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High-Order Methods for Parabolic Equations in Multiple Space Dimensions for Option Pricing Problems

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The model problem

Convection-diffusion-reaction equation

$$\frac{\partial u}{\partial t} = Lu, \qquad (x,t) \in \Omega_d \times \Omega_t,$$

$$Lu = \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} c_i \frac{\partial u}{\partial x_i} + bu,$$

$\Omega_d \times \Omega_t$ rectangular domain with suitable initial and boundary data

Spatial discretization

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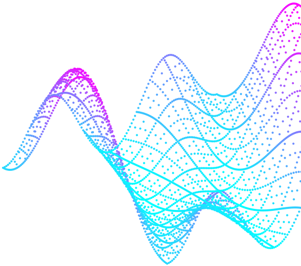
Semi discrete system of Ordinary Differential Equations

$$U'(t) = F(t)U(t), \qquad t \geq 0,$$

with initial value $U(0) = U_0 \in \mathbb{R}^m$ and discretization matrix $F(t) \in \mathbb{R}^{m \times m}$ with $m \in \mathbb{R}$.

Outline

1. Spatial Discretization: Approximation of the spatial operator L
   - High-Order Finite Differences
   - Pseudo-Spectral Methods
   - Sparse Grid Combination Technique
2. Time Discretization: Alternating Direction Implicit (ADI) schemes
   - Stability Analysis
3. Application to financial engineering partial differential equation
   - Basket-Options in the Black-Scholes model
   - European Plain-Vanilla options under Stochastic Volatility



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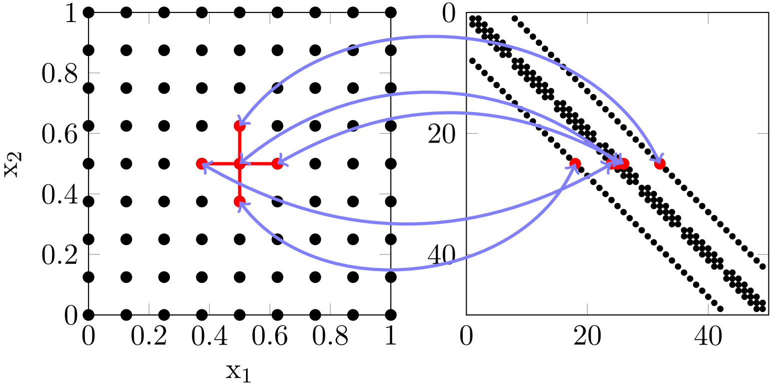

Figure: 2nd order FD scheme for 2-d heat equation on the grid $\Omega_{(3,3)}$.

Spatial Discretization
Standard finite differences

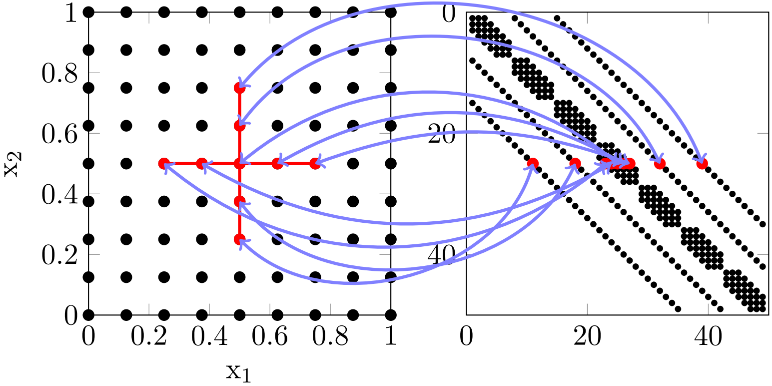

Figure: 4th order FD scheme for 2-d heat equation on the grid $\Omega_{(3,3)}$

Spatial Discretization - High-Order-Compact Finite Differences

- Exploit the structure of the PDE to derive a fourth order accurate discretization on the compact stencil
- In financial engineering: [Düring et al., 2014, Düring and Fournié, 2012, Düring and Heuer, 2015], [Düring and Miles, 2017, Hendricks et al., 2017]

Spatial Discretization - High-Order-Compact Finite Differences

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- Exploit the structure of the PDE to derive a fourth order accurate discretization on the compact stencil
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- We decompose the discretization matrix

$$\mathrm{FU(t)} = \mathrm{F_0 U(t)} + \mathrm{F_1 U(t)} + \mathrm{F_2 U(t)} + \ldots + \mathrm{F_d U(t)}$$

- $\mathrm{F_0}$ stems from all mixed derivatives
- $\mathrm{F_i}$ stems from the contribution of the i-th coordinate direction for $\mathrm{i} = 1, 2, \ldots, \mathrm{d}$

- We consider unidirectional contributions F_i for $i = 1, 2, \dots, d$

$$a_{ii}(x_{l,j}) \frac{\partial^2 u}{\partial x_i^2}(x_{l,j}) + c_i(x_{l,j}) \frac{\partial u}{\partial x_i}(x_{l,j}) = g(x_{l,j})$$

for $i = 1, ..., d$ and some arbitrary smooth right hand side g.

- Inserting the finite difference operators we obtain

$$a_{ii}\delta_i^2 u(x_{l,j}) - a_{ii}\frac{h_i^2}{12}\frac{\partial^4 u}{\partial x_i^4}(x_{l,j}) - a_{ii}\frac{h_i^4}{360}\frac{\partial^6 u}{\partial x_i^6}(x_{l,j})$$
$$+ c_i\delta_i^0 u(x_{l,j}) - c_i\frac{h_i^2}{6}\frac{\partial^3 u}{\partial x_i^3}(x_{l,j}) - c_i\frac{h_i^4}{120}\frac{\partial^5 u}{\partial x_i^5}(x_{l,j}) + \mathcal{O}(h_i^6) = g(x_{l,j})$$

Observation: leading error term is of order two
$\Rightarrow$ fourth-order compact approximation if the third and fourth derivative is approximated with second order accuracy on the compact stencil.

$$\frac{\partial^3 u}{\partial x_i^3} = \frac{1}{a_{ii}}\frac{\partial g}{\partial x_i} - \left(\frac{1}{a_{ii}}\frac{\partial a_{ii}}{\partial x_i} + \frac{c_i}{a_{ii}}\right)\frac{\partial^2 u}{\partial x_i^2} - \frac{1}{a_{ii}}\frac{\partial c_i}{\partial x_i}\frac{\partial u}{\partial x_i},$$

$$\begin{aligned}\frac{\partial^4 u}{\partial x_i^4} = \frac{1}{a_{ii}}\frac{\partial^2 g}{\partial x_i^2} - \left(\frac{c_i}{a_{ii}^2} + \frac{2}{a_{ii}^2}\frac{\partial a_{ii}}{\partial x_i}\right)\frac{\partial g}{\partial x_i} + \Bigg(\frac{c_i^2}{a_{ii}^2} + \frac{3c_i}{a_{ii}^2}\frac{\partial a_{ii}}{\partial x_i} + \frac{2}{a_{ii}^2}\left[\frac{\partial a_{ii}}{\partial x_i}\right]^2 \\ - \frac{2}{a_{ii}}\frac{\partial c_i}{\partial x_i} - \frac{1}{a_{ii}}\frac{\partial^2 a_{ii}}{\partial x_i^2}\Bigg)\frac{\partial^2 u}{\partial x_i^2} \\ + \left(\frac{c_i}{a_{ii}^2}\frac{\partial c_i}{\partial x_i} + \frac{2}{a_{ii}^2}\frac{\partial a_{ii}}{\partial x_i}\frac{\partial c_i}{\partial x_i} - \frac{1}{a_{ii}}\frac{\partial^2 c_i}{\partial x_i^2}\right)\frac{\partial u}{\partial x_i}\end{aligned}$$

