Unsteady Inflow Boundary Conditions for Compressible Laminar and Turbulent Flows

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Abstract

A method to impose unsteady inflow boundary conditions in compressible flow calculations is proposed. It is based on a modification of the Navier Stokes Characteristic Boundary Conditions (NSCBC) method [13] to extend it to multidimensional Navier Stokes equations. Its performance is compared to classical boundary conditions for the injection of a two dimensional vortex and a turbulent field in a domain. It is shown that when the present method is used, unsteady inflow conditions can be imposed with no distorsion or noise induction.

Keywords: compressible Navier–Stokes equations, characteristic boundary conditions, inflow boundary conditions, direct numerical simulation, turbulence injection.

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Adequate boundary conditions are an essential ingredient of successful compressible Navier–Stokes (NS) calculations. This is especially true in the domain of direct numerical simulation, where high order non–dissipative schemes can not damp errors generated by boundary conditions.

In direct simulations of turbulent fields, defining an inlet through which turbulence can be injected permits to avoid the simulation of transition to turbulence and to maintain turbulence in otherwise decaying fields. Different physical boundary conditions have been proposed to handle subsonic injection (the supersonic case poses no special problem [11, 10]). These include non–reflecting boundary conditions [5, 1], by definition incompatible with the injection of a prescribed non stationary field, and imposition of u, v, w, T on the boundary plus a fifth condition (usually $\frac{\partial p}{\partial n} = 0$ [6]), which results in an overdetermination in the boundary conditions. A third approach consists in adding a "buffer zone" through which the flow is injected at supersonic speed and gradually decelerated to reach the original boundary at a subsonic speed [4]. The present paper proposes a systematic procedure to impose such inlet conditions, based on characteristic analysis [16, 13].

1 Characteristic treatment of boundary conditions for the Navier–Stokes equations

Characteristic treatments of boundary conditions for hyperbolic equations are based on the analysis of characteristic waves crossing boundaries to identify information leaving or flowing into the computational domain. Physical boundary conditions (imposed by the outside) can only be related to incoming waves while waves traveling from the inside of the domain to the outside are completely determined by the interior field.

Such an analysis, originally used for the Euler equations [15, 16], is the basis of the Navier–Stokes Characteristic Boundary Conditions (NSCBC) method [13]. For a bounding surface normal to the x direction, characteristic analysis of the hyperbolic part of Navier–Stokes equations permits to detect five waves W_1, W_2, W_3, W_4, W_5 crossing the boundaries at speeds: $\lambda_1 = u - c$, $\lambda_2 = u$, $\lambda_3 = u$, $\lambda_4 = u$ and $\lambda_5 = u + c$. Their amplitudes are related to the convective terms in NS equations through:

$$-\frac{\partial}{\partial t}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \end{bmatrix} = \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \\ \mathcal{L}_5 \end{bmatrix} = \begin{bmatrix} \lambda_1\left(\frac{\partial p}{\partial x} - \rho c\frac{\partial u}{\partial x}\right) \\ \lambda_2\left(c^2\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\right) \\ \lambda_3\frac{\partial v}{\partial x} \\ \lambda_4\frac{\partial w}{\partial x} \\ \lambda_5\left(\frac{\partial p}{\partial x} + \rho c\frac{\partial u}{\partial x}\right) \end{bmatrix} \tag{1}$$

where x is the direction normal to the boundary, ρ, p, u, v w the density, thermodynamic pressure and components of the velocity fields and c the speed of sound ($(\gamma p/\rho)^{1/2}$). Depending on the sign of the velocities λ_i these five waves represent information traveling from the interior of the domain to the exterior or vice versa.

In certain situations the physical boundary conditions can be directly imposed, namely when the values of the variables on the boundary are dictated by the properties of the surface (for example, imposed velocities). Other physical boundary conditions can not be directly written in terms of values of the variables, but have to be expressed as relations between them (for example, non–reflecting boundaries). Moreover, the number of physical boundary conditions is usually smaller than the number of variables of the problem and values of the variables which are not fixed by the physical boundary conditions must be obtained by solving the NS equations themselves on the boundary, in which inflow waves amplitudes ($\mathcal{L}_i$) must be specified in a compatible way with physical boundary conditions.

