Efficient Parallel iterative solvers for high order DG methods The curse of the polynomial degree

M. Maischak

Department of Mathematics
Brunel University

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Outline

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Model problem

$$-\Delta u = f \text{ in } \Omega,$$
$$u = g \text{ on } \Gamma.$$

We introduce the mesh $\mathcal{T}_h$ and the space

$$V_h = \{u_h : u_h|_K \in \mathcal{P}_p(K), \forall K \in \mathcal{T}_h\}$$

On the skeleton $\mathcal{E}(\mathcal{T}_h)$ we define

$$\{\{v\}\} = \frac{1}{2}(v^+ + v^-), \qquad \{\{\mathbf{q}\}\} = \frac{1}{2}(\mathbf{q}^+ + \mathbf{q}^-),$$
$$[[v]] = v^+\mathbf{n}_{K^+} + v^-\mathbf{n}_{K^-}, \qquad [[\mathbf{q}]] = \mathbf{q}^+ \cdot \mathbf{n}_{K^+} + \mathbf{q}^- \cdot \mathbf{n}_{K^-}.$$

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Find $u_h \in V_h$ such that

$$A_h(u_h, v) = F_h(v) \qquad \forall v \in V_h$$

$$\begin{aligned} A_h(u,v) = &\sum_{K\in\mathcal{T}_h} \int_K \nabla_h u \cdot \nabla_h v \, dx \\ &- \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \int_\kappa (\{\{\nabla_h v\}\} \cdot [[u]] + \{\{\nabla_h u\}\} \cdot [[v]]) ds \\ &+ \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \int_\kappa c[[u]] \cdot [[v]] ds, \end{aligned}$$

$$F_h(v) = \int_\Omega fv \, dx - \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \int_\kappa g\nabla_h v \cdot \mathbf{n} \, ds + \sum_{\kappa\in\mathcal{E}(\mathcal{T}_h)} \int_\kappa cgv \, ds$$

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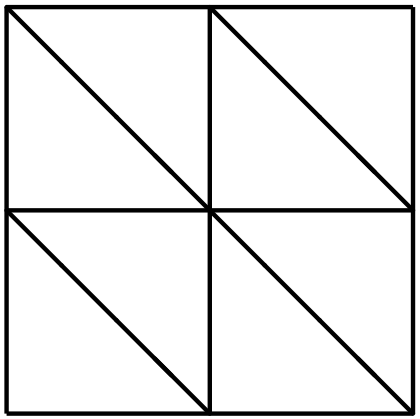
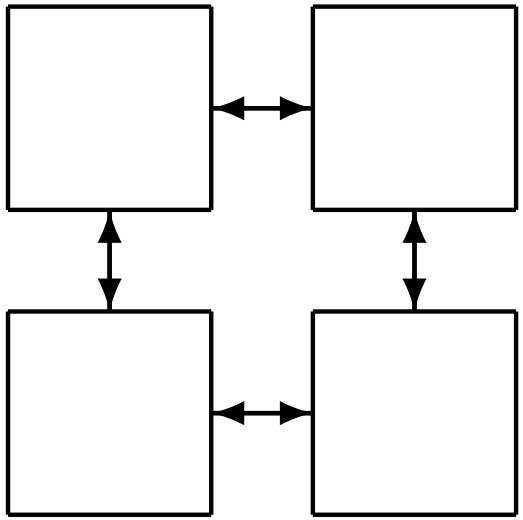
conforming method



DG method



dimension	d
number of elements	n
mesh size	$h \sim n^{-1/d}$
number of coefficients	$dof \sim n \cdot p^d$

Matrix vector multiplication costs

sparse	$\mathcal{O}(n)$
dense	$\mathcal{O}(n^2 \cdot p^{2d})$
sparse-p	$\mathcal{O}(n \cdot p^{2d})$