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High-Order-Compact Finite Differences Approximation to unidirectional convection-diffusion equation

$$\left(a_{ii} + \frac{h_i^2}{12}\frac{\partial^2 a_{ii}}{\partial x_i^2} - \frac{h_i^2 c_i}{12 a_{ii}}\frac{\partial a_{ii}}{\partial x_i} - \frac{h_i^2}{6 a_{ii}}\left[\frac{\partial a_{ii}}{\partial x_i}\right]^2 + \frac{h_i^2 c_i^2}{12 a_{ii}} + \frac{h_i^2}{6}\frac{\partial c_i}{\partial x_i}\right)\delta_i^2 u(x_{l,j})$$
$$+\left(c_i - \frac{h_i^2}{6 a_{ii}}\frac{\partial a_{ii}}{\partial x_i}\frac{\partial c_i}{\partial x_i} + \frac{h_i^2 c_i}{12 a_{ii}}\frac{\partial c_i}{\partial x_i} + \frac{h_i^2}{12}\frac{\partial^2 c_i}{\partial x_i^2}\right)\delta_i^0 u(x_{l,j}) + \mathcal{O}(h_i^4)$$
$$= g(x_{l,j}) + \frac{h_i^2}{12}\delta_i^2 g(x_{l,j}) + \left(\frac{h_i^2 c_i}{12 a_{ii}} - \frac{h_i^2}{6 a_{ii}}\frac{\partial a_{ii}}{\partial x_i}\right)\delta_i^0 g(x_{l,j})$$

or in matrix notation

$$A_{x_i} U = B_{x_i} G$$

Spatial Discretization - High-Order-Compact Finite Differences

- Semi-discrete scheme can be written as

$$\begin{aligned} \mathrm{U}'(\mathrm{t}) &= \mathrm{F_0 U} + \mathrm{F_1 U} + \ldots + \mathrm{F_d U} \\ &= \mathrm{F_0 U} + \mathrm{B}_{\mathrm{x_1}}^{-1} \mathrm{A}_{\mathrm{x_1}} \mathrm{U} + ... + \mathrm{B}_{\mathrm{x_d}}^{-1} \mathrm{A}_{\mathrm{x_d}} \mathrm{U} \\ &\qquad + \mathcal{O}(\mathrm{h}_1^4) + ... + \mathcal{O}(\mathrm{h}_\mathrm{d}^4) + \sum_{\mathrm{i,j}} \mathcal{O}(\mathrm{h}_\mathrm{i}^4 \mathrm{h}_\mathrm{j}^4) \end{aligned}$$

- Mixed derivatives can be approximated via standard fourth order stencils and collected in the matrix $\mathrm{F_0}$.

Spatial Discretization - Pseudo-Spectral Methods

1. An interpolant of the data is computed.
2. The interpolant is differentiated once (twice) to obtain an estimate of the first (second) derivative.

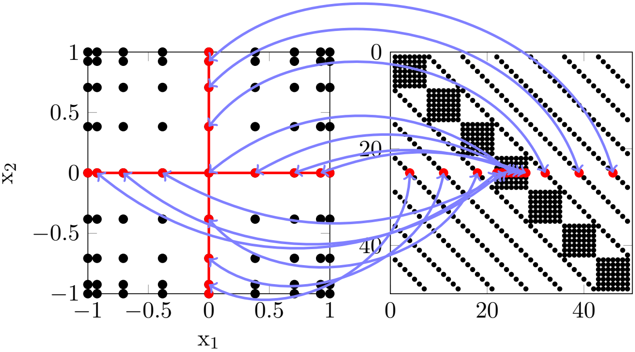

Figure: Chebyshev spectral scheme for 2-d heat equation on the grid $\Omega_{(3,3)}$

Theorem [Battles and Trefethen, 2004]

Let $u, u', \dots, u^{(m-1)}$ be absolutely continuous for some $m \geq 1$, and let $u^{(m)}$ be a function of bounded variation. Then

$$|u(x) - (P_N u)(x)| = \mathcal{O}(N^{-m})$$

as $N \to \infty$ for all $x \in [-1, 1]$.

Theorem [Battles and Trefethen, 2004]

If u is analytic and bounded in the Bernstein ellipse of foci ± 1 with semimajor and semiminor axis lengths summing to r, then the Chebyshev interpolant with $N + 1$ Chebyshev-Gauss-Lobatto nodes fulfills

$$|u(x) - (P_N u)(x)| = \mathcal{O}(r^{-N})$$

as $N \to \infty$ for all $x \in [-1, 1]$.

Spatial Discretization - The Curse of Dimensionality

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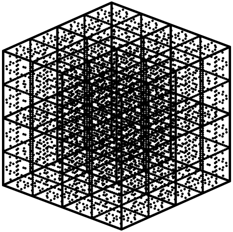
- In grid based methods the degrees of freedom grows with $\mathcal{O}(\mathrm{h}^{-\mathrm{d}}) = \mathcal{O}(\mathrm{N}^{\mathrm{d}})$.
- Already for problems with a moderate number of spatial dimensions this is a severe problem, e.g. $\Omega_{(6,6)}$ has 4,225 grid nodes, while $\Omega_{(6,6,6,6)}$ has 17,850,625 grid nodes.
- With sparse grids the growth of the degrees of freedoms can be reduced to $\mathcal{O}(\mathrm{h}^{-1}\log_2(\mathrm{h}^{-1})^{\mathrm{d}-1})$.



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- The method is based on the error splitting structure of the underlying numerical scheme.
- We consider a two-dimensional problem on the unit square $\Omega_2 = [0,1]^2$ and assume a numerical approximation u_l on Ω_l with $l = (l_1, l_2) \in \mathbb{N}_0^2$, with mesh widths $h = (h_1, h_2) = (2^{-l_1}, 2^{-l_2})$
- Error splitting structure of the numerical scheme

$$u - u_l = h_1^2 w_1(h_1) + h_2^2 w_2(h_2) + h_1^2 h_2^2 w_{1,2}(h_1, h_2).$$

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This structure can now be exploited by combining them in such a way that low order terms cancel out.

- hierarchical surplus of the numerical solution

$$\delta(\mathrm{u_l}) = \mathrm{u_l} - \mathrm{u_{l-e_1}} - \mathrm{u_{l-e_2}} + \mathrm{u_{l-e_1-e_2}},$$

  where $\mathrm{e}_1 = (1,0)$ and $\mathrm{e}_2 = (0,1)$.