For characteristic methods such as [16, 5, 13, 1], once values can be specified for the $\mathcal{L}_i$ corresponding to incoming waves, the physical variables on the boundary can be obtained by integration of the NS equations. In some situations, physical conditions can directly suggest the adequate values of the $\mathcal{L}_i$ variables, this is the approach used in [16] and [5]. Other situations may imply more complex relations between the $\mathcal{L}_i$.

Poinsot and Lele [13] propose a systematic approach to find the values of the $\mathcal{L}_i$. A local one dimensional inviscid (LODI) approximation of the NS equations is used to relate the amplitudes of incoming and outcoming waves and the variations of variables in the boundary (Eq. (24–36) in [13]), so that any physical boundary condition can be written in terms of $\mathcal{L}_i$ corresponding to incoming waves, while outgoing $\mathcal{L}_i$'s are determined by the interior solution. The values obtained for the $\mathcal{L}_i$'s together with all terms computed from the inside (including thus viscous and transverse terms) plus viscous boundary conditions are then used to advance in time the Navier–Stokes equations in the boundary. This method ensures compatibility of the numerical boundary conditions with the physical boundary conditions through the NS equations used in the domain. In fact, as the LODI relations are approximate, exact compatibility is ensured only with the inviscid one dimensional NS equations. It has been applied to subsonic injection in one dimensional situations [2], but when a two or three dimensional field is being injected through the boundary the error introduced by supposing that the LODI relations hold can be significant.

We propose here to extend the NSCBC method and write the compatibility relations as a three dimensional inviscid approximation, that is, to include the transverse convective terms in the determination of relations between the variables, so that compatibility with the three–dimensional

NS equations is ensured:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ p\\ u\\ v\\ w\end{bmatrix}+\begin{bmatrix}\frac{\mathcal{L}_2+(\mathcal{L}_5+\mathcal{L}_1)/2}{c^2}\\ \frac{\mathcal{L}_5+\mathcal{L}_1}{2}\\ \frac{\mathcal{L}_5-\mathcal{L}_1}{2\rho c}\\ \mathcal{L}_3\\ \mathcal{L}_4\end{bmatrix}+\begin{bmatrix}v&0&0&\rho&0\\ 0&v&0&\gamma p&0\\ 0&0&v&0&0\\ 0&1/\rho&0&v&0\\ 0&0&0&0&v\end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix}\rho\\ p\\ u\\ v\\ w\end{bmatrix}+\begin{bmatrix}w&0&0&0&\rho\\ 0&w&0&0&\gamma p\\ 0&0&w&0&0\\ 0&0&0&w&0\\ 0&1/\rho&0&0&w\end{bmatrix}\frac{\partial}{\partial z}\begin{bmatrix}\rho\\ p\\ u\\ v\\ w\end{bmatrix}=0 \tag{2}$$

and, equivalently, for temperature:

$$\frac{\partial T}{\partial t}+\frac{T}{\rho c^2}\left[-\mathcal{L}_2+\frac{(\gamma-1)}{2}(\mathcal{L}_1+\mathcal{L}_5)\right]+T(\gamma-1)\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=0 \tag{3}$$

The resulting method is an extension of NSCBC in which different compatibility relations are retained to allow turbulence injection. It maintains the fundamental behavior of that method, in terms of well posedness, stability or accuracy. The reader is referred to [13] to find the detailed theoretical basis of the method as well as implementation and performance studies in simple reference cases. The NSCBC technique guarantees the well posedness of the Euler part of Navier–Stokes equations (as the number and kind of conditions coincides with theoretical results ([14, 3]), but no formal proof of well-posedness has been derived for the complete NS equations. Well-posedness and stability of boundary conditions have been studied by many authors [14, 7, 12, 9, 3, 8] and are still the subject of intense research. This could be an interesting additional study, but is out of the aims of the present paper. Numerical results have already shown however, that the proposed modification did not alter the quality of the results in terms of well posedness or stability.