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Basis functions (classic) ADLP

$$\mathcal{L}_0(x) = \frac{1}{2}(1-x),$$
$$\mathcal{L}_1(x) = \frac{1}{2}(1+x),$$
$$\mathcal{L}_n(x) = \frac{1}{2n-1}(L_n(x) - L_{n-2}(x)) = \int_{-1}^{x} L_{n-1}(y)\,dy, \quad n \geq 2$$

The antiderivatives of Legendre polynomials have the important and simplifying property

$$\mathcal{L}_n(\pm 1) = 0, \quad n \geq 2$$

C1-polynomials

$$C_0(x) = \frac{1}{4}(1-x)^2(2+x), \qquad C_1(x) = \frac{1}{4}(1+x)^2(2-x),$$
$$C_2(x) = \frac{1}{4}(1+x)(1-x)^2, \qquad C_3(x) = \frac{1}{4}(1+x)^2(x-1),$$
$$C_n(x) = (1-x^2)\mathcal{L}_{n-2}(x), \quad n \geq 4$$

The C1-polynomials have the properties

$$C_0(-1) = 1, C_0(1) = 0, C_0'(-1) = 0, C_0'(1) = 0,$$
$$C_1(-1) = 0, C_1(1) = 1, C_1'(-1) = 0, C_1'(1) = 0,$$
$$C_2(-1) = 0, C_2(1) = 0, C_2'(-1) = 1, C_2'(1) = 0,$$
$$C_3(-1) = 0, C_3(1) = 0, C_3'(-1) = 0, C_3'(1) = 1,$$
$$C_n(\pm 1) = C_n'(\pm 1) = 0, \quad n \geq 4$$

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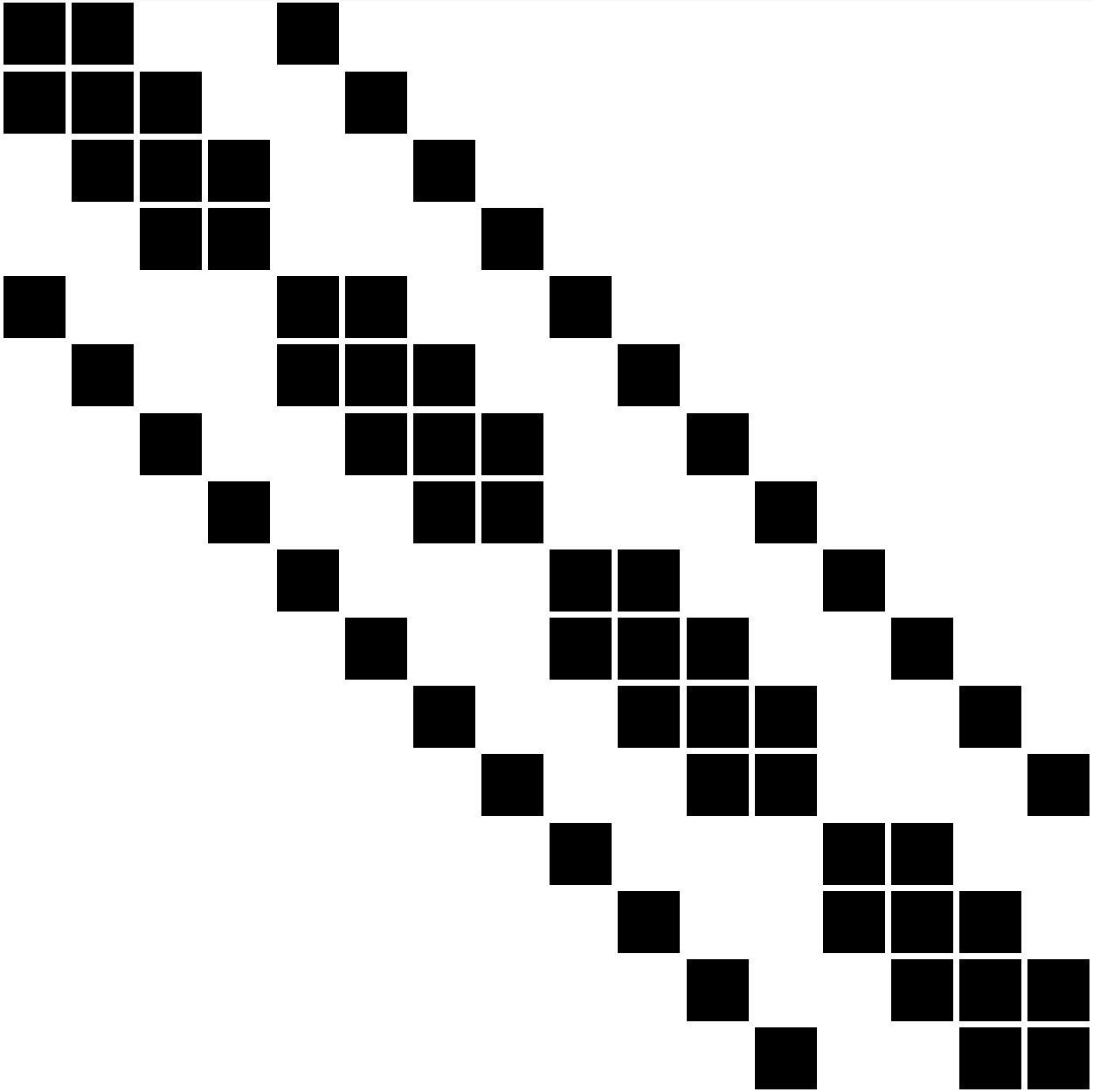