- Inserting the error splitting, we obtain

$$\begin{aligned}
\delta(\mathrm{u}-\mathrm{u_l}) &= \mathrm{h}_1^2\mathrm{w}_1(\mathrm{h}_1) + \mathrm{h}_2^2\mathrm{w}_2(\mathrm{h}_2) + \mathrm{h}_1^2\mathrm{h}_2^2\mathrm{w}_{1,2}(\mathrm{h}_1,\mathrm{h}_2) \\
&- 4\,\mathrm{h}_1^2\mathrm{w}_1(2\mathrm{h}_1) - \mathrm{h}_2^2\mathrm{w}_2(\mathrm{h}_2) - 4\,\mathrm{h}_1^2\mathrm{h}_2^2\mathrm{w}_{1,2}(2\mathrm{h}_1,\mathrm{h}_2) \\
&- \mathrm{h}_1^2\mathrm{w}_1(\mathrm{h}_1) - 4\mathrm{h}_2^2\mathrm{w}_2(2\mathrm{h}_2) - 4\,\mathrm{h}_1^2\mathrm{h}_2^2\mathrm{w}_{1,2}(\mathrm{h}_1,2\mathrm{h}_2) \\
&+ 4\,\mathrm{h}_1^2\mathrm{w}_1(2\mathrm{h}_1) + 4\,\mathrm{h}_2^2\mathrm{w}_2(2\mathrm{h}_2) + 16\,\mathrm{h}_1^2\mathrm{h}_2^2\mathrm{w}_{1,2}(2\mathrm{h}_1,2\mathrm{h}_2) \\
&= \mathrm{h}_1^2\mathrm{h}_2^2\mathrm{w}_{1,2}(\mathrm{h}_1,\mathrm{h}_2) - 4\mathrm{h}_1^2\mathrm{h}_2^2\mathrm{w}_{1,2}(2\mathrm{h}_1,\mathrm{h}_2) - 4\mathrm{h}_1^2\mathrm{h}_2^2\mathrm{w}_{1,2}(\mathrm{h}_1,2 \\
&\qquad + 16\mathrm{h}_1^2\mathrm{h}_2^2\mathrm{w}_{1,2}(2\mathrm{h}_1,2\mathrm{h}_2) \\
&= \mathcal{O}(\mathrm{h}_1^2\mathrm{h}_2^2) = \mathcal{O}(2^{-2\mathrm{l}_1}2^{-2\mathrm{l}_2}) = \mathcal{O}(2^{-2|\mathrm{l}|_1})
\end{aligned}$$

Spatial Discretization - Sparse Grid Combination Technique

- Combine all solutions with a high surplus (information gain).
- Combined sparse grid solution is the sum of all surpluses with $|\mathrm{l}|_1 \leq \mathrm{n}$ for $\mathrm{n} \in \mathbb{N}_0$

$$\mathrm{u}_\mathrm{n}^\mathrm{s} = \sum_{|\mathrm{l}|_1 \leq \mathrm{n}} \delta \mathrm{u}_\mathrm{l}.$$

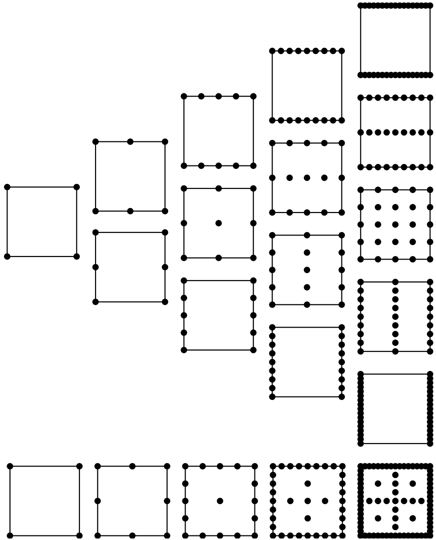
- Upper error bound can be found by incorporating the surpluses of all sub-solutions, which are not used to compute $\mathrm{u}_\mathrm{n}^\mathrm{s}$. We have

$$\|\mathrm{u}_\mathrm{n}^\mathrm{s} - \mathrm{u}\| \leq \sum_{|\mathrm{l}|_1 > \mathrm{n}} \|\delta \mathrm{u}_\mathrm{l}\| = \sum_{|\mathrm{l}|_1 > \mathrm{n}} \mathcal{O}(2^{-2|\mathrm{l}|_1}) = \sum_{\mathrm{i} > \mathrm{n}} \mathcal{O}((\mathrm{i}+1)2^{-2\mathrm{i}}) = \mathcal{O}(\mathrm{n}2^{-2\mathrm{n}}).$$

Let $\mathrm{h} = 2^{-\mathrm{n}}$, then error bound is $\|\mathrm{u}_\mathrm{n}^\mathrm{s} - \mathrm{u}\| \leq \mathcal{O}(\mathrm{h}^2 \log_2(\mathrm{h}^{-1}))$.

Spatial Discretization - Sparse Grid Combination Technique

- The number of grid points on each sub-grid grows with $\mathcal{O}(2^{\mathrm{n}})$.
- At each level there are $\mathrm{n}+1$ grids.
- Thus we have $\mathcal{O}(\mathrm{n} \cdot 2^{\mathrm{n}})$ grid nodes in the combined solution.
- Let $\mathrm{h} = 2^{-\mathrm{n}}$, we have $\mathcal{O}(\mathrm{h}^{-1} \log_2(\mathrm{h}^{-1}))$ grid points compared to $\mathcal{O}(\mathrm{h}^{-2})$ nodes in the full grid.



The same ideas can be carried over to the general d−dimensional case for numerical schemes with algebraic order of accuracy m.

Definition: Sparse grid combination technique

The sparse grid combination formula at level $n \in \mathbb{N}$ is given by

$$u_n^s = \sum_{q=0}^{d-1} \binom{d-1}{q} \sum_{|l|_1 = n-q} u_l.$$

- $\|u - u_n^s\| \le \mathcal{O}(n^{d-1} 2^{-m \cdot n}) = \mathcal{O}(h^{-m} \log_2(h^{-1})^{d-1})$
- $\mathcal{O}(n^{d-1} 2^n) = \mathcal{O}(h^{-1} \log_2(h^{-1})^{d-1})$ grid nodes

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- Sparse grid combination technique relies on the error splitting assumption

$$u - u_l = \sum_{k=1}^{d} \sum_{\substack{\{j_1,\ldots,j_k\} \\ \subseteq \{1,\ldots,d\}}} w_{j_1,\ldots j_k}(.; h_{j_1}, \ldots, h_{j_k}) h_{j_1}^m \cdots h_{j_k}^m.$$

Question: for which schemes does this error splitting hold?

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In the case of linear FD schemes the error splitting has been analyzed by Reisinger [Reisinger, 2013].
Assumptions:

1. The scheme has a pointwise truncation error of the form

$$(\mathrm{L}-\mathrm{L}_\mathrm{l})\mathrm{u}(\mathrm{x}_{\mathrm{l},\mathrm{j}}) = \sum_{\mathrm{k}=1}^{\mathrm{d}} \sum_{\substack{\{\mathrm{j}_1,\ldots,\mathrm{j}_\mathrm{k}\} \\ \subseteq\{1,\ldots,\mathrm{d}\}}} \tau_{\mathrm{j}_1,\ldots\mathrm{j}_\mathrm{k}}(\mathrm{x}_{\mathrm{l},\mathrm{j}};\mathrm{h}_{\mathrm{j}_1},\ldots,\mathrm{h}_{\mathrm{j}_\mathrm{k}})\mathrm{h}_{\mathrm{j}_1}^{\mathrm{m}}\cdots\mathrm{h}_{\mathrm{j}_\mathrm{k}}^{\mathrm{m}},$$

   for $\mathrm{x}_{\mathrm{l},\mathrm{j}} \in \Omega_\mathrm{l}$.
2. Stability of the discretization scheme.
3. Sufficiently smooth initial data and compatible boundary data, such that the mixed derivatives of required order are bounded.

- In the case of second-order accuracy the mixed derivatives of fourth order have to be bounded

$$\frac{\partial^{|\alpha|_1}}{\partial \mathrm{x}_1^{\alpha_1} \dots \partial \mathrm{x}_\mathrm{d}^{\alpha_\mathrm{d}}} \quad \text{with } \alpha_\mathrm{i} \in \{0, 1, ..., 4\},$$

  see [Reisinger, 2013]

- In the case of fourth-order accuracy the mixed derivatives of sixth order have to be bounded

$$\frac{\partial^{|\alpha|_1}}{\partial \mathrm{x}_1^{\alpha_1} \dots \partial \mathrm{x}_\mathrm{d}^{\alpha_\mathrm{d}}} \quad \text{with } \alpha_\mathrm{i} \in \{0, 1, ..., 6\},$$

  see [Hendricks et al., 2017a].

