Numerical Analysis of Dynamic Centrality

Philip A. Knight[1]

[1]Department of Mathematics and Statistics
University of Strathclyde

Sparse Days, September 2017

Static Centrality

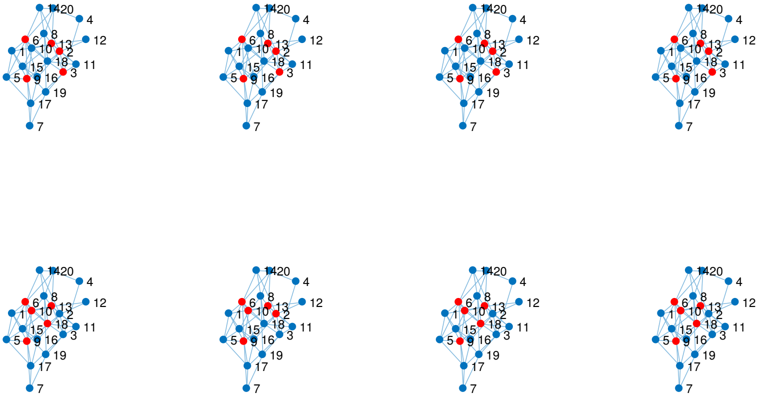
- Centrality measures are widely used in network theory.
- First introduced by Camille Jordan.
- A theory developed in social sciences from 1950s – 1980s
- **Classical measures:** Degree, closeness, betweenness.
- **Spectral measures:** Katz, eigenvector, Pagerank, subgraph centrality.

Katz Centrality

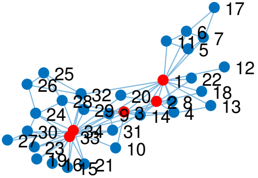
$$\mathbf{k} = (I - \alpha A)^{-1}\mathbf{e}$$

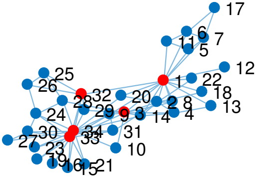
- Introduced by Katz in 1953.
- α can take any value outside spectrum of A^{-1}.
- Useful range of α lies in $(0, 1/\lambda_1(A))$.
- $(I - \alpha A)^{-1}\mathbf{e} = \mathbf{e} + \alpha A\mathbf{e} + \alpha^2 A^2\mathbf{e} + \cdots$ counts weighted sum of walks.
- As $\alpha \to 0$, $\mathbf{k}$ converges to degree centrality.
- As $\alpha \to 1/\lambda_1$, $\mathbf{k}$ converges to eigenvector centrality.

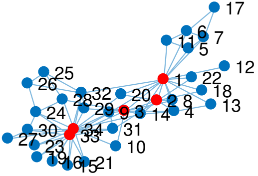
Example 1: Central Nodes

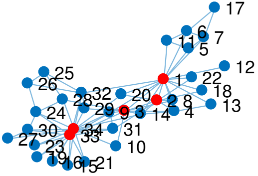


Example 2: Karate









Dynamic Networks

- In many applications we are interested in a network which evolves with time.
- Assume there is a constant number of nodes n but that some edges disappear/appear as time passes.
- We end up with a sequence of adjacency matrices $A^{[1]}$, $A^{[2]}, \ldots A^{[M]}$.
- Many properties of static networks can be generalised.
- We'll make use of **dynamic walks**, **paths** and **distances**.

Dynamic Centrality

- Dynamic closeness and betweenness defined in terms of dynamic paths and distances.
- Dynamic degree centrality: $\sum_{m=1}^{M} A^{[m]}\mathbf{e}$.
- Self-induced eigenvector centrality: Perron vector of $\sum_{m=1}^{M} A^{[m]}$.
- Alternatively, scale each adjacency matrix by degree.
- Or work with the Perron vector of

$$\begin{bmatrix} \epsilon f(A^{[1]}) & I & & & \\ I & \epsilon f(A^{[2]}) & I & & \\ & \ddots & \ddots & \ddots & \\ & & I & \epsilon f(A^{[M-1]}) & I \\ & & & I & \epsilon f(A^{[M]}) \end{bmatrix}.$$

Dynamic Katz Centrality

$$(I - \alpha A^{[1]})^{-1} \cdots (I - \alpha A^{[m]})^{-1}\mathbf{e}.$$

- Could use variable α.
- Respects time arrow.
- It makes sense to look at

  $$\mathbf{e}^T (I - \alpha A^{[1]})^{-1} \cdots (I - \alpha A^{[m]})^{-1},$$

  too.
- There's an assumption that walks of arbitrary length are possible at any time step.
- As $\alpha \longrightarrow 0$, dynamic Katz matches dynamic degree.
- But limit as $\alpha \longrightarrow \alpha_{\max}$ no longer connected to eigenvectors.