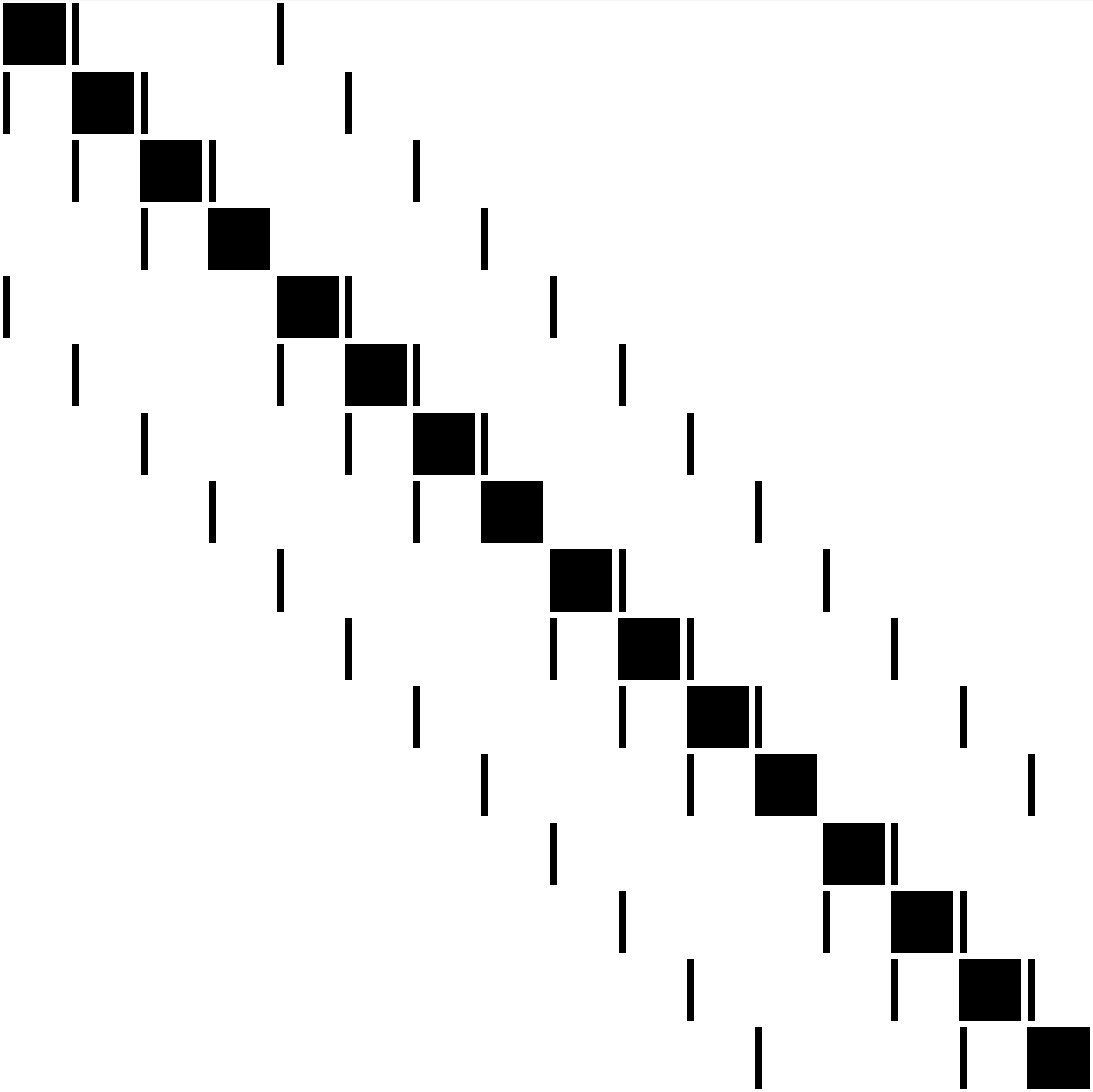

Using C1- basis functions save asymptotically up to

80% of memory and operations in 2d
86% of memory and operations in 3d

Data transfer via edges/faces, using C1 functions reduces data transfer costs from $\mathcal{O}(p^d)$ to $\mathcal{O}(p^{d-1})$.
Parallelization using MPI
To take advantage of reduced data transfer, distribute data element wise.

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Parallel matrix vector multiplication

ADLP functions

$$\tau_{MV} = \frac{5n \cdot p^{2d}}{\#proc} + \#proc \cdot n^{1/d} p^{d}$$

C1 funtions

$$\tau_{MV} = \frac{n \cdot p^{2d} + 4 \cdot n \cdot 4p^{d}}{\#proc} + \#proc \cdot n^{1/d} p^{d-1}$$

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Iterative solver (CG)

Reduce the number of iterations

- Wirebasket basis functions (e.g. locally discrete harmonic)
- Preconditioner, e.g. 2-level additive Schwarz preconditioner, compatible with distributed element data structure

2-level Hhqp additive Schwarz preconditioner

P.F.Antonietti, P. Houston, I. Smears (2015): A note on optimal spectral bounds for nonoverlapping domain decomposition preconditioners for hp-version discontinuous Galerkin methods

$$\kappa(P_{ad}) \leq C_{\#}^2(N_S + 2), \qquad C_{\#} = C_\sigma \frac{Hp^2}{hq}$$

2-level additive Schwarz preconditioner,
N_S fine grid blocks with pol. degree p,
1 coarse grid block with pol. degree q.