Spatial Discretization - Error Splitting

- Besides these key properties also the error structure has to be preserved by the interpolation technique used to combine the sub-solutions.

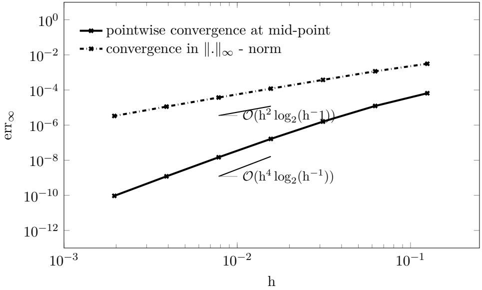

Figure: Convergence at the mid point and in the maximum norm.

Spatial Discretization - Error Splitting

- In the case of a fourth order accuracy a tensor-product based cubic spline interpolant preserves the error structure.
- Proof via separation of the errors into interpolation error (I) and the interpolation of the error of the numerical solution (II)

$$u(x) - (P_N u_l)(x) = \underbrace{u(x) - \big(P_N u\big)(x)}_{\mathrm{I}} + \underbrace{\big(P_N(u - u_l)\big)(x)}_{\mathrm{II}}.$$

- The application of cubic spline interpolation results in higher regularity requirements

$$\frac{\partial^{|\alpha|_1}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}} \quad \text{with } \alpha_i \in \{0, 1, ..., 10\}.$$

Spatial Discretization - Error Splitting

- To our best knowledge the sparse grid combination technique has not been used in the case of pseudo-spectral methods.
- Shen and Yu [Shen and Yu, 2010, Shen and Yu, 2012] construct a spectral sparse grid for elliptic problems based on nested, spectrally accurate quadratures.
- We consider the test problems given in [Shen and Yu, 2010]

$$-\Delta u = f \qquad \text{for } x \in \Omega_d = [-1, 1]^d.$$

Spatial Discretization - Error Splitting

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$$-\Delta u = f \qquad \text{for } x \in \Omega_d = [-1,1]^d$$

with solutions

$$u_1(x) = \prod_{i=1}^{d} \sin(k\pi \tfrac{x_i+1}{2}), \qquad u_2(x) = \sum_{i=1}^{d} \phi_k(x_i) \prod_{i\neq j} \sin(\pi \tfrac{x_i+1}{2}),$$

$$u_3(x) = \prod_{i=1}^{d} g_k(x_i), \qquad u_4(x) = \prod_{i=1}^{d} \left(h_k(x_i) - \tfrac{x_i+1}{2} \right),$$

where

$$\phi_k(x_i) = e^{\sin(k\pi \frac{x_i+1}{2})} - 1$$

$$g_k(x_i) = (1 - x_i^2)(1 + x_i)\log(1 + x_i + 10^{-k})$$

$$h_k(x_i) = \begin{cases} 0, & x_i \leq 0 \\ x_i^k, & x_i > 0 \end{cases}$$

and $k \in \mathbb{N}$.

- We assume in the analytic case

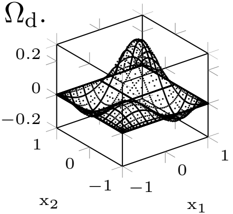
$$u(x) - (P_N u_l)(x) = \sum_{k=1}^{d} \sum_{\substack{\{j_1,j_2,\dots,j_k\} \\ \subset\{1,2,\dots,d\}}} r^{-N_{j_1}} \cdot \ldots \cdot r^{-N_{j_k}} \gamma_{j_1,j_2,\dots,j_k}(x; N_{j_1}, \ldots, N$$

with bounded functions γ.

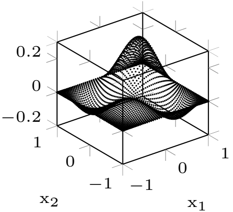
- We expect a hierarchical surplus of order

$$\delta u_l(x) = \mathcal{O}(r^{-N_1} \cdot r^{-N_2} \cdots r^{-N_d}) = \mathcal{O}(r^{-\sum_{i=1}^{d} N_i}) = \mathcal{O}(r^{-\sum_{i=1}^{d} 2^{l_i}})$$

for all $x \in \Omega_d$.



(a) $u_3, k = 3$



(b) $u_3, k = 3$

Spatial Discretization - Error Splitting

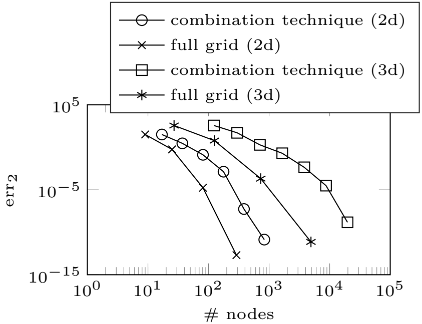
- $\delta u_{(3,3)} = \mathcal{O}(r^{-16})$, $\delta u_{(2,4)} = \mathcal{O}(r^{-20})$ $\rightarrow$ splitting structure is not appropriate for the combination technique.

l_1, l_2	1	2	3	4	5
1	0.05320007109	3.01091817426	0.67948036244	0.00090058675	0.00000425864
2	3.01091817426	5.81390129215	1.20139326430	0.02748897172	0.00101213360
3	0.67948036244	1.20139326430	0.06600992745	0.00176267546	0.00006409891
4	0.00090058675	0.02748897172	0.00176267546	0.00007605425	0.00000253873
5	0.00000425864	0.00101213360	0.00006409891	0.00000253873	0.00000017619

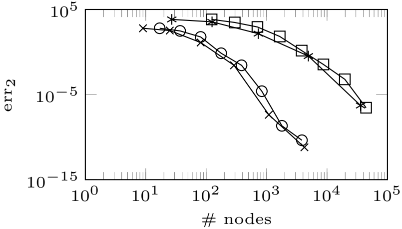
Table: Hierarchical surplus of the spectral method for case 3 with $k = 3$ and $d = 2$.

l_1, l_2	1	2	3	4	5
1	0	15.20688403880	3.15596490099	0.71863369303	0.17283592883
2	15.20688403880	4.89281826359	0.53018892670	0.10161973130	0.02346582984
3	3.15596490099	0.53018892670	0.16327550147	0.02257005970	0.00528871129
4	0.71863369303	0.10161973130	0.02257005970	0.00068325977	0.00010098876
5	0.17283592883	0.02346582984	0.00528871128	0.0001009887	0.00000821986

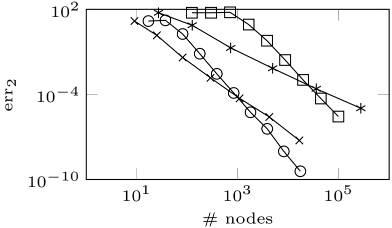
Table: Hierarchical surplus of the spectral method for case 4 with $k = 3$ and $d = 2$.