In what follows a test of the capability of the extended method for an unsteady subsonic inlet is presented. A laminar case in which a vortex is injected into the computing domain is first presented. This case was selected as an illustrative example of the behavior of the present method as compared to existing ones. A second test case shows its performance in imposing a subsonic turbulent inflow.

2 Characteristic boundary conditions for subsonic injection

In the case of a subsonic inlet along the x axis, the speeds of characteristic waves across the boundary are:

$$\lambda_1 = u - c < 0; \qquad \lambda_2 = u > 0; \qquad \lambda_3 = u > 0; \qquad \lambda_4 = u > 0; \qquad \lambda_5 = u + c > 0.$$

Four waves ($\mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4$ and $\mathcal{L}_5$) enter the domain while only one ($\mathcal{L}_1$) is outgoing. This means that four boundary conditions are necessary and sufficient to close the system, the fifth variable being completely determined by them and the interior solution. (Note that in the case of a supersonic inlet as $u > c$ the five waves would be traveling from the exterior to the interior of the domain, and would thus correspond to five physical boundary conditions. That is why there is no special problem for boundary conditions for supersonic injection).

Characteristic analysis can help in determining which variables can be imposed and what combinations result in a well posed problem. For example, imposing at the same time the values of u and p in the boundary is inconsistent, as this would fix $\mathcal{L}_1$, which should be determined by the interior field. Proper choices for the imposed variables are the three components of velocity and T or two components of the velocity plus two thermodynamic variables. In the first case, velocity and temperature being totally controlled, pressure and density would fluctuate, while in the second case the component of velocity will fluctuate. Thus to define an inlet through which a flow is to be injected with given velocities one should impose the values of u, v, w and T in the boundary.

To determine the fifth variable (ρ or p) a common methodology [6] is to impose:

$$\partial p/\partial x = 0. \tag{4}$$

This method is simple to implement but does not ensure compatibility with the NS equations. It may affect the calculation of the velocity fields or create non physical acoustic noise.

The present method uses a numerical condition for the fifth variable that is compatible with the rest of conditions from Navier Stokes equations in terms of wave amplitudes. Mass conservation on the boundary can be written in terms of waves as:

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right) = -\frac{1}{c^2}\left[\mathcal{L}_2 + \frac{1}{-}\left(\mathcal{L}_5 + \mathcal{L}_1\right)\right] - \frac{\partial \rho v}{\partial y} - \frac{\partial \rho w}{\partial z}$$

$\mathcal{L}_1$ corresponds to an outgoing wave and is to be calculated from inside the domain as $\mathcal{L}_1 = \lambda_1\left(\frac{\partial p}{\partial x} - \rho c \frac{\partial u}{\partial x}\right)$ (Eq. (1)), which, in a finite-differences calculation, means calculating the derivative at the boundary with an upwind scheme. $\mathcal{L}_2$ and $\mathcal{L}_5$ can be determined through the relations (2) and (3) in terms of the imposed values of the variables $u = U(t), v = V(t), w = W(t)$ and $T = T^0(t)$ and the outgoing wave as:

$$\begin{aligned}
\mathcal{L}_5 &= \mathcal{L}_1 - 2\rho c\left(\tfrac{dU}{dt} - v\tfrac{\partial u}{\partial y} - w\tfrac{\partial u}{\partial z}\right) \\
\mathcal{L}_2 &= \tfrac{1}{2}\left(\gamma - 1\right)\left(\mathcal{L}_5 + \mathcal{L}_1\right) + \rho\left(\gamma - 1\right)\left[c^2\left(\tfrac{\partial v}{\partial y} + \tfrac{\partial w}{\partial z}\right) + C_p\left(v\tfrac{\partial T}{\partial y} + w\tfrac{\partial T}{\partial z} + \tfrac{dT^0}{dt}\right)\right]
\end{aligned}$$

where C_p is the specific heat capacity at constant pressure.