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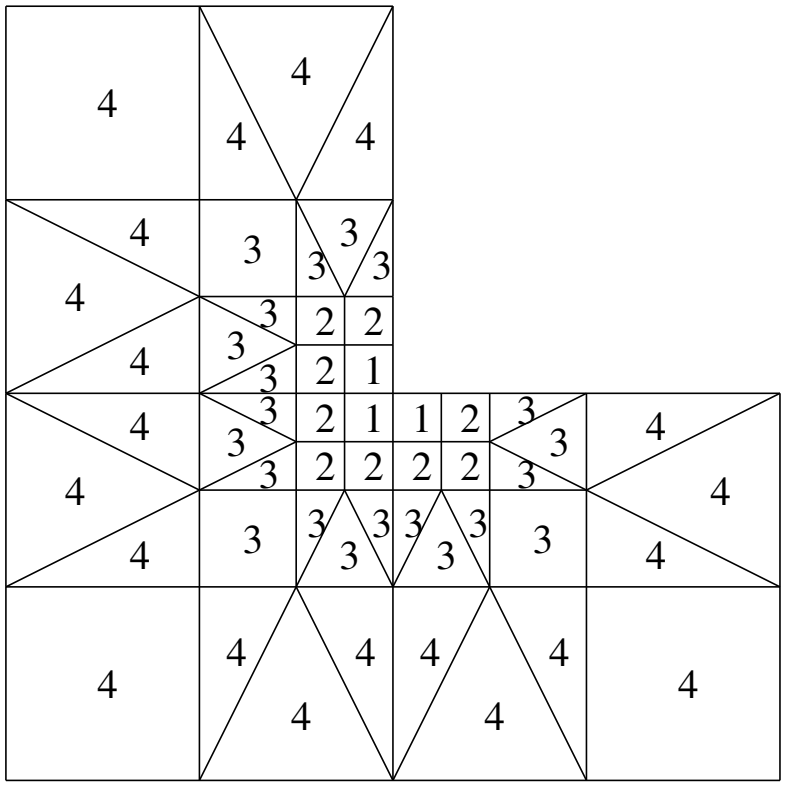
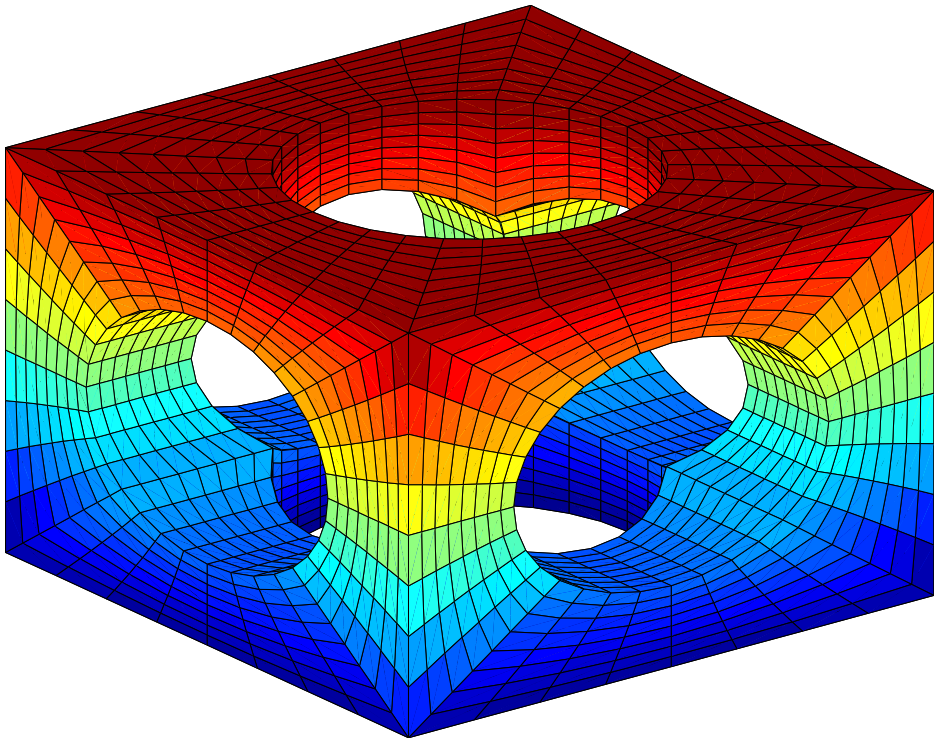
hp-geometrical refined mesh