Dynamic Subgraph Centrality

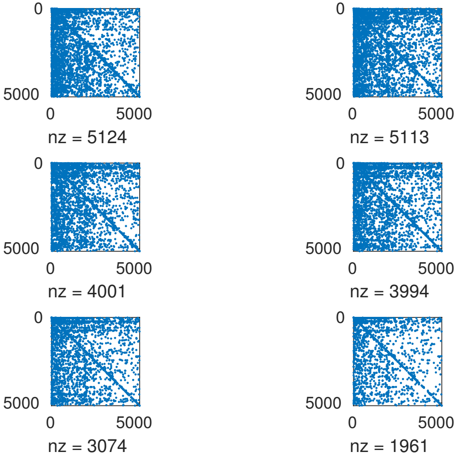
$$Q = f(A^{[1]})f(A^{[2]})\cdots f(A^{[M]}).$$

- Define a centrality measure by computing a statistic associate with Q.
- Time's arrow respected.
- $Q\mathbf{e}$, $\mathbf{diag}(Q)$, Perron vector of Q.
- For $f(A)$ we could choose A, $(I-A)^{-1}$, e^A, ...
- We advocate the Perron vector of $f(A) = I + A$ as a natural extension of eigenvector centrality.

Example 3: Random Dynamic Graph

Degree	Katz	Katz v	$f(A) = A$	$(I + A)$
17	5	5	5	5
13	17	17	7	7
3	7	9	2	17
12	13	2	9	12
11	9	7	12	2

Example 4: Kiel



Example 4: Kiel

Degree	EV	Katz v	$f(A) = A$	$(I + A)$
31	343	228	343	228
228	31	31	1094	31
38	340	343	2683	1094
767	1027	297	31	343
119	1444	4929	1564	2683

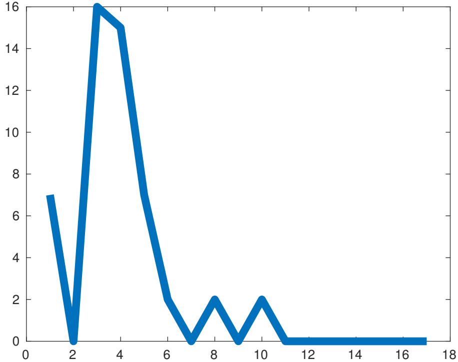
Too Much Time On My Hands

- Suppose $A^{[m]}$ is fixed. What do we find using Q?

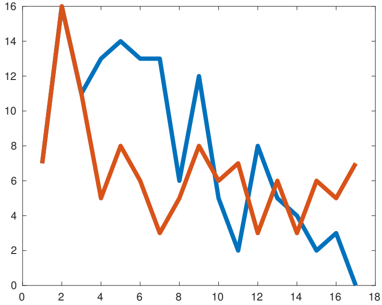
Too Much Time On My Hands

- Suppose $A^{[m]}$ is fixed. What do we find using Q?
- For variable Q a similar problem manifests itself.
- For relatively small m, $Q_m = f(A^{[1]}) \cdots f(A^{[m]})$ is effectively rank 1.
- $Q_m \approx \mathbf{u}\mathbf{v}^T$.
- $Q_{m+1}\mathbf{e} \approx \mathbf{u}\mathbf{v}^T Q_{m+1}\mathbf{e} = \beta_{m+1}\mathbf{u}$.
- Ranking only takes into account first few time steps.

Example 4: Kiel



Example 4: Kiel



Katz	EV	EVα
343	343	31
31	340	228
2552	31	119
1481	2552	38
1027	1481	343