Thus ρ can be updated at the boundary by integrating:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\frac{1}{c^2}\gamma\left[\mathcal{L}_1 - \rho c\left(\frac{dU}{dt} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\right] - \frac{\rho}{T}\left(\frac{dT^0}{dt} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) \\
&- \rho\gamma\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \left(v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right)
\end{aligned} \tag{5}$$

where all terms are to be calculated making use of interior or parallel to the boundary derivatives except for the temporal derivatives of the imposed variables, that are imposed by the injected field.

If the LODI relations had been used to obtain the values of $\mathcal{L}_2$ and $\mathcal{L}_5$ the resulting equation for ρ would have been:

$$\frac{\partial \rho}{\partial t} = -\frac{1}{c^2}\left[\gamma\left(\mathcal{L}_1 - \rho c\frac{dU}{dt}\right) + \frac{\rho c^2}{T}\frac{dT^0}{dt}\right] - \frac{\partial \rho v}{\partial y} - \frac{\partial \rho w}{\partial z}, \tag{6}$$

that, based on one dimensional relations, results in the multidimensional case in a misinterpretation of dU/dt and dT/dt, the total variation of the variables, as corresponding to normal convection terms. If the transverse terms ($\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$) are important this can lead to a value of ρ on the boundary that is not compatible with the Navier Stokes equations.

To illustrate the effect of using equations (4), (5) and (6) as boundary conditions, a simulation of the injection of a two dimensional vortex into a domain has been performed using the three methods. In figure 1 the resulting vorticity, pressure and dilatation fields are plotted. The boundary condition based in the LODI relations given by Eq. (6) (labeled NSCBC 1D) introduces deformations in the injected field while the boundary condition proposed in Eq. (5) (labeled NSCBC 3D) appears to permit the proper injection of the vortex. When condition (4) ($dp/dx = 0$) is used, high levels of dilatation are observed in the inlet section $x = 0$ and wiggles appear in the field indicating inconsistency of this condition with the NS equations solved in the interior field.

In a second simulation, an acoustic wave is initialized in the domain, propagating towards the boundary simultaneously with the vortex injection. In figure 2 the initial pressure and u fields corresponding to the centerline of the 2D domain and their evolution using the NSCBC technique (condition 5) and condition 4 are plotted. Spurious wiggles are detected in the fields that correspond to this last condition, that are symptomatic of inadequate boundary conditions.

3 Application to the injection of subsonic turbulence

To show the feasibility of a subsonic compressible simulation in which turbulence is maintained by continuous injection through one of the boundaries a two–dimensional case has been selected, the three dimensional one being equivalent. A two dimensional homogeneous isotropic turbulent field with a prescribed energy spectrum and $Re_T = 60$ was created in a $L \times L$ domain and injected into a 2D domain initially at rest with dimensions $5L \times L$ at a subsonic convective speed $u_i = c/10$.

Figure 3 shows the instantaneous vorticity field in the calculation domain obtained by injection after times corresponding to one ($t = L/u_i$), three ($t = 3L/u_i$) and five complete initial fields having been injected ($t = 5L/u_i$). The initial $L \times L$ vorticity field is also represented on top of the figure: turbulence is injected with no distortion into the computing domain and decays spatially because of viscous effects.

In order to be able to quantitatively validate the method, a temporal periodic simulation in which the 2D turbulent field was let to evolve and decay naturally was run. The time evolution of spatial statistical quantities of this homogeneous isotropic freely decaying turbulence was then compared to the time averaged statistics of the spatial simulation in which the homogeneous isotropic turbulence is continuously injected and decays in space (using Taylor's assumption). If injection is properly made and no noise is introduced by the boundary conditions, the decay in time of quantities as the kinetic energy, dissipation vorticity and velocity variance in homogeneous isotropic turbulence should coincide with the spatial decay of the time statistics of the injection simulation.

Figure 4 shows the results of such a comparison, in which the freely decaying isotropic turbulence results are labeled as "temporal decay" and the spatially developing injected turbulence results labeled as "spatial decay", and the x axis represents time in the first case and an "equivalent time", calculated as the distance to the inlet divided by injection speed, in the second case. Statistics of the spatially evolving field have been averaged along a period of time corresponding to the injection of a complete initial field (L/u_i). The comparison shows that the injected turbulent field suffers no distortion and maintains its statistical properties and thus its evolution along the convection direction is similar to the evolution of homogeneous isotropic turbulence in time.