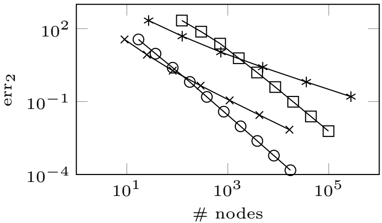
(c) $u_1, k = 1$



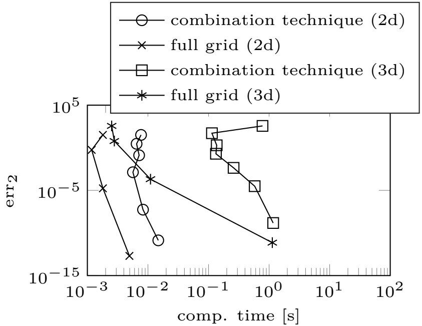
(d) $u_2, k = 2$



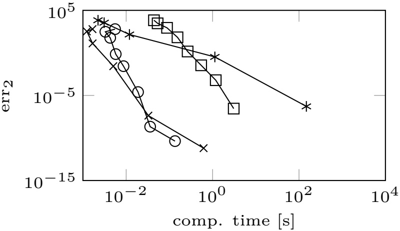
(e) $u_3, k = 3$



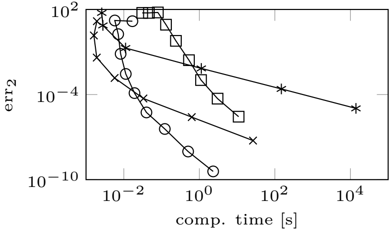
(f) $u_4, k = 3$



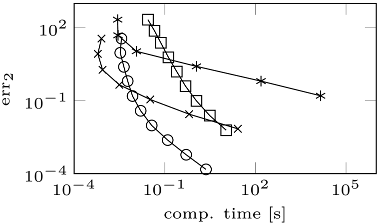
(g) $u_1, k = 1$



(h) $u_2, k = 2$



(i) $u_3, k = 3$



(j) $u_4, k = 3$

Time Discretization - Alternating Direction Implicit Schemes

Time Discretization - ADI Schemes

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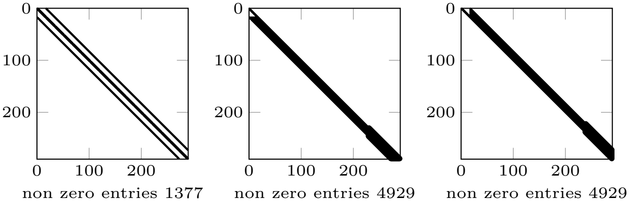
- Semi discrete system of ODEs

$$U'(t) = F(t)U(t), \qquad t \geq 0,$$

- Discretization in time via 'standard' techniques

$$U_{n+1} = U_n + (1-\theta)\Delta_t F(n\Delta_t)U_n + \theta F((n+1)\Delta_t)U_{n+1},$$

$\theta = 0$: explicit Euler, $\theta = 1$: implicit Euler, $\theta = 0.5$: Crank-Nicolson.



(a) $I - \theta\Delta_t F(n\Delta_t)$ (b) L (c) U

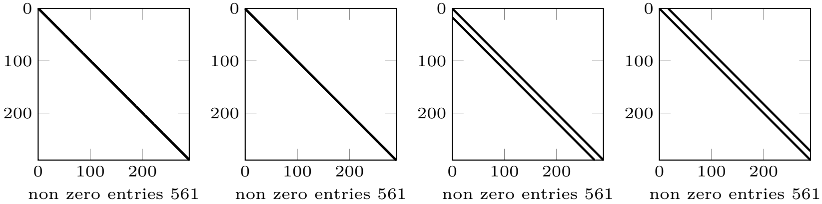
Time Discretization - ADI Schemes

- The spatial discretization matrix is decomposed into

$$F(t) = F_0(t) + F_1(t) + \ldots + F_d(t),$$

where F_0 stems from all mixed derivatives and F_i from each unidirectional contribution of coordinate direction $i = 1, ..., d$.

- With the help of ADI time stepping the equation system can be solved as a sequence of one-dimensional problems.



(a) L_1 (b) U_1 (c) L_2 (d) U_2

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[Douglas, 1962] Douglas scheme (DO):

$$\begin{cases} Y_0 &= U_n + \Delta_t F(t) U_n, \\ Y_i &= Y_{i-1} + \theta \Delta_t \left(F_i(t) Y_i - F_i(t) U_n \right) \ \text{for } i = 1, ..., d \\ U_{n+1} &= Y_d. \end{cases}$$

- Order two in time if $F_0 = 0$, order one otherwise.

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[Craig and Sneyd, 1988] Craig-Sneyd scheme (CS):

$$\begin{cases} Y_0 & = U_n + \Delta_t F(t) U_n, \\ Y_i & = Y_{i-1} + \theta \Delta_t \left(F_i(t) Y_i - F_i(t) U_n \right) \text{ for } i = 1, ..., d \\ \tilde{Y}_0 & = Y_0 + \frac{1}{2} \Delta_t \left(F_0 Y_d - F_0 U_n \right) \\ \tilde{Y}_i & = \tilde{Y}_{i-1} + \theta \Delta_t \left(F_i(t) \tilde{Y}_i - F_i(t) U_n \right) \text{ for } i = 1, ..., d \\ U_{n+1} & = \tilde{Y}_d. \end{cases}$$

- Order two in time iff $\theta = 0.5$.

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[in't Hout and Welfert, 2009] Modified Craig-Sneyd scheme (MCS):

$$\begin{cases} Y_0 &= U_n + \Delta_t F(t) U_n, \\ Y_i &= Y_{i-1} + \theta \Delta_t \left(F_i(t) Y_i - F_i(t) U_n\right) \text{ for } i = 1, ..., d \\ \hat{Y}_0 &= Y_0 + \theta \Delta_t \left(F_0 Y_d - F_0 U_n\right) \\ \tilde{Y}_0 &= \hat{Y}_0 + (\tfrac{1}{2} - \theta) \Delta_t \left(F(t) Y_d - F(t) U_n\right) \\ \tilde{Y}_i &= \tilde{Y}_{i-1} + \theta \Delta_t \left(F_i(t) \tilde{Y}_i - F_i(t) U_n\right) \text{ for } i = 1, ..., d \\ U_{n+1} &= \tilde{Y}_d. \end{cases}$$

- Order two in time for any $\theta > 0$.

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[Hundsdorfer, 2002] Hundsdorfer-Verwer scheme (HV):

$$
\begin{cases}
Y_0 & = U_n + \Delta_t F(t) U_n, \\
Y_i & = Y_{i-1} + \theta \Delta_t \left(F_i(t) Y_i - F_i(t) U_n \right) \text{ for } i = 1, \dots, d \\
\tilde{Y}_0 & = Y_0 + \frac{1}{2} \Delta_t \left(F(t) Y_d - F(t) U_n \right) \\
\tilde{Y}_i & = \tilde{Y}_{i-1} + \theta \Delta_t \left(F_i(t) \tilde{Y}_i - F_i(t) Y_d \right) \text{ for } i = 1, \dots, d \\
U_{n+1} & = \tilde{Y}_d,
\end{cases}
$$

- Order two in time for any $\theta > 0$.

Time Discretization - ADI Schemes

- Stability analysis within the von Neumann framework [in't Hout and Mishra, 2011, in't Hout and Mishra, 2013, in't Hout and Welfert, 2009, in't Hout and Welfert, 2007, Lanser et al., 2001, Mishra, 2016].
- Convection-diffusion equation with constant coefficients

$$\frac{\partial u}{\partial t} = \mathrm{div}(A\nabla u) + c \cdot \nabla u$$

  with symmetric and positive semi-definite matrix $A = (a_{ij})$, and $c = (c_1, c_2, ..., c_d)^\top$.
- Consider the FD scheme given in one step form

$$U_{n+1} = RU_n,$$

  where R denotes the iteration matrix.

