Permuting Spiked Matrices to Triangular Form in the Linear Programming *LU* Update

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Sparse Days, Toulouse 2017

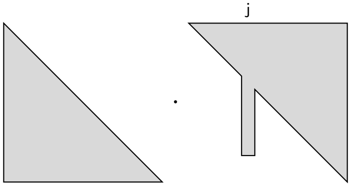
Setting

Given $B = LU$ and $B' = B + (\boldsymbol{a} - B\boldsymbol{e}_j)\boldsymbol{e}_j^T$.

Task: Compute factorized form of B'

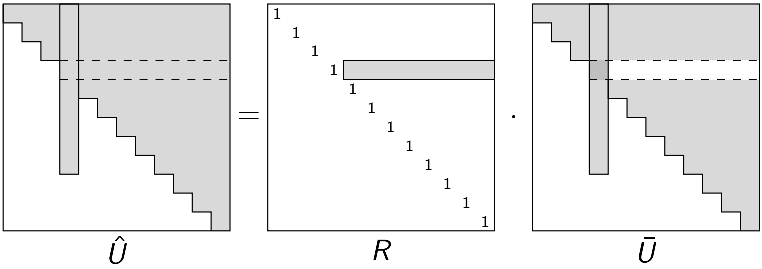
Ansatz: Let $L\hat{\boldsymbol{a}} = \boldsymbol{a}$ and $\hat{U} = U + (\hat{\boldsymbol{a}} - U\boldsymbol{e}_j)\boldsymbol{e}_j^T$. Then

$B' = L\hat{U} =$



Forrest-Tomlin Update

Decompose



- $\bar{U}$ is symmetrically permuted upper triangular
- each update accumulates a *row eta matrix* R

Goal

An LU update scheme that

- permutes $\hat{U}$ to upper triangular form if possible,
- falls back to the Forrest-Tomlin update otherwise.

We will have to distinguish two cases:

1. $\hat{U}_{jj} \neq 0 \quad (\implies$ permutation can only be symmetric)
2. $\hat{U}_{jj} = 0 \quad (\implies$ permutation can only be unsymmetric)

Sparse Matrices and Graphs

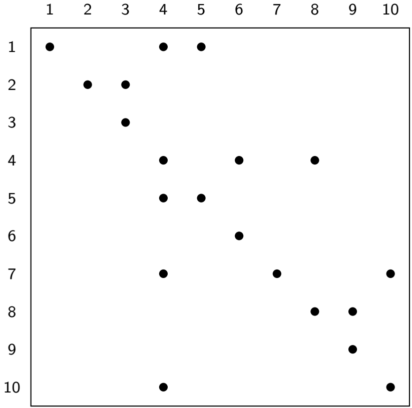
Given $A \in \mathbb{R}^{m\times m}$. Define $G_A = (V, E)$ by

$$V = \{1, \ldots, m\}, \quad (i, j) \in E \iff A_{ij} \neq 0.$$

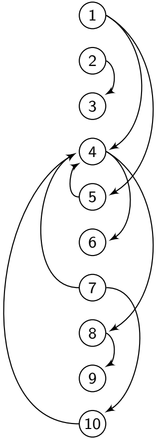
Assume that the diagonal of A is zero-free. Then

$$A \text{ is permuted triangular} \quad \iff \quad G_A \text{ is acyclic.}$$

Example

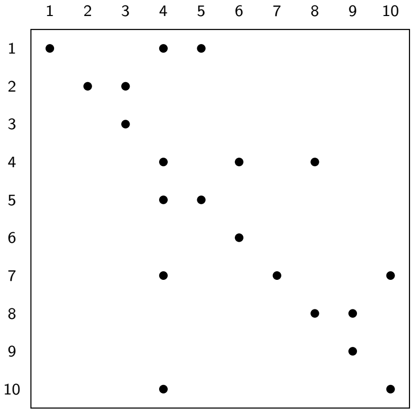


Nonzero pattern of $\hat{U}$

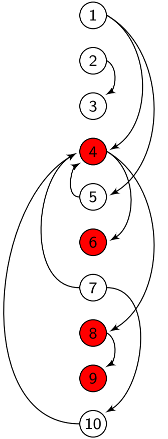


Directed graph $G_{\hat{U}}$

Example

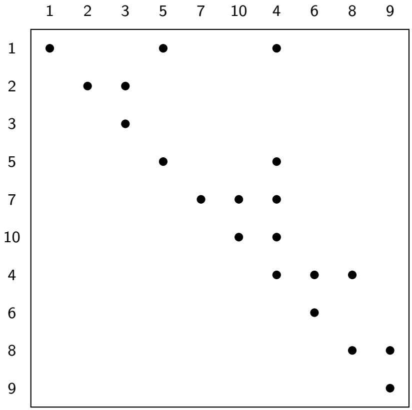


Nonzero pattern of $\hat{U}$

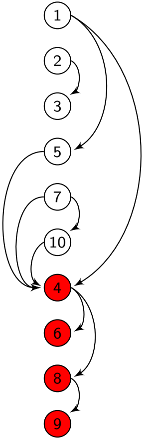


Directed graph $G_{\hat{U}}$

Example



Nonzero pattern of $\hat{U}$



Directed graph $G_{\hat{U}}$

Diagonal element of spike is nonzero

Assume that the diagonal of $\hat{U}$ is zero-free and that the spike is in column j. Then $\hat{U}$ is permuted triangular iff