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List of Figures

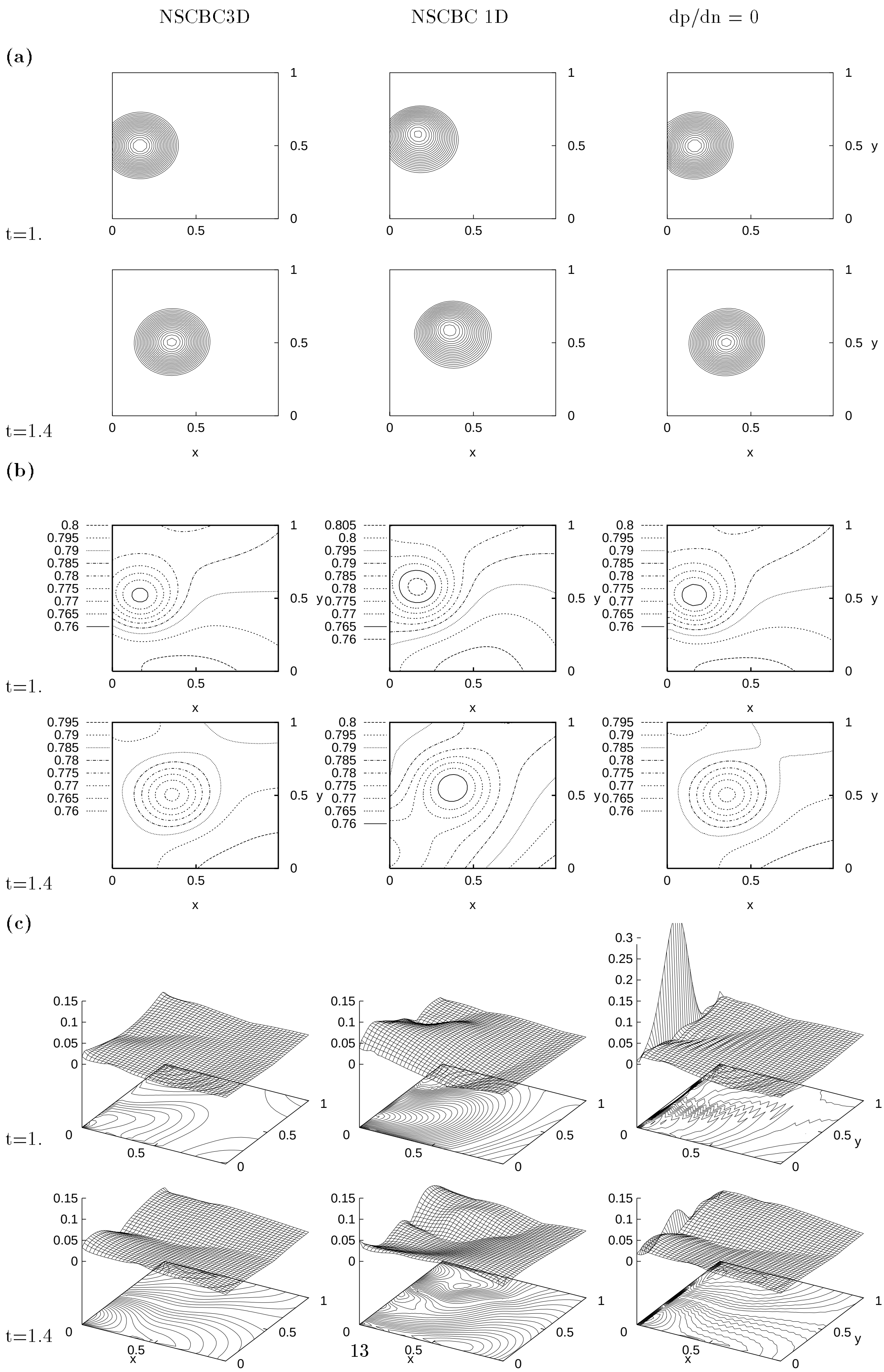

Figure 1: FIG 1