Theorem 1 in [Hendricks et al., 2016a]

- Diffusion equation with periodic BCs in 2-d or 3-d
- Symmetric positive semi-definite coefficient matrix A

HO-ADI schemes are unconditionally stable with lower bound on θ:

HO Douglas scheme

$$\theta \geq \frac{1}{2} \text{ if d} = 2 \qquad \theta \geq \frac{2}{3} \text{ if d} = 3$$

HO Craig-Sneyd scheme

$$\theta \geq \frac{1}{2} \text{ if d} = 2, 3$$

HO modified Craig-Sneyd scheme

$$\theta \geq \frac{1}{3} \text{ if d} = 2 \qquad \theta \geq \frac{6}{13} \text{ if d} = 3$$

HO Hundsdorfer-Verwer scheme

$$\theta \geq \frac{1}{2+\sqrt{2}} \text{ if d} = 2 \qquad \theta \geq \frac{3}{4+2\sqrt{3}} \text{ if d} = 3$$

Theorem 2 in [Hendricks et al., 2016a]

- Diffusion equation with periodic BCs with $\mathrm{d} \geq 2$
- Symmetric positive semi-definite coefficient matrix A

HO-ADI schemes need to fulfill lower bound on θ for unconditional stability:

HO Douglas scheme

$$\theta \geq \frac{1}{2}\mathrm{d}(1-\tfrac{1}{\mathrm{d}})^{\mathrm{d}-1},$$

HO Craig-Sneyd scheme

$$\theta \geq \max\left\{\frac{1}{2}, \frac{1}{2}\mathrm{d}(1-\tfrac{1}{\mathrm{d}})^{\mathrm{d}}\right\},$$

HO modified Craig-Sneyd scheme

$$\theta \geq \frac{1}{2}\frac{\mathrm{d}}{1+(\frac{\mathrm{d}}{\mathrm{d}-1})^{\mathrm{d}-1}},$$

HO Hundsdorfer-Verwer scheme

$$\theta \geq \frac{1}{2}\mathrm{d}\mathrm{a}_{\mathrm{k}},$$

where a_{k} is the unique solution $\mathrm{a} \in \left(0, \frac{1}{2}\right)$ of $2\mathrm{a}\left(1+\frac{1-\mathrm{a}}{\mathrm{d}-1}\right)^{\mathrm{d}-1} - 1 = 0.$

- Both Theorems can be proven within the von Neumann framework by considering the scalar stability functions

$$
\begin{aligned}
r_{DO}(z_0, z_1, ..., z_d) &= 1 + \tfrac{z}{p}, \\
r_{CS}(z_0, z_1, ..., z_d) &= 1 + \tfrac{z}{p} + \tfrac{1}{2}\tfrac{z_0\, z}{p^2}, \\
r_{MCS}(z_0, z_1, ..., z_d) &= 1 + \tfrac{z}{p} + \theta\tfrac{z_0\, z}{p^2} + (\tfrac{1}{2} - \theta)\tfrac{z^2}{p^2}, \\
r_{HV}(z_0, z_1, ..., z_d) &= 1 + 2\tfrac{z}{p} - \tfrac{z}{p^2} + \tfrac{1}{2}\tfrac{z^2}{p^2}.
\end{aligned}
$$

  with $p = (1 - \theta z_1) \cdot ... \cdot (1 - \theta z_d)$ and $z = z_0 + z_1 + ... + z_d$, where the eigenvalue z_i stems from F_i for $i = 0, 1, ..., d$.

- And exploiting, that for the eigenvalues it holds $z_i \in \mathbb{R}$ for all i, $z_i \le 0$ for $i = 1, \dots, d$, $z_0 + z_1 + \dots z_d \le 0$ and $|z_0| \le \sum_{i \ne j} \sqrt{z_i z_j}$, see [Hendricks et al., 2016a].

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Lemma 1 in [Hendricks et al., 2017]

Let $d = 2$ and the HO-ADI schemes be applied to the convection-diffusion problem with symmetric positive semi-definite coefficient matrix A. Then it holds for the eigenvalues

$$\mathrm{Re}(z_1) \leq 0, \qquad \mathrm{Re}(z_2) \leq 0 \qquad \text{and} \qquad |z_0| \leq 2\sqrt{\mathrm{Re}(z_1) \cdot \mathrm{Re}(z_2)}. \tag{1}$$

Lemma 1 in [Hendricks et al., 2017]

Let $d = 2$ and the HO-ADI schemes be applied to the convection-diffusion problem with symmetric positive semi-definite coefficient matrix A. Then it holds for the eigenvalues

$$\mathrm{Re}(z_1) \le 0, \qquad \mathrm{Re}(z_2) \le 0 \qquad \text{and} \qquad |z_0| \le 2\sqrt{\mathrm{Re}(z_1)\cdot \mathrm{Re}(z_2)}. \tag{1}$$

Let (1) hold and further

Let $z_0, z_1, z_2 \in \mathbb{C}$ and $\theta \ge \frac{1}{2}$, then $|r_{\mathrm{DO}}| \le 1$ and $|r_{\mathrm{CS}}| \le 1$.

— [in't Hout and Welfert, 2007]

Suppose $|r_{\mathrm{MCS}}(z_0, z_1, z_2)| \le 1$ for all $z_0 \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$ holds, then $\theta \ge \frac{2}{5}$.

— [in't Hout and Mishra, 2011]

Let $z_0 = 0$, then $|r_{\mathrm{HV}}| \le 1$ for arbitrary $z_1, z_2 \in \mathbb{C}$ if and only if $\theta \ge \frac{1}{2} + \frac{1}{6}\sqrt{3}$.

— [Lanser et al., 2001]

- Stability regions for HO ADI scheme coincide with the stability regions of their second-order FD counterparts.
- For convection-diffusion problems with more than two spatial dimensions stability cannot be guaranteed. However, HO ADI schemes show stable behavior for moderate convection in numerical experiments, see [Hendricks et al., 2017].
- Pseudo-spectral ADI methods show a stable behavior in numerical experiments, see [Hendricks et al., 2018].

Application to Option Pricing

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- Multi-dimensional Black-Scholes PDE with $\mathrm{d} \in \mathbb{N}$ assets

$$\frac{\partial \mathrm{u}}{\partial \mathrm{t}} + \frac{1}{2} \sum_{\mathrm{i,j}=1}^{\mathrm{d}} \sigma_\mathrm{i} \sigma_\mathrm{j} \rho_{\mathrm{i,j}} \mathrm{s_i s_j} \frac{\partial^2 \mathrm{u}}{\partial \mathrm{s_i} \partial \mathrm{s_j}} + \sum_{\mathrm{i}=1}^{\mathrm{d}} \mathrm{r s_i} \frac{\partial \mathrm{u}}{\partial \mathrm{s_i}} - \mathrm{ru} = 0,$$

$$\mathrm{u}(\mathrm{s}_1, \mathrm{s}_2, \ldots, \mathrm{s_d}, \mathrm{T}) = \left(\mathrm{K} - \sum_{\mathrm{i}=1}^{\mathrm{d}} \mathrm{s_i} \right)^+,$$

in the space-time cylinder $\Omega_\mathrm{d} \times \Omega_\mathrm{t}$ with $\Omega_\mathrm{d} = [0, \infty)^\mathrm{d}$, $\Omega_\mathrm{t} = [0, \mathrm{T}]$.

