgstab ($10^{-7}$), ILU(0.001)

$$\Delta u - Ru\nabla u = x(1-x)y(1-y), \ \ R = 100$$

| matrix / precond | CG its / no update | CG its / L-based update | CG its / U-based update |
|---|---|---|---|
| $A^1$ / $M^1$ | 4 | 4 | 4 |
| $A^2$ / $M^1$ | 8 | 5 | 5 |
| $A^3$ / $M^1$ | 12 | 7 | 10 |
| $A^4$ / $M^1$ | 16 | 8 | 13 |
| $A^5$ / $M^1$ | 19 | 10 | 13 |
| $A^6$ / $M^1$ | 23 | 13 | 16 |
| $A^7$ / $M^1$ | 32 | 16 | 21 |
| $A^8$ / $M^1$ | 44 | 25 | 21 |
| $A^9$ / $M^1$ | 50 | 28 | 23 |
| $A^{10}$ / $M^1$ | 48 | 27 | 23 |
| $A^{1-11}$ / $M^{1-11}$ | $4 \pm 2$ | $4 \pm 2$ | $4 \pm 2$ |

# 4. Reuse of matrix: Triangular updates IV.

Triangular update versus no update

The 2D nonlinear convection-diffusion problem (Kelley, 1995); 5-point finite diferences; uniform grid $70 \times 70$; first 8 systems; ILUT(0.1,5)

$$\Delta u - Ru\nabla u = 2000x(1-x)y(1-y), \; R = 500$$

| matrix / precond | CG its / no update | CG its / triangular update |
|---|---|---|
| $A^1$ / $M^1$ | 25 | 25 |
| $A^2$ / $M^1$ | 98 | 30 |
| $A^3$ / $M^1$ | 90 | 27 |
| $A^4$ / $M^1$ | 135 | 30 |
| $A^5$ / $M^1$ | 179 | 35 |
| $A^6$ / $M^1$ | 229 | 36 |
| $A^7$ / $M^1$ | 275 | 36 |
| $A^8$ / $M^1$ | 345 | 53 |
| $A^{1-8}$ / $M^{1-8}$ | $25 \pm 10$ | $25 \pm 10$ |

# 4. Reuse of matrix: Triangular updates V.

Triangular update versus no update

The 2D nonlinear convection-diffusion problem (Kelley, 1995); 5-point finite diferences; uniform grid $70 \times 70$; first 8 systems; ILUT(0.1,5)

$$\Delta u - Ru\nabla u = 2000x(1-x)y(1-y), \; R = 100$$

| matrix / precond | CG its / no update | CG its / triangular update |
|---|---|---|
| $A^1$ / $M^1$ | 25 | 25 |
| $A^2$ / $M^1$ | 33 | 26 |
| $A^3$ / $M^1$ | 46 | 26 |
| $A^4$ / $M^1$ | 67 | 28 |
| $A^5$ / $M^1$ | 80 | 27 |
| $A^6$ / $M^1$ | 113 | 27 |
| $A^7$ / $M^1$ | 117 | 27 |
| $A^8$ / $M^1$ | 147 | 28 |
| $A^{1-8}$ / $M^{1-8}$ | $25 \pm 10$ | $25 \pm 10$ |

# 4. Reuse of matrix: Triangular updates VI.

Triangular update versus no update

The 2D nonlinear convection-diffusion problem (Kelley, 1995); 5-point finite diferences; uniform grid $70 \times 70$; first 8 systems; ILUT(0.1,5)

$$\Delta u - Ru\nabla u = 2000x(1-x)y(1-y), \; R = 50$$

| matrix / precond | CG its / no update | CG its / triangular update |
|---|---|---|
| $A^1$ / $M^1$ | 25 | 25 |
| $A^2$ / $M^1$ | 33 | 26 |
| $A^3$ / $M^1$ | 47 | 27 |
| $A^4$ / $M^1$ | 58 | 27 |
| $A^5$ / $M^1$ | 83 | 27 |
| $A^6$ / $M^1$ | 88 | 28 |
| $A^7$ / $M^1$ | 119 | 28 |
| $A^8$ / $M^1$ | 114 | 27 |
| $A^{1-8}$ / $M^{1-8}$ | $25 \pm 10$ | $25 \pm 10$ |

# 5. Conclusions

- Nonsymmetric preconditioners in the form of decompositions can be successfully updated by algebraic techniques.
- Shown to be efficient in solving nonlinear problems
- Also pattern recycling can help
- A lot of possible combinations with matrix-free environment
- What we did not show
  - additional scaling by $diag(L/U)$ improves the triangular update
  - mixing $L$ and $U$ based preconditioners; preprocessing for using $L$ or $U$
- A lot of possibilities for improvements under investigation
  - e.g., combination of rank-1 and rank-n updates
  - e.g., sparse updates of incomplete factorizations

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Special issue of LAA to celebrate 80th birthday of Prof. Miroslav Fiedler announced in the latest NA-digest, will cover LAA topics + his interests (i.e., also CSC interests); deadline for submissions: end of 2005