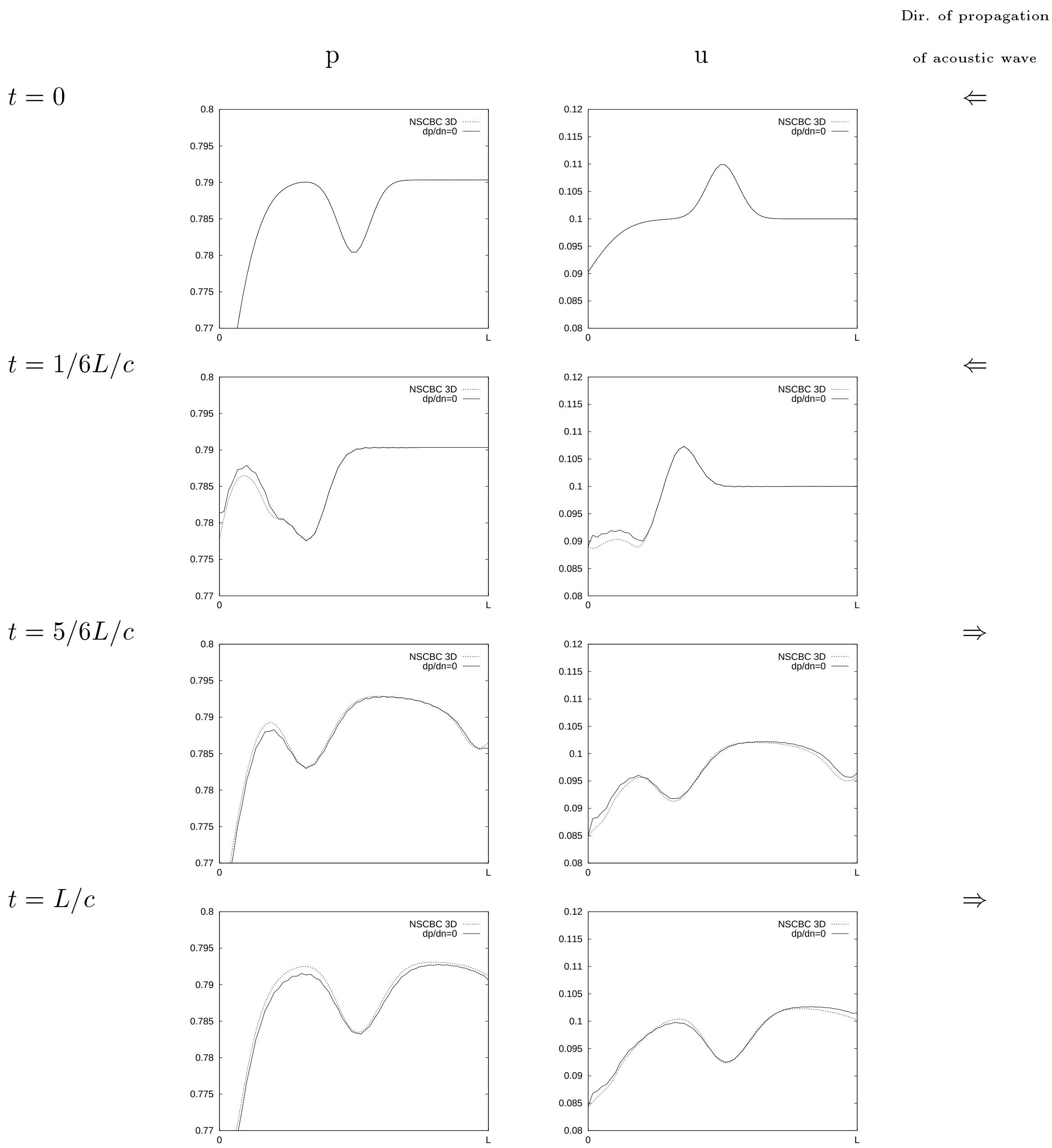
Dir. of propagation
of acoustic wave
p
u
t = 0
t = 1/6L/c
t = 5/6L/c
t = L/c
NSCBC 3D
dp/dn=0
0
L

Figure 2: .