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- Logarithmic transformation $x_i = \log(s_i)$ for $i = 1, ..., d$, $\tau = T - t$, $u = e^{r\tau} u$ yields

$$\frac{\partial u}{\partial \tau} - \frac{1}{2} \sum_{i,j=1}^{d} \rho_{ij}\sigma_i\sigma_j \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{i=1}^{d} \left(r - \tfrac{1}{2}\sigma_i^2\right) \frac{\partial u}{\partial x_i} = 0,$$

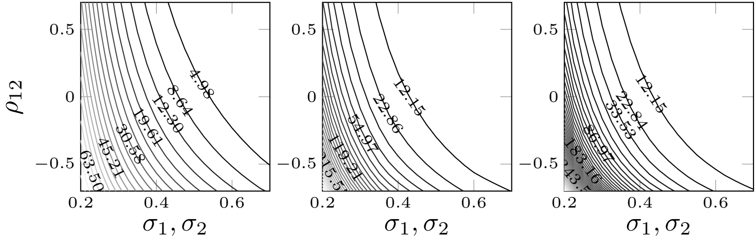
- Payoff transforms to $u(x_1, ..., x_d, 0) = \left(K - \sum_{i=1}^{d} e^{x_i}\right)^+$

- HO ADI scheme in a sparse grid setting
- We obtain the following discretization matrices

$$
\begin{aligned}
A_{x_i} =& \left(1/2\sigma_i^2 + \frac{h_i^2(r - 1/2\sigma_i^2)^2}{6\sigma_i^2}\right) \cdot I_{N_1} \otimes ... \otimes I_{N_{i-1}} \otimes D_{FD_i}^2 \otimes I_{N_{i+1}} \otimes \ldots \otimes I_{N_d} \\
&+ \left(r - 1/2\sigma_i^2\right) \cdot I_{N_1} \otimes ... \otimes I_{N_{i-1}} \otimes D_{FD_i} \otimes I_{N_{i+1}} \otimes \ldots \otimes I_{N_d}, \\
B_{x_i} =& I_{N_1 \cdot N_2 \cdot \ldots \cdot N_d} + \frac{h_i^2}{12} \cdot I_{N_1} \otimes ... \otimes I_{N_{i-1}} \otimes D_{FD_i}^2 \otimes I_{N_{i+1}} \otimes \ldots \otimes I_{N_d} \\
&+ \left(\frac{h_i^2(r - 1/2\sigma_i^2)}{6\sigma_i^2}\right) \cdot I_{N_1} \otimes ... \otimes I_{N_{i-1}} \otimes D_{FD_i} \otimes I_{N_{i+1}} \otimes \ldots \otimes I_{N_d}.
\end{aligned}
$$

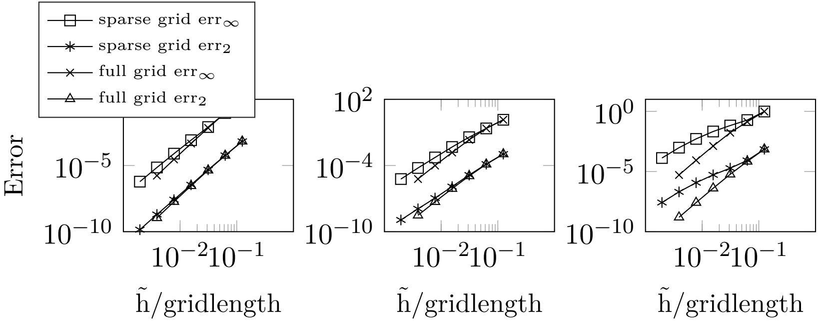
	σ_1	σ_2	σ_3	ρ_{12}	ρ_{13}	ρ_{23}
A	0.6	0.6	0.6	0.2	0.2	0.2
B	0.4	0.4	0.4	0.2	0.2	0.2
C	0.6	0.6	0.6	-0.5	0.5	-0.25

Table: Parameter sets for numerical experiments

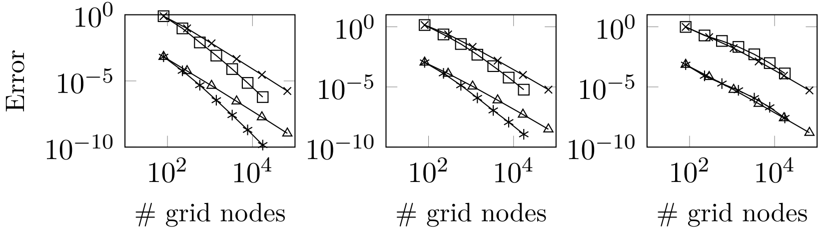


(a) 33^2 grid nodes. (b) 65^2 grid nodes. (c) 129^2 grid nodes.

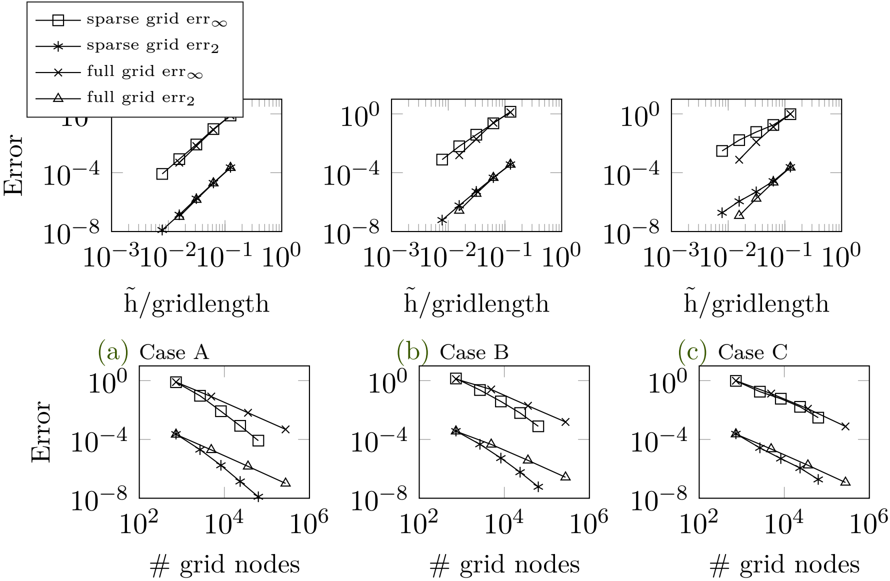
Figure: Maximum of the mixed fourth derivative for a decreasing mesh width h in the 2-d case.



(a) Case A (b) Case B (c) Case C



(d) Case A (e) Case B (f) Case C



(a) Case A (b) Case B (c) Case C

(d) Case A (e) Case B (f) Caset C

Numerical examples - Stochastic volatility models

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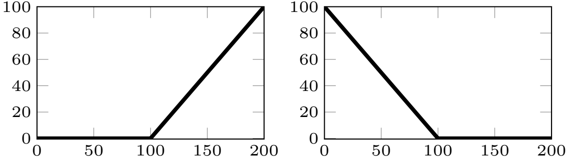
- Heston-Hull-White PDE

$$\begin{aligned}\frac{\partial u}{\partial t} =& \frac{1}{2}s^2 v\frac{\partial^2 u}{\partial s^2} + \frac{1}{2}\sigma_1^2 v\frac{\partial^2 u}{\partial v^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 u}{\partial r^2} \\ &+ \rho_{12}\sigma_1 sv\frac{\partial^2 u}{\partial s\partial v} + \rho_{13}\sigma_2 s\sqrt{v}\frac{\partial^2 u}{\partial s\partial r} + \rho_{23}\sigma_1\sigma_2\sqrt{v}\frac{\partial^2 u}{\partial v\partial r} \\ &+ rs\frac{\partial u}{\partial s} + \kappa(\eta - v)\frac{\partial u}{\partial v} + a_r(b_r - r)\frac{\partial u}{\partial r} - ru,\end{aligned}$$

for inverse time $t \in [0, T]$, asset $s \in [0, \infty)$, volatility $v \in [0, \infty)$ and risk-free interest rate $r \in (-\infty, \infty)$.