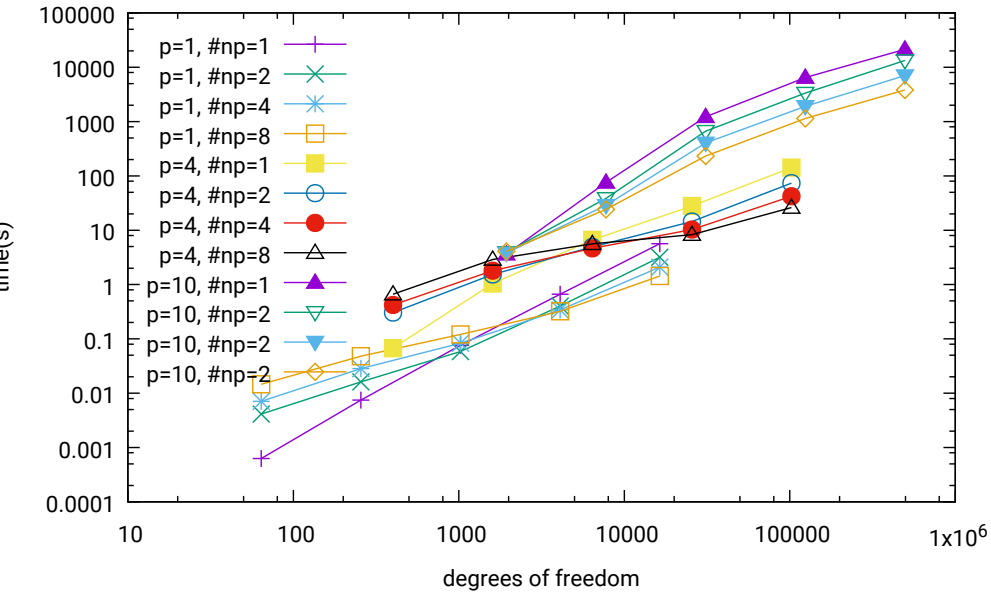


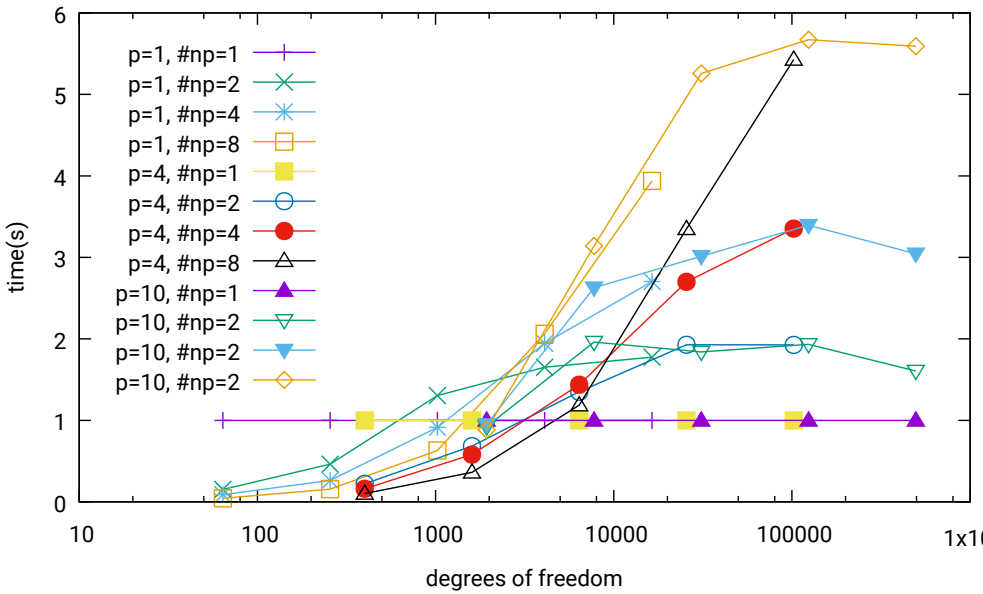


Cluster

12 computing nodes
20 cores, 2.4GHz, 64 GByte per node
1GBit Ethernet

ADLP functions, no preconditioning





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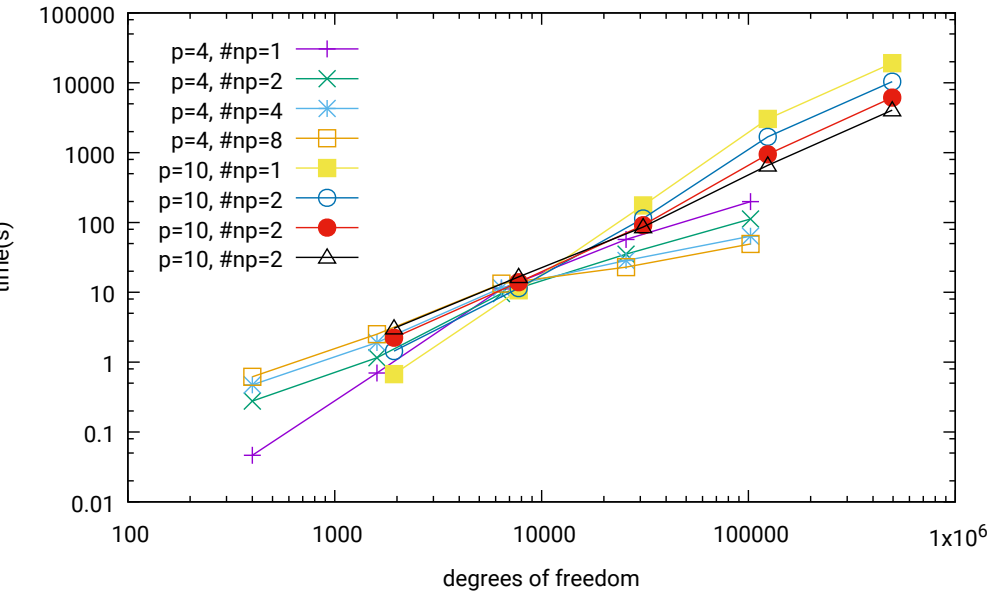
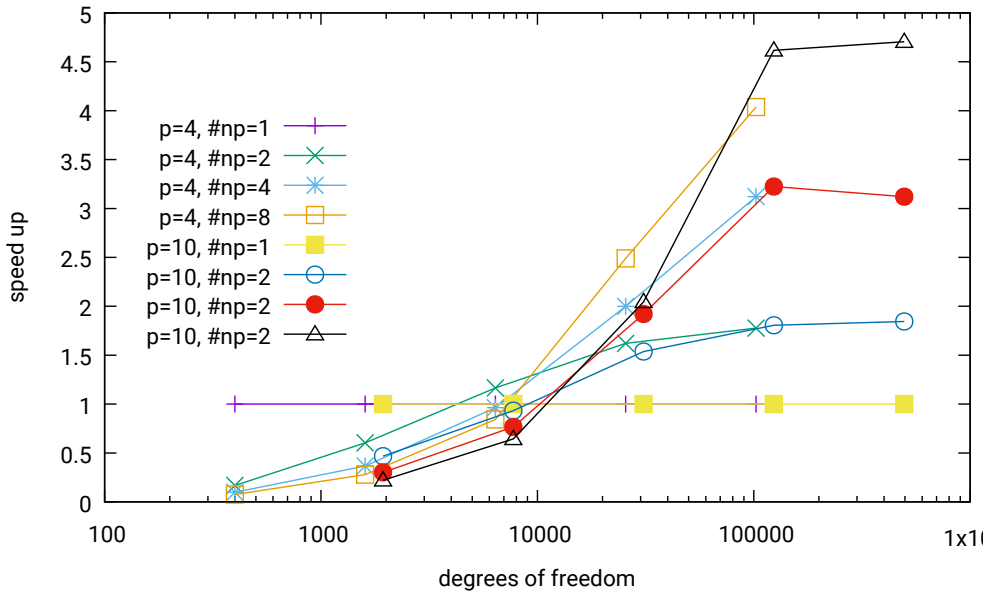
C1 functions, no preconditioning





C1 functions, preconditioner, p=10

FDoF	CDoF	#np	Iterations	Solver (s)	speedup
495616	4096	1	480	591.43825	1
495616	4096	2	480	305.07621	1.938657393
495616	4096	4	480	167.29093	3.535387424
495616	4096	8	480	95.92294	6.165764093

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