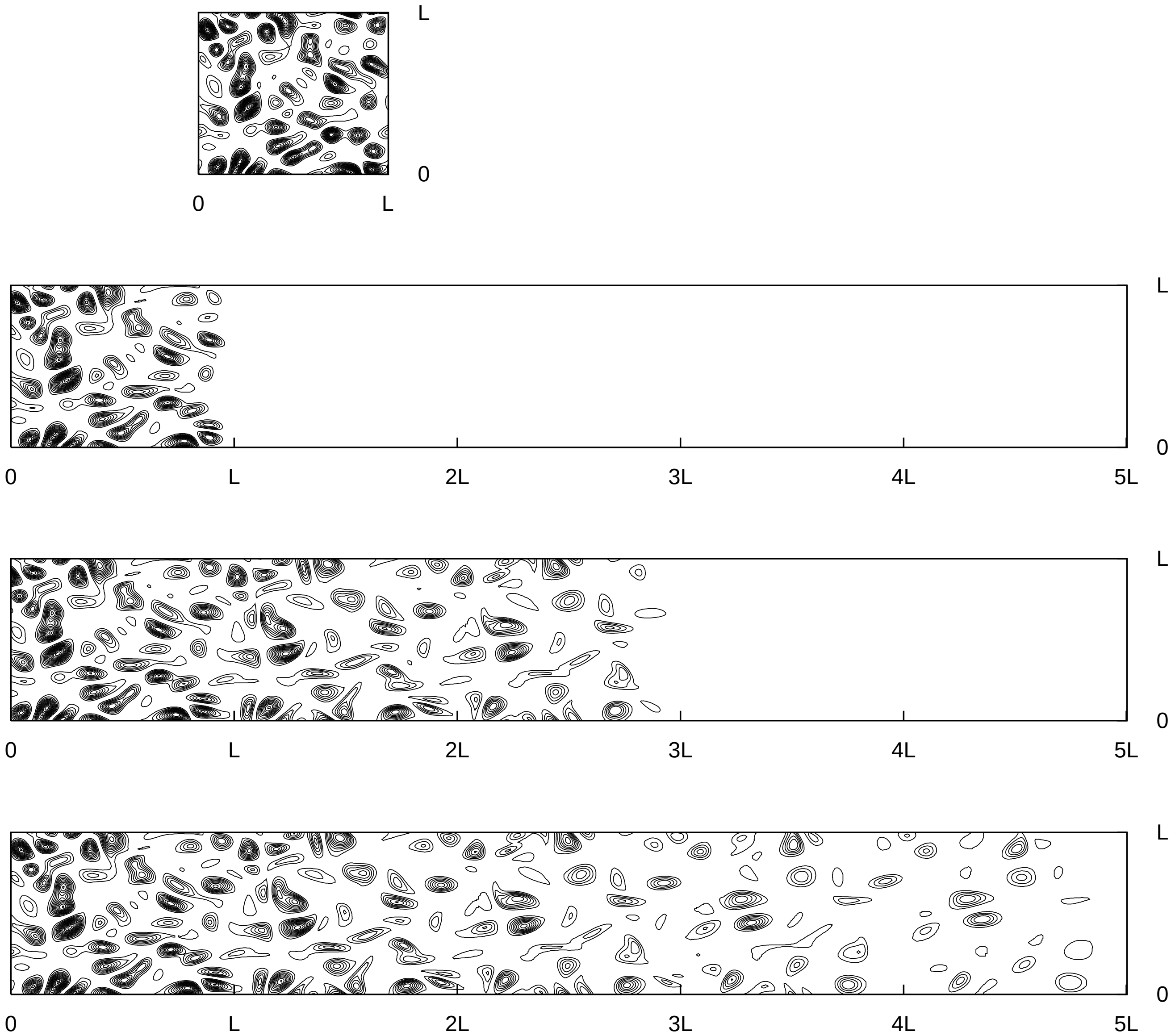

Figure 3: .