Numerical examples - Stochastic volatility models

- For plain-vanilla European options the first derivative of the initial condition has a discontinuity at the strike price K.



(a) Call option (b) Put option

- But no discontinuity in direction of the volatility v and interest rate r.
- Exploit structure by Hybrid Finite Difference/Pseudo-Spectral method:
  - Standard finite differences in s direction.
  - Pseudo-spectral scheme in v and r direction.

Application to the Heston-Hull-White PDE

- Grid transformation in stock direction
  [Tavella and Randall, 2000, in't Hout and Foulon, 2010]

$$\psi_s(x) = \frac{c_1 + \sinh^{-1}(\frac{K-x}{\alpha})}{c_1 - c_2}$$

  where

$$c_1 = \sinh^{-1}(\frac{s_{min}-K}{\alpha}),$$
$$c_2 = \sinh^{-1}(\frac{s_{max}-K}{\alpha}).$$

- Transformation maps $[s_{min}, s_{max}]$ to $[0, 1]$ and clusters grid points around the strike price K.

Application to the Heston-Hull-White PDE

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- Grid transformation in direction of the volatility and interest rate to map the finite interval $[a, b]$ to $[-1, 1]$

$$\psi_1(x) = \frac{2}{b-a}x + \frac{a+b}{a-b}.$$

- In a second step the clustering can be done.

$$\psi_2(x) = e \sinh\left(\frac{1}{2}(x-1)\left(\sinh^{-1}\left(\frac{1-d}{e}\right) + \sinh^{-1}\left(\frac{d+1}{e}\right)\right) + \sinh^{-1}\left(\frac{1-d}{e}\right)\right) + d,$$

  where the parameter $d \in [-1, 1]$ determines the region of clustering and $e > 0$ the degree of non-uniformity of the grid spacing.

- Complete transformation is given by the composition

$$\psi = \psi_2 \circ \psi_1.$$

Application to the Heston-Hull-White PDE

- Heston-Hull-White

$$
\begin{aligned}
\frac{\partial u}{\partial t} =& \frac{1}{2}s^2 v\left[\psi_s'(s)^2\frac{\partial^2 u}{\partial x_1^2} + \psi_s''(s)\frac{\partial u}{\partial x_1}\right] + \frac{1}{2}\sigma_1^2 v\left[\psi_v'(v)^2\frac{\partial^2 u}{\partial x_2^2} + \psi_v''(v)\frac{\partial u}{\partial x_2}\right]\\
&+ \frac{1}{2}\sigma_2^2\left[\psi_r'(r)^2\frac{\partial^2 u}{\partial x_3^2} + \psi_r''(r)\frac{\partial u}{\partial x_3}\right]\\
&+ \rho_{12}\sigma_1 s v\psi_s'(s)\psi_v'(v)\frac{\partial^2 u}{\partial x_1\partial x_2} + \rho_{13}\sigma_2 s\sqrt{v}\psi_s'(s)\psi_r'(r)\frac{\partial^2 u}{\partial x_1\partial x_3}\\
&+ \rho_{23}\sigma_1\sigma_2\sqrt{v}\psi_v'(v)\psi_r'(r)\frac{\partial^2 u}{\partial x_2\partial x_3}\\
&+ rs\psi_s'(s)\frac{\partial u}{\partial x_1} + \kappa(\eta - v)\psi_v'(v)\frac{\partial u}{\partial x_2} + a_r(b_r - r)\psi_r'(r)\frac{\partial u}{\partial x_3} - ru,
\end{aligned}
$$

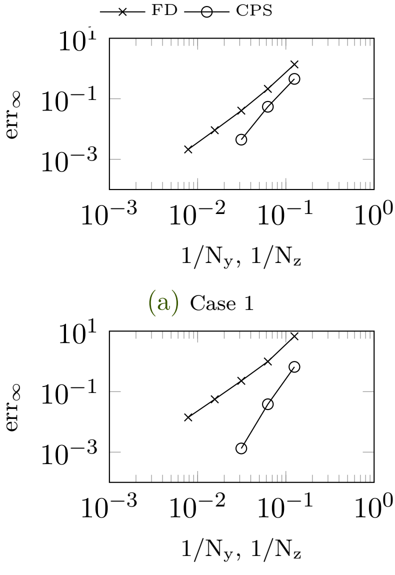
where $s = \psi_s^{-1}(x_1)$, $v = \psi_v^{-1}(x_2)$ and $r = \psi_r^{-1}(x_3)$ with $(x_1, x_2) \in \tilde{\Omega} = [0,1] \times [-1,1]$ and $(x_1, x_2, x_3) \in \tilde{\Omega} = [0,1] \times [-1,1]^2$.

Numerical examples - Stochastic volatility models

	Case 1	Case 2	Case 3	Case 4
K	100	100	100	100
T	1	1	3	0.5
σ_1	0.3	0.04	0.2928	0.5
ρ_{12}	-0.9	-0.6	-0.7571	-0.5
κ	1.5	3	0.6067	2
η	0.04	0.12	0.0707	0.02
r	0.025	0.04	0.03	0.01
a_r	0.00883	0.2	0.05	0.15
b_r	0.025	0.05	0.055	0.101
σ_2	0.00631	0.06	0.03	0.1
ρ_{13}	0 (0.6)	0 (0.2)	0 (0.6)	0 (-0.3)
ρ_{23}	0 (-0.7)	0 (0.4)	0 (-0.2)	0 (0.2)

Table: Scenarios for numerical tests
[in't Hout and Foulon, 2010, Haentjens and in't Hout, 2012].

Convergence in volatility, interest rate direction (HHW)



(a) Case 1 (b) Case 2

(c) Case 3 (d) Case 4

Convergence in volatility, interest rate direction (HHW)

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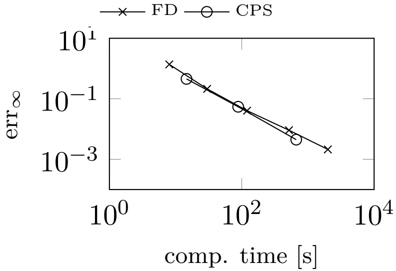
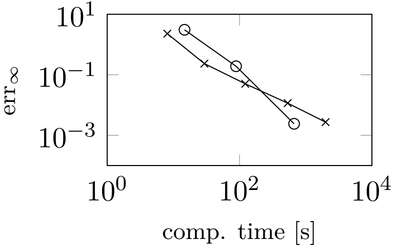
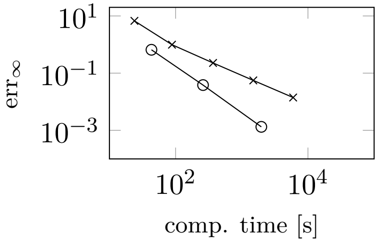


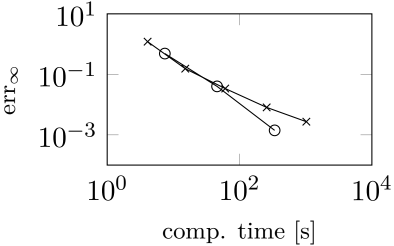
(a) Case 1



(b) Case 2



(c) Case 3



(d) Case 4

Conlusion

- Derived and analyzed numerical schemes to price European options in a multivariate Black-Scholes setting and under stochastic volatility.
- Applied High-order-compact and Pseudo-spectral discretizations in space.
- Proposed Hybrid schemes to efficiently exploit the regularity in the certain coordinate directions.
- ADI time stepping methods to reduce the computational complexity.
- Showed that the stability bounds of central second-order finite difference ADI and HO ADI schemes coincide.
- Sparse grid combination technique to reduce the number of degrees of freedoms.

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