$$\hat{\mathcal{U}}_j \cap \mathsf{Reach}_{\hat{U}}(j) = \{j\}.$$

Upper triangular form is obtained by moving rows and columns with indices in $\mathsf{Reach}_{\hat{U}}(j)$ in topological order to the end.

Diagonal element of spike is zero

- find Q such that $\hat{U}Q$ has a zero-free diagonal
  (requires to compute one *alternating augmenting path*)
- $\hat{U}Q$ is spiked upper triangular with multiple spike columns
- find P (if it exists) such that $P\hat{U}QP^T$ is upper triangular

Implementation

Maintain the factorization in the form

$$B = LR^1 \cdots R^\nu U,$$

$L(\bar{p}, \bar{p})$ unit lower triangular, $U(p, q)$ upper triangular.

triangular solve with sparse RHS

- rowwise and columwise storage of U
- mappings $i = \texttt{pmap}(j)$ and $j = \texttt{qmap}(i)$ if (i, j) is pivot element

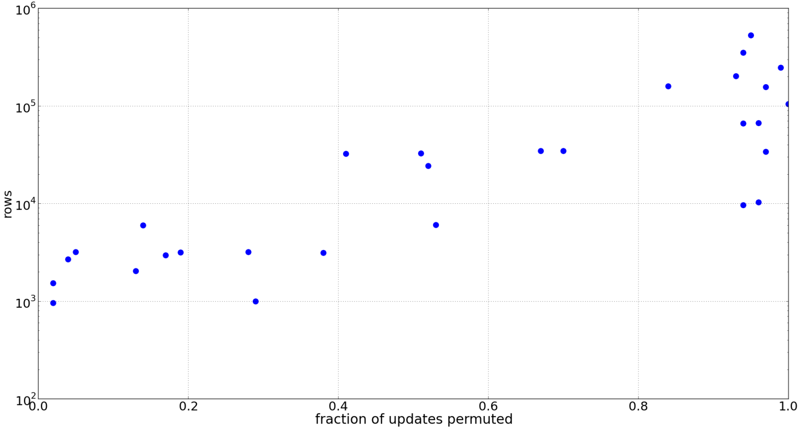
triangular solve with dense RHS

- requires to maintain p, q
- moving indices to the end of p, q leaves gaps; occasional garbage collection

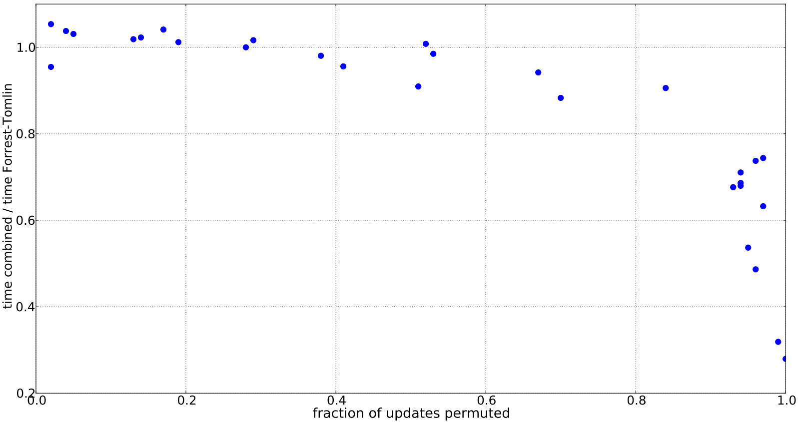
Numerical Comparison

- compare combined update to Forrest-Tomlin update
- 29 LPs with different structural properties
- sequence of basis matrices from CPLEX dual simplex
- for each basis compute FTRAN, BTRAN and perform update

Numerical Comparison



Numerical Comparison



Remark

- our implementation is less efficient in solving $U\boldsymbol{x} = \boldsymbol{b}$ for a dense $\boldsymbol{b}$ than the original Forrest-Tomlin method
  (not included in the numerical comparison)
- a general-purpose LP solver may
  - use the combined update method on very sparse LPs,
  - use the original Forrest-Tomlin method otherwise;

References

- Technical Report ERGO 17-002
  Permuting spiked matrices to triangular form and its application to the Forrest-Tomlin update
- BASICLU software package
  http://www.maths.ed.ac.uk/ERGO/BASICLU/

- R. Fukasawa and L. Poirrier,
  Permutations in the factorization of simplex bases