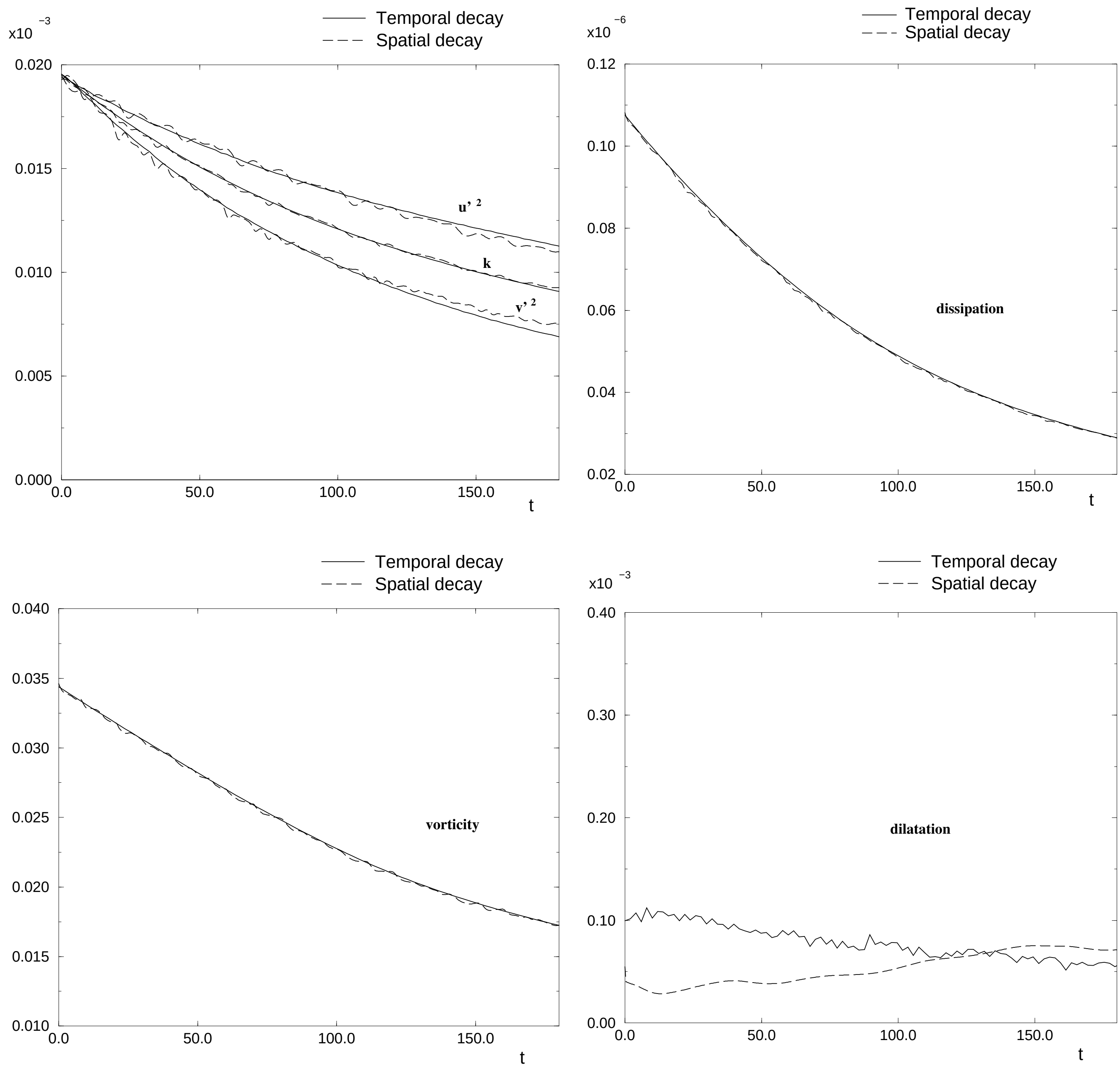
Temporal decay
Spatial decay
x10 −3
u' 2
k
v' 2
t
x10 −6
dissipation
t
vorticity
t
x10 −3
dilatation
t

Figure 4: .